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ERGODIC MEASURES ON COMPACT METRIC SPACES FOR ISOMETRIC ACTIONS BY INDUCTIVELY COMPACT GROUPS

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Abstract. We obtain a partial converse of Vershik's description of ergodic probability measures on a compact metric space with respect to an isometric action by an inductively compact group. This allows us to identify, in this setting, the set of ergodic probability measures with the set of weak limit points of orbital measures. We also show that for a general action of an inductively compact group, the weak limit of orbital measures can fail to be ergodic.

1. Introduction

Let

$$\mathcal{K}(1) \subset \cdots \subset \mathcal{K}(n) \subset \cdots \subset \mathcal{K}(\infty)$$

be an increasing chain of topological groups such that $\mathcal{K}(n)$'s are compact groups and $\mathcal{K}(\infty)$ is the corresponding inductive limit group

$$\mathcal{K}(\infty) = \varinjlim \mathcal{K}(n).$$

Such a group $\mathcal{K}(\infty)$ is called an inductively compact group. Vershik and Kerov developed a very successful method in the study of invariant measures of an inductively compact group; see Kerov [2], Olshanski and Vershik [3]. See also Thoma [4] for his groundbreaking work on the infinite symmetric group.

In this note, we will investigate the ergodic probability measures on a compact metric space with respect to an *isometric action* by an inductively compact group.

Let $\mathcal{X}$ be a separable metric complete space. Denote the metric on $\mathcal{X}$ by $d_{\mathcal{X}}(\cdot, \cdot)$. Assume that $\mathcal{K}(\infty)$ acts on $\mathcal{X}$ by Borel actions. By a result of Vershik, all ergodic probability measures on $\mathcal{X}$ are the weak limits of orbital measures. Let us state this more precisely. Denote by m_n the normalized Haar measure on $\mathcal{K}(n)$. For any point $x \in \mathcal{X}$, we denote by μ_n^x the unique $\mathcal{K}(n)$-invariant probability measure on the orbit

$$\mathcal{K}(n) \cdot x = \{y \in \mathcal{X} : y = g \cdot x \text{ for some } g \in \mathcal{K}(n)\}.$$

Denote by $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}_{\mathrm{erg}}^{\mathcal{K}(\infty)}(\mathcal{X})$ the set of all Borel probability measures on $\mathcal{X}$ and the set of all ergodic $\mathcal{K}(\infty)$-invariant probability measures on $\mathcal{X}$ respectively.

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Recall that a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Borel probability measures on $\mathcal{X}$ converges weakly to a Borel probability measure $\nu \in \mathcal{P}(\mathcal{X})$ if for any bounded continuous function f, we have

$$\int_{\mathcal{X}} f d\nu = \lim_{n\to\infty} \int_{\mathcal{X}} f d\nu_n.$$

In this situation, we denote $\nu_n \Longrightarrow \nu$ as $n \to \infty$.

Definition 1.1 (Limit of orbital measures)**.** Define

$$\mathscr{ORB}^{\mathcal{K}(\infty)}(\mathcal{X}) \subset \mathcal{P}(\mathcal{X})$$

to be the set of probability measures μ such that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of positive integers verifying

$$\mu_{n_k}^{x_k} \Longrightarrow \mu.$$

Theorem 1.2 (Vershik [5, Theorem 1])**.** *Let μ be an ergodic $\mathcal{K}(\infty)$-invariant Borel probability measure on $\mathcal{X}$. Then for μ-almost every point $x \in \mathcal{X}$, we have $\mu_n^x \Longrightarrow \mu$ as $n \to \infty$. In particular,*

$$\mathcal{P}_{\mathrm{erg}}^{\mathcal{K}(\infty)}(\mathcal{X}) \subset \mathscr{ORB}^{\mathcal{K}(\infty)}(\mathcal{X}).$$

The purpose of this note is to give a partial converse of Theorem 1.2.

