

VOLUME RENORMALIZATION FOR SINGULAR YAMABE METRICS

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1. INTRODUCTION

In [LN], Loewner and Nirenberg introduced what is now known as the singular Yamabe problem. The case of interest here can be formulated as follows. Given a smooth compact Riemannian manifold-with-boundary (M^{n+1}, \bar{g}) , find a defining function u for $\partial M = \Sigma$ so that the scalar curvature of the metric $g = u^{-2}\bar{g}$ on \dot{M} satisfies $R_g = -n(n+1)$. This problem has an obvious conformal invariance: the rescaling $\hat{\bar{g}} = \Omega^2\bar{g}$, where $0 < \Omega \in C^\infty(M)$, induces the rescaling $\hat{u} = \Omega u$ leaving g unchanged. Hence the datum for the problem can be considered as a conformal class of metrics on M , and its solution produces from the datum a canonical conformally compact metric in the conformal class on \dot{M} .

It follows from results in [LN], [AM], [ACF] that for $n \geq 2$ there always exists a unique solution u . However, u might not be smooth up to ∂M . In [ACF], Andersson, Chruściel and Friedrich showed that u has an asymptotic expansion involving powers of r and $\log r$, where r is a smooth defining function for Σ . Moreover, the expansion to order $n+2$ is locally determined, and there is a single conformally invariant, locally determined density on Σ , the coefficient of the first $\log r$ term $r^{n+2} \log r$ in the expansion, which, if nonzero, obstructs smoothness of u .

In [GoW2], [GoW3], [GGHW], Gover and Waldron and collaborators have developed the singular Yamabe problem as a tool for studying the geometry of a hypersurface in a conformal manifold, in the process applying and further developing their boundary calculus for conformally compact manifolds ([GoW1]). A starting point for these investigations was the observation that when $n = 2$ and \bar{g} is Euclidean, the obstruction identified in [ACF] is the Willmore invariant of Σ , i.e. the variational derivative of the Willmore energy with respect to variations of Σ . This led them to interpret the obstruction in general as a higher-dimensional generalization of the Willmore invariant. In [GoW2], they raised the question of whether in higher dimensions, there is a conformally invariant energy, generalizing the Willmore energy, whose variational derivative is the singular Yamabe obstruction. The main purpose of this paper is to show that such an energy can be constructed by renormalizing the volume of the singular Yamabe metric. After we described this work to Gover and Waldron and showed them our proof, they posted [GoW4], which

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also discusses volume renormalization for singular Yamabe metrics and proves the same result.

The volume renormalization process carried out here is the generalization to singular Yamabe metrics of volume renormalization for Poincaré-Einstein metrics, introduced in the physics literature and described in [G1]. We consider the asymptotics as $\epsilon \rightarrow 0$ of $\text{Vol}_g(\{r > \epsilon\})$, where r is the distance to Σ in the metric \bar{g} . The volume form of g has a pointwise expansion (2.9), (2.10) whose coefficients $v^{(k)}$ are invariants of the hypersurface Σ in the Riemannian manifold (M, \bar{g}) . These coefficients generalize to the singular Yamabe problem the Poincaré-Einstein renormalized volume coefficients which have been the subject of recent investigations ([GJ], [CF], [J1], [G2], [CFG], [J2]). Upon integration, the pointwise expansion produces an expansion (2.11) for $\text{Vol}_g(\{r > \epsilon\})$ whose coefficients (2.12) are constant multiples of the integrals of the $v^{(k)}$. Our energy is the coefficient of the $\log \frac{1}{\epsilon}$ term in the renormalized volume expansion. It is the integral of $v^{(n)}$ and is invariant under conformal rescalings of \bar{g} . In [GoW4], a version of Q -curvature is defined for the singular Yamabe setting, and the energy is shown to equal the integral of the Q -curvature. The fact that the variation of the energy with respect to Σ is the log term coefficient in the solution u is an analog of the result of [HSS], [GH] that in the Poincaré-Einstein case with even-dimensional boundary, the metric variation of the log term coefficient in the volume expansion is a constant multiple of the ambient obstruction tensor, the log term coefficient in the expansion of the Poincaré-Einstein metric itself.

There is a similar renormalization for the area of minimal submanifolds of Poincaré-Einstein spaces, also introduced and used extensively in the physics literature. Forthcoming work with Nicholas Reichert will analyze in some detail the corresponding energy in this setting, introduced in [GrW], and, among other things, will prove the analogous result: for even-dimensional submanifolds, the submanifold variation of the log term coefficient in the renormalized area expansion agrees with the obstruction to smoothness for minimal extension. Unlike the extension problems for Poincaré-Einstein metrics and minimal submanifolds, in which log terms occur only for even-dimensional boundaries, smoothness for the singular Yamabe problem is generically obstructed in all dimensions $n > 1$ ([GoW3]). Another distinction is that in the singular Yamabe problem, the conformal rescaling happens in the same space in which the extension problem is posed rather than on the boundary at infinity.