Theorem 1.3. *Let $\mathcal{X}$ be a compact metric space. Assume that $\mathcal{K}(\infty)$ acts on $\mathcal{X}$ by isometric isomorphisms. If $\mu \in \mathcal{P}(\mathcal{X})$ is a weak limit point of the sequence $(\mu_n^{x_n})_{n\in\mathbb{N}}$, where x_n's are points in $\mathcal{X}$, then μ is ergodic.*

Corollary 1.4. *Let $\mathcal{X}$ be a compact metric space. Assume that $\mathcal{K}(\infty)$ acts on $\mathcal{X}$ by isometric isomorphisms. Then*

$$\mathcal{P}_{\mathrm{erg}}^{\mathcal{K}(\infty)}(\mathcal{X}) = \mathscr{ORB}^{\mathcal{K}(\infty)}(\mathcal{X}). \tag{1.1}$$

Remark 1.5. Denote by H the space of infinite Hermitian matrices. The group $U(\infty)$ of infinite unitary group acts on H by conjugations. As a by-product of Olshanski and Vershik's approach to Pickrell's classification theorem [3, Corollary 4.2], the following relation holds:

$$\mathcal{P}_{\mathrm{erg}}^{U(\infty)}(H) = \mathscr{ORB}^{U(\infty)}(H). \tag{1.2}$$

It would be interesting to understand this coincidence. At this moment, however, the author does not see a direct approach for obtaining (1.2).

We also show that, in general, the assumption that the action is isometric in Theorem 1.3 cannot be dropped.

Theorem 1.6. *There exists an inductively compact group $\mathcal{K}(\infty)$, a compact metric space $\mathcal{X}$, on which the group $\mathcal{K}(\infty)$ acts by homeomorphisms, and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathcal{X}$, such that none of the probability measures on $\mathcal{X}$ that are weak limits of $(\mu_n^{x_n})_{n\in\mathbb{N}}$ are ergodic.*

2. Proof of Theorem 1.3

Our proof is based on the following Reverse Martingale type result from [1]. For introducing this result, we need more notation. Given any bounded measurable function $f : \mathcal{X} \to \mathbb{R}$, we may define, for any positive integer $n \in \mathbb{N}$, a function $A_n f$ by

$$(A_n f)(x) := \int_{\mathcal{X}} f d\mu_n^x = \int_{\mathcal{K}(n)} f(g \cdot x) m_n(dg). \tag{2.1}$$

Let ν be any $\mathcal{K}(\infty)$-invariant probability measure. By using the Reverse Martingale theorem, one may prove that the limit

$$(A_\infty^\nu f)(x) := \lim_{n\to\infty} (A_n f)(x) \tag{2.2}$$

exists for ν-almost every $x \in \mathcal{X}$. Moreover, ν is ergodic if and only if there exists a dense subset $\Psi \subset L_1(\mathcal{X}, \mu)$ such that for any $\psi \in \Psi$, one has

$$(A_\infty^\nu \psi)(x) = \int_{\mathcal{X}} \psi d\nu, \text{ for } \nu\text{-almost every } x \in \mathcal{X}. \tag{2.3}$$

See [1, Propositions 6, 7 and 8] for the details of the above statements.

Let us now proceed to the proof of Theorem 1.3. We start with the following two lemmas.

Lemma 2.1. *The continuous functions in the sequence $(A_n f)_{n\in\mathbb{N}}$ are equicontinuous; that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $y, z \in \mathcal{X}$ satisfy $d_{\mathcal{X}}(y, z) \le \delta$, then*

$$\sup_{n\in\mathbb{N}} |A_n f(y) - A_n f(z)| \le \varepsilon.$$

Proof. Since f is a continuous function on the compact space $\mathcal{X}$, it is uniformly continuous. Thus for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $y, z \in \mathcal{X}$ satisfy $d_{\mathcal{X}}(y, z) \le \delta$, then $|f(y) - f(z)| \le \varepsilon$. Thus under the above condition on y, z, since $\mathcal{K}(\infty)$ acts on $\mathcal{X}$ by isometric isomorphisms, for any $g \in \mathcal{K}(\infty)$, we have

$$d_{\mathcal{X}}(g \cdot y, g \cdot z) = d_{\mathcal{X}}(y, z) \le \delta.$$

Hence $|f(g \cdot y) - f(g \cdot z)| \le \varepsilon$. It follows that for any $n \in \mathbb{N}$, we have

$$|A_n f(y) - A_n f(z)| \le \left| \int_{\mathcal{K}(n)} f(g \cdot y) m_n(dg) - \int_{\mathcal{K}(n)} f(g \cdot z) m_n(dg) \right| \le \varepsilon.$$

This proves the equicontinuity of the family $\{A_n f : n \in \mathbb{N}\}$. □

Lemma 2.2. *Let $\mathcal{X}$ be a compact metric space. Assume that $\mathcal{K}(\infty)$ acts on $\mathcal{X}$ by isometric isomorphisms. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$. Assume that as n goes to infinity, we have*

$$x_n \to x_0 \text{ and } \mu_n^{x_n} \Longrightarrow \mu.$$

Then $\mu_n^{x_0} \Longrightarrow \mu$ as $n \to \infty$.

Proof. Let f be any continuous function on $\mathcal{X}$. By assumption, we have

$$\lim_{n\to\infty} \int_{\mathcal{X}} f d\mu_n^{x_n} = \int_{\mathcal{X}} f d\mu.$$

For any $\varepsilon > 0$, let $\delta > 0$ be chosen such that whenever $d_{\mathcal{X}}(x, y) \le \delta$, then $|f(x) - f(y)| \le \varepsilon$. There exists n_0, such that whenever $n \ge n_0$, we have $d_{\mathcal{X}}(x_n, x_0) \le \delta$. Using the assumption that the group action is isometric, we obtain that for any $n \ge n_0$ and any $g \in \mathcal{K}(\infty)$,

$$|f(g \cdot x_n) - f(g \cdot x_0)| \le \varepsilon.$$

Consequently, for any $n \geq n_0$, we have

$$\left| \int_{\mathcal{X}} f d\mu_n^{x_n} - \int_{\mathcal{X}} f d\mu_n^{x_0} \right| \leq \int_{\mathcal{K}(n)} |f(g \cdot x_n) - f(g \cdot x_0)| \, m_n(dg) \leq \varepsilon.$$

It follows that

$$\limsup_{n \to \infty} \left| \int_{\mathcal{X}} f d\mu_n^{x_n} - \int_{\mathcal{X}} f d\mu_n^{x_0} \right| = 0.$$

Hence the relation

$$\lim_{n \to \infty} \int_{\mathcal{X}} f d\mu_n^{x_0} = \int_{\mathcal{X}} f d\mu$$

holds for any continuous function f on $\mathcal{X}$, that is,

$$\mu_n^{x_0} \Longrightarrow \mu \text{ as } n \to \infty.$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 1.3. Let μ be a probability measure on $\mathcal{X}$ which is a weak limit point of the sequence $(\mu_n^{x_n})_{n \in \mathbb{N}}$, where x_n's are points in $\mathcal{X}$. By assumption, there exists a subsequence of positive integers $n_1 < n_2 < \cdots < \cdots$ such that

$$\mu_{n_i}^{x_{n_i}} \Longrightarrow \mu \text{ as } i \to \infty. \tag{2.4}$$

Since $\mathcal{X}$ is compact, by passing to a further subsequence if necessary, we may assume that there exists $x_0 \in \mathcal{X}$ such that

$$\lim_{i \to \infty} x_{n_i} = x_0. \tag{2.5}$$

By Lemma 2.2, the relations (2.4) and (2.5) imply that

$$\mu_{n_i}^{x_0} \Longrightarrow \mu \text{ as } i \to \infty.$$

Note that if we replace the chain $\mathcal{K}(1) \subset \cdots \mathcal{K}(n) \subset \cdots$ by the chain $\mathcal{K}(n_1) \subset \cdots \mathcal{K}(n_i) \subset \cdots$, then both chains define the same inductive limit group $\mathcal{K}(\infty)$. By this observation, without loss of generality, we may thus assume that

$$\mu_n^{x_0} \Longrightarrow \mu \text{ as } n \to \infty.$$

First note that if $x \in \mathcal{K}(\infty) \cdot x_0$, then

$$\mu_n^{x} \Longrightarrow \mu \text{ as } n \to \infty. \tag{2.6}$$

Indeed, this follows from the elementary fact that once $x \in \mathcal{K}(N) \cdot x_0$, then for any $n \geq N$, we have $\mu_n^x = \mu_n^{x_0}$. By definition of weak convergence of probability measures, for any bounded continuous function $f : \mathcal{X} \to \mathbb{R}$, we have

$$\lim_{n \to \infty} \int_{\mathcal{X}} f(y) \mu_n^x(dy) = \int_{\mathcal{X}} f(y) \mu(dy) \quad (x \in \mathcal{K}(\infty) \cdot x_0).$$