Note about terminology. In discussing renormalized volume generally, we use *energy* for the coefficient of the log term in the volume expansion, *renormalized volume* for the constant term, and *anomaly* for the difference of the renormalized volumes corresponding to different choices of conformal representatives. The energy is conformally invariant and is the integral of a local scalar invariant of the geometry determined by fixing a metric in the conformal class. The renormalized volume is global and in general is not conformally invariant. The anomaly is the integral of a locally determined nonlinear differential operator which depends on local background geometry applied to the conformal factor. In the case of a constant conformal factor, the anomaly reduces to a multiple of the energy. But in general, the anomaly contains more information. For pure conformal geometry, the linearized anomaly determines a particular integrand (namely $v^{(n)}$) for the energy, so fixes divergence terms. Anomalies associated to submanifolds contain still more

information: since the rescaling occurs in the full space but the integration is over the submanifold, normal derivatives of the conformal factor appear in the anomaly as well.

In §2 we review the asymptotics of solutions of the singular Yamabe problem, discuss the renormalized volume expansion, and describe how Poincaré-Einstein volume renormalization is a special case. In §3 we formulate and prove that the variation of the energy is the singular Yamabe obstruction. In §4 we carry out the calculations outlined in §2 far enough to identify the first two renormalized volume coefficients $v^{(1)}$, $v^{(2)}$ for general n , and the energy \mathcal{E} and the anomaly for $n = 2$. We also compare the $n = 2$ energy and anomaly with the corresponding quantities derived in [GrW] for the renormalization of the area of a minimal submanifold of the corresponding Poincaré-Einstein space with boundary at infinity equal to Σ .

2. ASYMPTOTICS

Let (M^{n+1}, \bar{g}) , $n \geq 1$, be a Riemannian manifold-with-boundary and denote $\partial M = \Sigma$. We search for a defining function u of Σ so that $g = u^{-2}\bar{g}$ has constant scalar curvature $R_g = -n(n+1)$. Recall the conformal change of scalar curvature in the form:

$$(2.1) \quad R_{u^{-2}\bar{g}} = -n(n+1)|du|_{\bar{g}}^2 + 2nu\Delta_{\bar{g}}u + u^2R_{\bar{g}},$$

where $\Delta_{\bar{g}} = \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta$. So the singular Yamabe problem amounts to solving the equation

$$(2.2) \quad n(n+1) = n(n+1)|du|_{\bar{g}}^2 - 2nu\Delta_{\bar{g}}u - u^2R_{\bar{g}}.$$

The normal exponential map $\exp : [0, \delta)_r \times \Sigma \rightarrow M$ relative to \bar{g} is a diffeomorphism onto a neighborhood of Σ , with respect to which \bar{g} takes the form

$$(2.3) \quad \bar{g} = dr^2 + h_r$$

for a one-parameter family of metrics h_r on Σ . So r is the \bar{g} -distance to Σ and h_0 the induced metric. We use α, β as indices for objects on M , i, j for objects on Σ , and 0 for the r factor. Thus α corresponds to the pair $(i, 0)$ relative to the product identification induced by \exp . The derivatives $\partial_r^k h_r$ at $r = 0$ can be expressed in terms of the curvature of \bar{g} , its covariant derivatives, and the second fundamental form, which we denote L_{ij} . For instance, denoting ∂_r by $'$, one has at $r = 0$,

$$(2.4) \quad h'_{ij} = -2L_{ij}, \quad h''_{ij} = -2\bar{R}_{0i0j} + 2L_{ik}L_j{}^k.$$

For \bar{g} of the form (2.3), we have $\Delta_{\bar{g}}u = \partial_r^2u + \frac{1}{2}h^{ij}h'_{ij}\partial_ru + \Delta_{h_r}u$. Thus (2.2) becomes

$$n(n+1) = n(n+1)((\partial_ru)^2 + h^{ij}\partial_iu\partial_ju) - 2nu(\partial_r^2u + \frac{1}{2}h^{ij}h'_{ij}\partial_ru + \Delta_{h_r}u) - u^2R_{\bar{g}}.$$

Consider the formal asymptotics of u . Setting $r = 0$ and recalling that $u = 0$ when $r = 0$ and $u > 0$ for $r > 0$, one concludes that $\partial_ru = 1$ at $r = 0$. So write $u = r + r^2\varphi$. In terms of φ , the equation becomes

$$(2.5) \quad \begin{aligned} & (1+r\varphi)\left[r^2\varphi_{rr} + 4r\varphi_r + 2\varphi + \frac{1}{2}h^{ij}h'_{ij}(1+2r\varphi+r^2\varphi_r) + r^2\Delta_{h_r}\varphi\right] \\ & - \frac{n+1}{2}\left[2(r\varphi_r + 2\varphi) + r(r\varphi_r + 2\varphi)^2 + r^3h^{ij}\partial_i\varphi\partial_j\varphi\right] \\ & + \frac{1}{2n}r(1+r\varphi)^2R_{\bar{g}} = 0. \end{aligned}$$