In the notation (2.1), this means that

$$\lim_{n \to \infty} (A_n f)(x) = \int_{\mathcal{X}} f d\mu \quad (x \in \mathcal{K}(\infty) \cdot x_0). \tag{2.7}$$

Since the support $\mathrm{supp}(\mu_n^x)$ of the measure μ_n^x equals $\mathcal{K}(n) \cdot x$, we see that $\mathrm{supp}(\mu_n^x)$ is a subset of the closure $\mathrm{clos}(\mathcal{K}(\infty) \cdot x_0)$ of the orbit $\mathcal{K}(\infty) \cdot x_0$. By (2.6), we get

$$\mathrm{supp}(\mu) \subset \mathrm{clos}(\mathcal{K}(\infty) \cdot x_0). \tag{2.8}$$

By using the Arzelà-Ascoli theorem, (2.7) can actually be strengthened to

$$\lim_{n\to\infty}(A_n f)(x) = \int_{\mathcal{X}} f d\mu \quad (x \in \operatorname{clos}(\mathcal{K}(\infty)\cdot x_0)). \tag{2.9}$$

For the sake of completeness, let us give more details for the above assertion. First, if $x \in \operatorname{clos}(\mathcal{K}(\infty)\cdot x_0)$, then the sequence $((A_n f)(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges. Indeed, for any $\varepsilon > 0$, let $\delta > 0$ be chosen as in Lemma 2.1. We may find $y \in \mathcal{K}(\infty)\cdot x_0$ such that $d_{\mathcal{X}}(y, x) \le \delta$, hence

$$\sup_{n\in\mathbb{N}} |A_n f(y) - A_n f(x)| \le \varepsilon.$$

It follows that

$$\begin{aligned} |A_n f(x) - A_m f(x)| \le& |A_n f(x) - A_n f(y)| + |A_n f(y) - A_m f(y)| \\ &+ |A_m f(y) - A_m f(x)| \\ \le& 2\varepsilon + |A_n f(y) - A_m f(y)|. \end{aligned} \tag{2.10}$$

Note that by (2.7), $(A_n f(y))_{n\in\mathbb{N}}$ is a Cauchy sequence. Consequently, by (2.10), so is the sequence $((A_n f)(x))_{n\in\mathbb{N}}$. Now we have proved that the sequence of functions $(A_n f|_{\operatorname{clos}(\mathcal{K}(\infty)\cdot x_0)})$, all defined on a compact set $\operatorname{clos}(\mathcal{K}(\infty)\cdot x_0)$, is uniformly bounded and equicontinuous and converges pointwise; hence by the Arzelà-Ascoli theorem, it converges uniformly on $\operatorname{clos}(\mathcal{K}(\infty)\cdot x_0)$. This completes the proof of (2.9).

Finally, in view of (2.8), we have proved that

$$\lim_{n\to\infty}(A_n f)(x) = \int_{\mathcal{X}} f d\mu \text{ for } \mu\text{-almost every } x \in \mathcal{X}. \tag{2.11}$$

Take $\Psi = C(\mathcal{X})$, the set of continuous functions on $\mathcal{X}$. Since Ψ is dense in $L^1(\mathcal{X}, \mu)$, we may apply the characterization of ergodic measures (2.3) to conclude that μ is ergodic. □

3. Proof of Theorem 1.6

Let $\mathcal{K}(n) = \mathbb{Z}/2^n\mathbb{Z}$. Note that we have the natural inclusion

$$\mathcal{K}(n) = \mathbb{Z}/2^n\mathbb{Z} \simeq 2\mathbb{Z}/2^{n+1}\mathbb{Z} \subset \mathbb{Z}/2^{n+1}\mathbb{Z} = \mathcal{K}(n+1).$$