The Taylor expansion of φ can be derived by successive differentiation of this equation at $r = 0$. Just setting $r = 0$ gives

$$(2.6) \quad \varphi|_{r=0} = \frac{1}{4n} h^{ij} h'_{ij} = -\frac{1}{2n} H,$$

where $H = h^{ij} L_{ij}$ is the mean curvature. Applying ∂_r^k at $r = 0$ gives

$$(k-n)(k+2)\partial_r^k \varphi|_{r=0} = \text{lots},$$

where lots denotes an expression in lower order derivatives of φ which have already been determined. So $\partial_r^k \varphi|_{r=0}$ is formally determined for $1 \leq k \leq n-1$, and there is a potential obstruction in solving for $\partial_r^n \varphi|_{r=0}$ which can be resolved by including a term in the expansion of φ of the form $r^n \log r$. It follows that we can uniquely determine functions \mathcal{L} and $u^{(k)}$, $2 \leq k \leq n+1$, on Σ so that if we set

$$(2.7) \quad u = r + u^{(2)} r^2 + \cdots + u^{(n+1)} r^{n+1} + \mathcal{L} r^{n+2} \log r,$$

then $g = u^{-2} \bar{g}$ satisfies

$$(2.8) \quad R_g = -n(n+1) + O(r^{n+2} \log r).$$

The log term coefficient \mathcal{L} is the singular Yamabe obstruction, a scalar field on Σ . One sees easily that under a conformal change $\hat{\bar{g}} = \Omega^2 \bar{g}$, \mathcal{L} transforms by $\hat{\mathcal{L}} = (\Omega|_\Sigma)^{-n-1} \mathcal{L}$.

The function u given by (2.7) has been defined near Σ in terms of the product identification determined by the exponential map. Below we consider the asymptotics of the global quantity $\text{Vol}_g(\{r > \epsilon\})$. As our primary interest is in locally determined quantities near Σ , in such global considerations it will suffice to take u to be any positive function on M with an asymptotic expansion which agrees mod $O(r^{n+2})$ with (2.7). According to [ACF], the exact solution u for which $R_g = -n(n+1)$ has this property.

The volume form of $g = u^{-2} \bar{g}$ is given by

$$(2.9) \quad dv_g = u^{-n-1} dv_{\bar{g}} = r^{-n-1} (1+r\varphi)^{-n-1} dv_{h_r} dr = r^{-n-1} (1+r\varphi)^{-n-1} \sqrt{\frac{\det h_r}{\det h_0}} dv_{h_0} dr.$$

We can expand

$$(2.10) \quad (1+r\varphi)^{-n-1} \sqrt{\frac{\det h_r}{\det h_0}} = 1 + v^{(1)} r + v^{(2)} r^2 + \cdots + v^{(n)} r^n + O(r^{n+1} \log r).$$

The coefficients $v^{(k)} \in C^\infty(\Sigma)$ are the singular Yamabe renormalized volume coefficients. It follows that

$$(2.11) \quad \text{Vol}_g(\{r > \epsilon\}) = \int_{r>\epsilon} dv_g = c_0 \epsilon^{-n} + c_1 \epsilon^{-n+1} + \cdots + c_{n-1} \epsilon^{-1} + \mathcal{E} \log \frac{1}{\epsilon} + V + o(1)$$

with

$$(2.12) \quad c_k = \frac{1}{n-k} \int_{\Sigma} v^{(k)} dv_{h_0}, \quad 0 \leq k \leq n-1, \quad \mathcal{E} = \int_{\Sigma} v^{(n)} dv_{h_0}.$$

\mathcal{E} is the singular Yamabe energy of Σ , and V is the renormalized volume of (M, g) with respect to the representative metric \bar{g} . We will see by direct calculation in §4 that if $n = 1$, then $\mathcal{L} = 0$ and $v^{(1)} = 0$, and so also $\mathcal{E} = \int_{\Sigma} v^{(1)} dv_{h_0} = 0$.

Proposition 2.1. \mathcal{E} is invariant under conformal changes of \bar{g} .

Proof. Let $\widehat{g} = e^{2\omega} \bar{g}$ be a conformally related metric, and \widehat{r} the distance to Σ with respect to \widehat{g} . Then $\widehat{r} = e^{\Upsilon} r$ for some smooth function Υ . We need to show that the log term coefficients \mathcal{E} in the volume expansions (2.11) for \bar{g} and \widehat{g} agree. We will derive an expression for the difference of the volume expansions from which this is immediate and which we will use in §4 to calculate the anomaly.

Use the normal exponential map of \bar{g} to identify M near Σ with $[0, \delta)_r \times \Sigma$ as above, and denote points of Σ by x . For fixed x , we can solve the relation $\widehat{r} = e^{\Upsilon(x,r)} r$ for r as a function of \widehat{r} : $r = \widehat{r} b(x, \widehat{r})$, where $b(x, \widehat{r})$ is a smooth nonvanishing function. Set $\widehat{\epsilon}(x, \epsilon) = \epsilon b(x, \epsilon)$. Then $\widehat{r} > \epsilon$ is equivalent to $r > \widehat{\epsilon}(x, \epsilon)$. Recalling (2.9), (2.10), we have

$$\begin{aligned}
 \text{Vol}_g(\{r > \epsilon\}) - \text{Vol}_g(\{\widehat{r} > \epsilon\}) &= \int_{r > \epsilon} dv_g - \int_{\widehat{r} > \epsilon} dv_g \\
 &= \int_{\Sigma} \int_{\epsilon}^{\widehat{\epsilon}} \sum_{0 \leq k \leq n} v^{(k)}(x) r^{-n-1+k} dr dv_{h_0} + o(1) \\
 (2.13) \quad &= \sum_{0 \leq k \leq n-1} \epsilon^{-n+k} \int_{\Sigma} \frac{v^{(k)}(x)}{-n+k} (b(x, \epsilon)^{-n+k} - 1) dv_{h_0} \\
 &\quad + \int_{\Sigma} v^{(n)}(x) \log b(x, \epsilon) dv_{h_0} + o(1).
 \end{aligned}$$

Clearly this expression has no $\log \frac{1}{\epsilon}$ term as $\epsilon \rightarrow 0$. \square

The anomaly $V - \widehat{V}$ measures the failure of conformal invariance of V under the rescaling $\widehat{g} = e^{2\omega} \bar{g}$. Clearly $V - \widehat{V}$ is the constant term in the expansion in ϵ of the last line of (2.13). It follows from this characterization using the kind of analysis that we use in §4 that $V - \widehat{V}$ can be expressed as the integral over Σ of a polynomial expression in ω and its derivatives whose coefficients depend on derivatives of curvature and second fundamental form for \bar{g} .

A Poincaré-Einstein metric g has constant scalar curvature, so is the singular Yamabe metric in its conformal class. If g is written near Σ in asymptotically hyperbolic normal form $g = r^{-2}(dr^2 + h_r)$, and we choose $\bar{g} = dr^2 + h_r$ near Σ , then the associated singular Yamabe defining function u is equal to $r \bmod O(r^{n+2})$, and the volume renormalization expansion (2.11), (2.12) reduces to the usual Poincaré-Einstein volume renormalization. For Poincaré-Einstein metrics, the Taylor expansion of h_r to order n is determined by h_0 and is even in r , and the singular Yamabe renormalized volume coefficients $v^{(k)}$ and energy \mathcal{E} reduce to the corresponding Poincaré-Einstein coefficients for h_0 . If n is odd, the energy and anomaly vanish. The obstruction \mathcal{L} vanishes in all dimensions since $u = r \bmod O(r^{n+2})$ is smooth (the global term in the expansion of h_r at order n and the log term for n even arising from the obstruction tensor do not affect \mathcal{L} since they are trace-free). We note, however, that even for Poincaré-Einstein metrics, the identification of the variation of \mathcal{E} in Theorem 3.1 is a different result from that of [HSS], [GH]. The functional \mathcal{E} is the same in both cases, but Theorem 3.1 varies Σ with the conformal class of g fixed, while [HSS], [GH] vary the conformal infinity h_0 .

3. VARIATIONS

Let again (M, \bar{g}) be our Riemannian manifold, and now let $F_t : \Sigma \rightarrow M$, $0 \leq t < \delta$ be a variation of Σ , i.e. a smoothly varying one-parameter family of embeddings

with $F_0 = \text{Id}$. Set $\Sigma_t = F_t(\Sigma)$ and $\mathcal{E}_t = \int_{\Sigma_t} v_t^{(n)} dv_{\Sigma_t}$. Also set $\dot{F} = \partial_t F|_{t=0} \in \Gamma(TM|_\Sigma)$ and $\dot{\mathcal{E}} = \partial_t \mathcal{E}_t|_{t=0}$. Let \bar{n} denote the inward pointing \bar{g} -unit normal to Σ in M .

Theorem 3.1. *If $n \geq 1$, then*

$$\dot{\mathcal{E}} = (n+2)(n-1) \int_{\Sigma} \langle \dot{F}, \bar{n} \rangle_{\bar{g}} \mathcal{L} dv_{\Sigma}.$$

Remark 3.2. As noted above and derived in §4, both \mathcal{L} and \mathcal{E} vanish in case $n = 1$.

As in [GH], the main step in the proof is to express the variation of $\text{Vol}_g(\{r > \epsilon\})$ as a boundary integral, in which the log term in the expansion of g will appear. In both cases, the metrics have constant scalar curvature, so $-n(n+1) \text{Vol}_g(\{r > \epsilon\}) = \int_{r > \epsilon} R_g dv_g$. In the Poincaré-Einstein case, this is the Einstein-Hilbert action, which is critical for Einstein metrics. In the singular Yamabe case, the variations are within a conformal class, and the total scalar curvature is critical for constant scalar curvature metrics. In both cases, this criticality is used to write the variation as a boundary integral.