The corresponding group inductive limit $\mathcal{K}(\infty)$ is called the Prüfer 2-group and is usually denoted by $\mathbb{Z}(2^\infty)$. Obviously, $\mathbb{Z}(2^\infty)$ is a countable group, which is inductively compact but not compact. Let $\mathcal{X} = \{0,1\}^{\mathcal{K}(\infty)}$; then $\mathcal{X}$ is a compact metrizable space. For instance, we may fix the following metric on $\mathcal{X}$: if $x = (x(g))_{g\in\mathcal{K}(\infty)}$ and $y = (y(g))_{g\in\mathcal{K}(\infty)}$, then

$$d_{\mathcal{X}}(x, y) = \sum_{n=1}^{\infty} 4^{-n} |\{g \in \mathcal{K}(n) : x(g) \ne y(g)\}|,$$

where $|\{g \in \mathcal{K}(n) : x(g) \ne y(g)\}|$ denotes the cardinality of the finite set $\{g \in \mathcal{K}(n) : x(g) \ne y(g)\}$. The group $\mathcal{K}(\infty)$ acts naturally on $\{0,1\}^{\mathcal{K}(\infty)}$ by translations. This action is not an isometric action.

Let us fix the natural inclusions $\mathcal{K}(n) \subset \mathcal{K}(n+1)$. For any $n \in \mathbb{N}$, we have

$$\mathcal{K}(n+1) = \mathcal{K}(n) \sqcup (\mathcal{K}(n+1) \setminus \mathcal{K}(n)).$$

Hence $\mathcal{K}(n+1)$ is the union of two $\mathcal{K}(n)$-cosets in $\mathcal{K}(n+1)$. Now we set $x_n \in \mathcal{X}$ as follows: $x_n : \mathcal{K}(\infty) \to \{0,1\}$ is a function defined inductively by the following: on $\mathcal{K}(n)$, we set $x_n|_{\mathcal{K}(n)} = \mathbb{1}_{\mathcal{K}(n-1)}$; on $\mathcal{K}(n+1)$, we extend the definition of x_n such that its restriction on the nontrivial $\mathcal{K}(n)$-coset $g_n + \mathcal{K}(n)$ with $g_n \in \mathcal{K}(n+1) \setminus \mathcal{K}(n)$ is a translation of $x_n|_{\mathcal{K}(n)}$; if we have defined $x_n|_{\mathcal{K}(m)}$ for a positive integer $m \geq n$, then we may extend it to a function $x_n|_{\mathcal{K}(m+1)}$ such that the restriction of $x_n|_{\mathcal{K}(m+1)}$ on the nontrivial $\mathcal{K}(m)$-coset $g_m + \mathcal{K}(m)$ with $g_m \in \mathcal{K}(m+1) \setminus \mathcal{K}(m)$ is given by

$$x_n|_{g_m+\mathcal{K}(m)}(\cdot) = x_n|_{\mathcal{K}(m)}(\cdot - g_m).$$

It is not hard to see that the orbit $\mathcal{K}(n) \cdot x_n$ consists of two points (both viewed as $\{0,1\}$-valued functions defined on $\mathcal{K}(\infty)$):

$$\mathcal{K}(n) \cdot x_n = \{x_n, 1 - x_n\},$$

and we have

$$\mu_n^{x_n} = \frac{1}{2}(\delta_{x_n} + \delta_{1-x_n}),$$

where δ_{x_n} and δ_{1-x_n} are Dirac measures on x_n and $1 - x_n$ respectively. By construction, we have the following convergence in the space $\mathcal{X}$:

$$\lim_{n\to\infty} x_n = \bar{1} \text{ and } \lim_{n\to\infty}(1 - x_n) = \bar{0},$$

where $\bar{0}, \bar{1} \in \mathcal{X} = \{0,1\}^{\mathcal{K}(\infty)}$ are constant functions on $\mathcal{K}(\infty)$ taking values 0 and 1 respectively. Consequently,

$$\mu_n^{x_n} \Longrightarrow \frac{1}{2}(\delta_{\bar{1}} + \delta_{\bar{0}}) \text{ as } n \to \infty.$$

Since the singletons $\{\bar{0}\}$ and $\{\bar{1}\}$ are both $\mathcal{K}(\infty)$-invariant, the weak limit probability measure $\frac{1}{2}(\delta_{\bar{1}} + \delta_{\bar{0}})$ is not ergodic. This completes the proof of Theorem 1.6.

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