Proof. Again use the normal exponential map of \bar{g} to identify M near Σ with $\Sigma \times [0, \delta)_r$. Now $\dot{\mathcal{E}}$ is a linear functional of \dot{F} which depends only on the normal component $\langle \dot{F}, \bar{n} \rangle_{\bar{g}}$, since \mathcal{E} is independent of reparametrizations of Σ . Thus it suffices to take $F_t(x) = (x, tf(x))$ so that $\Sigma_t = \{r = tf(x)\}$ for some function f on Σ . Then $\langle \dot{F}, \bar{n} \rangle_{\bar{g}} = f$. Let r_t denote the geodesic distance to Σ_t . Then the approximate singular Yamabe defining function u_t for Σ_t analogous to (2.7) takes the form

$$(3.1) \quad u_t = r_t + u_t^{(2)} r_t^2 + \cdots + u_t^{(n+1)} r_t^{n+1} + \mathcal{L}_t r_t^{n+2} \log r_t$$

relative to the product decomposition of M determined by the exponential map of Σ_t . This u_t has the property that $g_t = u_t^{-2} \bar{g}$ satisfies $R_{g_t} = -n(n+1) + O(r_t^{n+2} \log r_t)$. Differentiating the volume expansion (2.11) gives

$$\text{Vol}_{g_t}(\{r_t > \epsilon\})' = \dot{c}_0 \epsilon^{-n} + \dot{c}_1 \epsilon^{-n+1} + \cdots + \dot{c}_{n-1} \epsilon^{-1} + \dot{\mathcal{E}} \log \frac{1}{\epsilon} + O(1).$$

So $\dot{\mathcal{E}}$ is the coefficient of $\log \frac{1}{\epsilon}$ in the expansion of $\text{Vol}_{g_t}(\{r_t > \epsilon\})'$.

Now

$$(3.2) \quad \begin{aligned} \text{Vol}_{g_t}(\{r_t > \epsilon\})' &= \left(\int_{r_t > \epsilon} dv_{g_t} \right)' \\ &= \left(\int_{r_t > \epsilon} dv_g \right)' + \int_{r > \epsilon} (u_t^{-(n+1)})' dv_{\bar{g}} \\ &= \left(\int_{r_t > \epsilon} dv_g \right)' - (n+1) \int_{r > \epsilon} \dot{u} u^{-1} dv_g. \end{aligned}$$

We identify the coefficient of $\log \frac{1}{\epsilon}$ in the expansion of each of the two terms on the last line of (3.2).

For the first term, observe that $\{r_t > \epsilon\}$ can alternately be written as $\{r > \psi(x, t, \epsilon)\}$ for a smooth function $\psi(x, t, \epsilon)$. In fact, $\psi(x, t, \epsilon)$ is the r -coordinate of $\exp_{\Sigma_t}((x, tf(x)), \epsilon)$, i.e. the r -coordinate of the point obtained by following for time ϵ the normal geodesic to Σ_t originating from the point $(x, tf(x))$. In particular,

$\psi(x, 0, \epsilon) = \epsilon$ and $(\partial\psi/\partial t)(x, 0, \epsilon) = f(x, \epsilon)$ for a smooth function $f(x, \epsilon)$ satisfying $f(x, 0) = f(x)$. Therefore for $\epsilon_0 > 0$ small and fixed and $\epsilon \ll \epsilon_0$, we have

$$\begin{aligned} \left(\int_{r_t > \epsilon} dv_g \right)^\cdot &= \left(\int_{\Sigma} \int_{\psi(x, t, \epsilon)}^{\epsilon_0} u_0(x, r)^{-n-1} \sqrt{\frac{\det h_r(x)}{\det h_0(x)}} dr dv_{h_0}(x) \right)^\cdot \\ &= - \int_{\Sigma} f(x, \epsilon) u_0(x, \epsilon)^{-n-1} \sqrt{\frac{\det h_\epsilon(x)}{\det h_0(x)}} dv_{h_0}(x). \end{aligned}$$

Now $f(x, \epsilon) \sqrt{\frac{\det h_\epsilon(x)}{\det h_0(x)}}$ is smooth in ϵ and equals $f(x)$ at $\epsilon = 0$. So recalling (2.7), it follows that the $\log \frac{1}{\epsilon}$ coefficient in $\left(\int_{r_t > \epsilon} dv_g \right)^\cdot$ is

$$(3.3) \quad -(n+1) \int_{\Sigma} f \mathcal{L} dh_0.$$

To analyze the second term, set $\omega_t = -\log(u_t/u_0)$ so that $u_t = e^{-\omega_t} u_0$ and $g_t = e^{2\omega_t} g$. The scalar curvature of g_t is given by

$$R_{e^{2\omega_t} g} = e^{-2\omega_t} [R_g - 2n\Delta_g \omega_t - n(n-1)|d\omega_t|_g^2].$$

Differentiating gives

$$(3.4) \quad (R_{g_t})^\cdot = -2(n\Delta_g \dot{\omega} + R_g \dot{\omega}).$$

Differentiation of (2.8) for g_t shows that $(R_{g_t})^\cdot = O(r^{n+1} \log r)$. Also

$$\begin{aligned} R_g \dot{\omega} &= -n(n+1)\dot{\omega} + O(\dot{\omega} r^{n+2} \log r) \\ &= -n(n+1)\dot{\omega} + O\left(\frac{\dot{u} r^{n+2} \log r}{u}\right) \\ &= -n(n+1)\dot{\omega} + O(r^{n+1} \log r). \end{aligned}$$

Therefore (3.4) gives

$$(n+1)\dot{\omega} = \Delta_g \dot{\omega} + O(r^{n+1} \log r).$$

Hence

$$\begin{aligned} (3.5) \quad -(n+1) \int_{r > \epsilon} \dot{u} u^{-1} dv_g &= (n+1) \int_{r > \epsilon} \dot{\omega} dv_g \\ &= \int_{r > \epsilon} \Delta_g \dot{\omega} dv_g + O(1) \\ &= \int_{r=\epsilon} \partial_{\nu_g} \dot{\omega} d\sigma_g + O(1), \end{aligned}$$

where ν_g is the outward-pointing g -unit normal and $d\sigma_g$ the induced area element on $\{r = \epsilon\}$. Since $g = u^{-2}(dr^2 + h_r)$, we have $\partial_{\nu_g} = -u\partial_r$ and $d\sigma_g = u^{-n} dv_{h_\epsilon}$. Hence

$$(3.6) \quad -(n+1) \int_{r > \epsilon} \dot{u} u^{-1} dv_g = \int_{r=\epsilon} (u^{-n} \dot{u}_r - u^{-n-1} u_r \dot{u}) dv_{h_\epsilon} + O(1).$$

Now \dot{r} is a smooth function of (x, r) which equals $-f(x)$ at $r = 0$. So it follows by differentiation of (3.1) that \dot{u} takes the form

$$\dot{u} = -f(x, r) - (n+2)f(x)\mathcal{L}(x)r^{n+1} \log r + O(r^{n+1}),$$

where $f(x, r)$ is a smooth function satisfying $f(x, 0) = f(x)$. (This $f(x, r)$ need not be the same function as the $f(x, \epsilon)$ which entered into the analysis of the first term above.) Differentiating with respect to r , we conclude that

$$\dot{u}_r = s(x, r) - (n+2)(n+1)f(x)\mathcal{L}(x)r^n \log r + O(r^n)$$

for a smooth function $s(x, r)$. From (2.7), it follows that

$$\begin{aligned} u^{-n} &= r^{-n} \left(\lambda(x, r) + O(r^{n+1} \log r) \right), \\ u^{-n-1} &= r^{-n-1} \left(\mu(x, r) - (n+1)\mathcal{L}(x)r^{n+1} \log r + O(r^{n+1}) \right), \end{aligned}$$

where λ and μ are smooth functions satisfying $\lambda(x, 0) = \mu(x, 0) = 1$. Hence the coefficient of $\log \frac{1}{\epsilon}$ in $\int_{r=\epsilon} u^{-n} \dot{u}_r dv_{h_\epsilon}$ is

$$(n+2)(n+1) \int_{\Sigma} f \mathcal{L} dv_{h_0},$$

and the coefficient of $\log \frac{1}{\epsilon}$ in $\int_{r=\epsilon} u^{-n-1} u_r \dot{u} dv_{h_\epsilon}$ is

$$(n+3) \int_{\Sigma} f \mathcal{L} dv_{h_0}.$$

Combining these in (3.6), it follows that the coefficient of $\log \frac{1}{\epsilon}$ in

$$-(n+1) \int_{r>\epsilon} i u^{-1} dv_g$$

is

$$(n^2 + 2n - 1) \int_{\Sigma} f \mathcal{L} dv_{h_0}.$$

Combining with (3.3) in (3.2) concludes the proof. \square

4. CALCULATIONS

Recall from §2 that $g = u^{-2}\bar{g}$ and that we write $u = r + r^2\varphi$. The Taylor expansion of φ is determined by successive differentiation of (2.5), and the coefficients $v^{(k)}$ are determined by the expansion (2.10). In this section we outline the calculation of $v^{(1)}$ and $v^{(2)}$ via this prescription. In particular, this identifies \mathcal{E} for $n = 2$. We also calculate the anomaly in the renormalized volume V for $n = 2$. Throughout this section we write $\bar{R}_{\alpha\beta\gamma\delta}$ and \bar{R} for the curvature tensor and scalar curvature of \bar{g} , and R_{ijkl} and R for the curvature tensor and scalar curvature of the induced metric $h = h_0$ on Σ .

Equation (2.6) identifies $\varphi|_{r=0}$. Differentiating (2.5) at $r = 0$ and substituting (2.4) gives

$$(4.1) \quad 3(n-1)\varphi_r|_{r=0} = \frac{1}{n}H^2 - |L|^2 - h^{ij}\bar{R}_{0i0j} + \frac{1}{2n}\bar{R}.$$

Now

$$(4.2) \quad \bar{R} = \bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta} = 2h^{ij}\bar{R}_{0i0j} + h^{ij}h^{kl}\bar{R}_{ikjl}.$$

The Gauss equation states $\bar{R}_{ikjl} = R_{ikjl} + L_{il}L_{jk} - L_{ij}L_{kl}$, so $h^{ij}h^{kl}\bar{R}_{ikjl} = R + |L|^2 - H^2$. Substituting this in (4.2) and solving for $h^{ij}\bar{R}_{0i0j}$ give

$$(4.3) \quad h^{ij}\bar{R}_{0i0j} = \frac{1}{2}(\bar{R} - R - |L|^2 + H^2).$$

Substituting (4.3) in (4.1) and then decomposing $L_{ij} = \mathring{L}_{ij} + \frac{1}{n}Hh_{ij}$ finally give

$$3(n-1)\varphi_r|_{r=0} = \frac{1-n}{2n}(\overline{R} + H^2) + \frac{1}{2}(R - |\mathring{L}|^2).$$

When $n = 1$, we have $R = 0$ and $\mathring{L} = 0$. So in this case this equation states $0 = 0$, which shows that $\mathcal{L} = 0$ for $n = 1$.

To calculate $v^{(1)}$ and $v^{(2)}$, first observe that for any 1-parameter family of metrics h_r ,

$$\sqrt{\frac{\det h_r}{\det h_0}} = 1 + \frac{1}{2}(\text{tr}_h h')r + \frac{1}{4}[\text{tr}_h h'' - |h'|^2 + \frac{1}{2}(\text{tr}_h h')^2]r^2 + \dots.$$

Substituting (2.4) and then (4.3) shows that this becomes

$$(4.4) \quad \sqrt{\frac{\det h_r}{\det h_0}} = 1 - Hr + \frac{1}{4}[R - \overline{R} - |L|^2 + H^2]r^2 + \dots.$$

The Taylor expansion of φ is determined mod $O(r^2)$ by (2.6) and (4.1). Using this to calculate the expansion of $(1+r\varphi)^{-n-1}$ and then multiplying by (4.4) and simplifying, one finds that

$$\begin{aligned} (1+r\varphi)^{-n-1} \sqrt{\frac{\det h_r}{\det h_0}} \\ = 1 + \frac{1-n}{2n}Hr + \left[\frac{n-5}{12(n-1)}(R - |\mathring{L}|^2) + \frac{n-2}{24n^2}((n-3)H^2 - 2n\overline{R}) \right]r^2 \\ + O(r^3). \end{aligned}$$

Thus

$$(4.5) \quad \begin{aligned} v^{(1)} &= \frac{1-n}{2n}H, \\ v^{(2)} &= \frac{n-5}{12(n-1)}(R - |\mathring{L}|^2) + \frac{n-2}{24n^2}((n-3)H^2 - 2n\overline{R}). \end{aligned}$$

Substituting the above expressions for $v^{(1)}$, $v^{(2)}$ into (2.12) gives formulae for c_1 , c_2 in (2.11).

One can consider volume expansions $\text{Vol}_g(\{\rho > \epsilon\})$ for other defining functions ρ of Σ . Changing from r to ρ is equivalent to changing the choice of background metric from \overline{g} to $\Omega^2\overline{g}$ with $\Omega = |d\rho|_{\overline{g}}$, since $|d\rho|_{\Omega^2\overline{g}} = 1$ so that ρ is the distance to Σ in the metric $\Omega^2\overline{g}$. So Proposition 2.1 implies that the coefficient of $\log \frac{1}{\epsilon}$ (the energy) is independent of the choice of ρ . If one takes $\rho = u$, then the coefficients of all the divergent terms are integrals of local invariants of \overline{g} , just like for $\rho = r$, since the Taylor expansion of u is locally determined by \overline{g} . In [GoW4], closed formulae are derived for all the coefficients $c_0, \dots, c_{n-1}, \mathcal{E}$ for a general defining function ρ . In the case $\rho = u$, the formulae are made explicit in terms of the geometry of \overline{g} for the coefficients c_1, c_2, c_3 .

Observe from (4.5) that $v^{(1)} = 0$ when $n = 1$. So also $\mathcal{E} = 0$ when $n = 1$. When $n = 2$ we have

$$v^{(2)} = \frac{1}{4}(|\mathring{L}|^2 - R).$$

So the singular Yamabe energy for $n = 2$ is

$$\mathcal{E} = \frac{1}{4} \int_{\Sigma} (|\mathring{L}|^2 - R) dv_{h_0}.$$

One recognizes this as a linear combination of the Willmore energy and the Euler characteristic of Σ .

Finally we derive the anomaly for the renormalized volume for $n = 2$. Let \bar{g} and $\widehat{g} = e^{2\omega}\bar{g}$ be conformally related metrics, and let V and \widehat{V} be the associated renormalized volumes for (M, g) . The difference $V - \widehat{V}$ is the constant term in the expansion of $\text{Vol}_g(\{r > \epsilon\}) - \text{Vol}_{\widehat{g}}(\{\widehat{r} > \epsilon\})$. Equation (2.13) gives a formula for this in terms of the function $b(x, \epsilon)$. We calculate enough of the Taylor expansion of $b(x, \epsilon)$ to evaluate the constant term in the expansion of the last line of (2.13) when $n = 2$.

The distance \widehat{r} for the metric \widehat{g} is determined by the eikonal equation $|d\widehat{r}|_{\widehat{g}}^2 = 1$. Using $\widehat{g} = e^{2\omega}\bar{g}$ and writing $\widehat{r} = e^\Upsilon r$, this can be written as $e^{2(\Upsilon-\omega)}|dr + rd\Upsilon|_{\bar{g}}^2 = 1$ or

$$2r\Upsilon_r + r^2[(\Upsilon_r)^2 + h_r^{ij}\Upsilon_i\Upsilon_j] = e^{2(\omega-\Upsilon)} - 1.$$

Setting $r = 0$ gives $\Upsilon(x, 0) = \omega(x, 0)$. Differentiating with respect to r gives $\Upsilon_r = \frac{1}{2}\omega_r$ at $r = 0$. Differentiating again gives $\Upsilon_{rr} = \frac{1}{3}[\omega_{rr} + \frac{1}{4}(\omega_r)^2 - \omega_i\omega^i]$ at $r = 0$. Thus

$$\Upsilon(x, r) = \omega(x, 0) + \frac{1}{2}\omega_r(x, 0)r + \frac{1}{6}[\omega_{rr} + \frac{1}{4}(\omega_r)^2 - \omega_i\omega^i]r^2 + O(r^3).$$

Now solve the equation $\widehat{r} = e^{\Upsilon(x, r)}r$ for r as a function of \widehat{r} : $r = \widehat{r}b(x, \widehat{r})$. It is elementary to carry this out to obtain

$$(4.6) \quad b(x, \widehat{r}) = e^{-\omega}[1 - \frac{1}{2}\omega_r e^{-\omega}\widehat{r} + (\frac{1}{3}(\omega_r)^2 + \frac{1}{6}\omega_i\omega^i - \frac{1}{6}\omega_{rr})e^{-2\omega}\widehat{r}^2] + O(\widehat{r}^3),$$

where ω and its derivatives are all evaluated at $(x, 0)$.

As noted above, $V - \widehat{V}$ is the constant term in the expansion of the last line of (2.13). Taking $n = 2$, this is the constant term in the expansion of

$$(4.7) \quad \int_{\Sigma} \left[-\frac{1}{2}\epsilon^{-2}b(x, \epsilon)^{-2} - \epsilon^{-1}v^{(1)}(x)b(x, \epsilon)^{-1} + v^{(2)}(x)\log b(x, \epsilon) \right] dv_h.$$

The constant term in $\log b(x, \epsilon)$ is clearly $-\omega(x, 0)$. Easy calculations manipulating the expansion (4.6) show that the coefficient of ϵ in the expansion of $b(x, \epsilon)^{-1}$ is $\frac{1}{2}\omega_r(x, 0)$ and the coefficient of ϵ^2 in the expansion of $b(x, \epsilon)^{-2}$ is $[\frac{1}{12}(\omega_r)^2 - \frac{1}{3}\omega_i\omega^i + \frac{1}{3}\omega_{rr}](x, 0)$. Putting this into (4.7) along with (4.5) for $n = 2$ and collecting the terms give the following.

Proposition 4.1. *When $n = 2$, the anomaly is given by*

$$V - \widehat{V} = -\frac{1}{8} \int_{\Sigma} \left[2(|\mathring{L}|^2 - R)\omega - H\omega_r + \frac{1}{3}(4\omega_{rr} - 4\omega_i\omega^i + (\omega_r)^2) \right] dv_h.$$

It is interesting to compare the singular Yamabe energy and anomaly with the corresponding quantities $\mathcal{E}_{\min \text{ area}}$ and $(V - \widehat{V})_{\min \text{ area}}$ arising from the renormalization of the area of the minimal submanifold of the Poincaré-Einstein space associated to \bar{g} whose boundary at infinity is equal to Σ . This energy was calculated in [GrW] to be

$$\mathcal{E}_{\min \text{ area}} = -\frac{1}{8} \int_{\Sigma} (H^2 + 4h^{ij}\overline{P}_{ij})dv_h,$$

and the corresponding anomaly was calculated in Proposition 2.2 of [GrW] to be

$$(V - \widehat{V})_{\min \text{ area}} = \frac{1}{8} \int_{\Sigma} \left[(H^2 + 4h^{ij}\overline{P}_{ij})\omega - 2H\omega_r + 2\omega_i\omega^i \right] dv_h.$$

Here $\overline{P}_{\alpha\beta} = \frac{1}{n-1} \left(\overline{R}_{\alpha\beta} - \frac{\overline{R}}{2n} \overline{g}_{\alpha\beta} \right)$ denotes the Schouten tensor of \overline{g} . These quantities can be compared to those for the singular Yamabe problem via the identity $H^2 + 4h^{ij}\overline{P}_{ij} = 2(|\mathring{L}|^2 + R)$ for $n = 2$. In particular,

$$\mathcal{E}_{\min \text{ area}} = -\mathcal{E} - 2\pi\chi(\Sigma).$$

But clearly there is not such a simple relation between the anomalies.

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