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UPPER BOUNDS FOR GK-DIMENSIONS OF FINITELY GENERATED P.I. ALGEBRAS

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Abstract. We prove that if A is characteristic zero algebra generated by k elements and satisfying a polynomial identity of degree d then it has GK-dimension less than or equal to $k\lfloor d/2\rfloor^2$. We conjecture that the stronger upper bound that the GK-dimension of A is less than or equal to $(k-1)\lfloor d/2\rfloor^2+1$ and prove it in a number of special cases.

1. Introduction

In [1] we proved that any finitely generated p.i. algebra has finite GK-dimension. This implies that there is a function $F(k,d)$ bounding the GK-dimension of any algebra generated by k elements and satisfying an identity of degree d. Procesi proved in [8] that the characteristic zero algebra generated by k generic $h\times h$ matrices $U_k(M_h(F))$, which satisfies the standard identity S_{2h}, has GK-dimension $(k-1)h^2+1$. This implies that $F(k,2h)\geq (k-1)h^2+1$. Turning to codimension theory, which we will review briefly below, every p.i. algebra A has a codimension sequence $\{c_n(A)\}_{n=0}^{\infty}$ which measures how many multilinear identities A satisfies. Berele and Regev proved in [3] that S_d is the weakest identity of degree d in the sense of having the largest codimension sequence, and Giambruno and Zaicev proved in [5] that the codimensions of S_d are asymptotic to those of $M_{\lfloor d/2\rfloor}(F)$. Putting this together we make the following conjecture.

Conjecture 1. *If the algebra A is generated by k elements and satisfies an identity of degree d, then the GK-dimension of A is at most $(k-1)\lfloor d/2\rfloor^2+1$. Moreover, if $m\leq\lfloor d/2\rfloor$ is as large as possible so that all of the identities of A are also identities for $m\times m$ matrices, then the GK-dimension is less than or equal to $(k-1)m^2+1$.*

It is interesting to compare our conjecture with Proposition 9.31 of [9] in which the authors prove that if A has Kemer index (t,s), then the GK-dimension is bounded by $s((k-1)t+1)$. The first Kemer index, t, is equal to the eventual arm length which is bounded by $\lfloor d/2\rfloor^2$, so our conjecture would strengthen Kanel-Belov and Rowen's theorem by removing the factor of s.

Although we cannot prove the conjecture even in characteristic zero, we can prove two upper bounds on GK-dimensions. The first is a weaker upper bound and the second is the conjecture in a number of special cases.

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Theorem 1.1. *Let A be a characteristic zero algebra generated by k elements and satisfying a polynomial identity of degree d. Then the GK-dimension of A is less than or equal to $k\lfloor d/2\rfloor^2$.*

Theorem 1.2. *Let A be a characteristic zero algebra generated by k elements and satisfying a polynomial identity of degree d, and assume $k \geq \lfloor d/4\rfloor + 1$. Then the GK-dimension of A will be less than or equal to $(k-1)\lfloor d/2\rfloor^2 + 1$ in these cases: (1) d is even, (2) A satisfies the standard identity S_d, or (3) A satisfies an identity not satisfied by $h \times h$ matrices, $h = \lfloor d/2\rfloor$. In this last case the GK-dimension will be strictly less.*

A few general comments before we begin the proofs: First, the proofs will use the theory of cocharacters and we will assume that the reader is familiar with this theory and for the reader not familiar with this theory, we recommend looking at [7]. For the reader unwilling to take our good advice, here are the bare bone definitions: Let V_n be the space of multilinear, degree n polynomials in $x_1, \ldots, x_n$ and let $I_n(A) \subseteq V$ be the identities of the algebra A in V_n. Then $c_n(A)$ is defined to be the dimension of the quotient $V_n/I_n(A)$. Moreover, the space V_n admits a natural action of the symmetric group $\mathcal{S}_n$ acting by $\sigma(x_i) = x_{\sigma^{-1}(i)}$. With respect to this action, $I_n(A)$ will be a submodule and so the quotient $V_n/I_n(A)$ will also be a module for $\mathcal{S}_n$. The $\mathcal{S}_n$-character of this module is called the n^{th} cocharacter of A. In characteristic zero it will decompose as a sum of irreducible characters $\sum m_\lambda \chi^\lambda$, where λ runs over the partitions of n.

Second, we will be using the standard big-O notation $f = O(g)$ to mean that $f(n) \leq Cg(n)$ for some constant $C > 0$, and the non-standard notation $f = L(t)$ to mean $f = O(n^t)$. Third, we assume without loss that A is a generic p.i. algebra on k generators, since every p.i. algebra is a homomorphic image of a generic one. Letting m_λ be the multiplicity of the irreducible character on λ in the cocharacter sequence of A, we proved in [1] that A has a growth sequence

$$\sum_{|\lambda|\leq n} m_\lambda d_k(\lambda), \tag{1}$$

where $d_k(\lambda)$ is the degree of the irreducible $GL(k)$-character corresponding to the partition λ. It is classical that $d_k(\lambda)$ equals the number of semistandard tableaux of shape λ in k letters.

2. Proof of Theorem 1.1

Since we want to estimate (1) our first task is to estimate $d_k(\lambda)$. Let $\Lambda'_e(n)$ be the set of partitions λ of n such that $n - (\lambda_1 + \cdots + \lambda_e)$ is bounded by some unspecified constant.

Lemma 2.1. *If $\lambda \in \Lambda'_e(n)$, then $d_k(\lambda) = L(ke - \binom{e+1}{2})$. In particular, if $k = e$, then $d_k(\lambda) = L(\binom{e}{2})$.*

Proof. We can assume that λ has height at most e since, letting $\mu = (\lambda_1, \ldots, \lambda_e)$, $d_k(\lambda) \leq d_k(\mu)k^{|\lambda-\mu|} = O(d_k(\mu))$.

For each semistandard k-tableau T of shape λ let x_{ij} be the number of i's in row j. Then by definition of semistandard $x_{ij} = 0$ unless $i \geq j$. Moreover,

$$\sum_i x_{ij} = \lambda_j$$

for each j. The number of solutions of this equation is $L(k-j)$ and so the total number of solutions is a polynomial of degree $\sum_{j=1}^{e}(k-j) = ke - \binom{e+1}{2}$. $\square$

We now need three theorems about $exp(A) =_{DEF} \lim_{n\to\infty}(c_n(A))^{1/n}$, the exponential rate of growth of the codimension sequence. The first is due to Berele and Regev from [3]; see Theorem 9.2.6 in [7].

Theorem 2.2. *If A satisfies an identity of degree $d \neq 3$, then the exponential rate of growth of the codimensions $exp(A)$ is at most $\lfloor d/2\rfloor^2$. Moreover, if A is the generic algebra for the standard identity S_{2n}, then $exp(A) = n^2$.*

Remark 2.3. If we restrict to finitely generated algebras, then the hypothesis of $d \neq 3$ is no longer needed. The only counterexample is the infinite Grassmann algebra and no finitely generated algebra is equivalent to it.

The second theorem is due to Giambruno and Zaicev from [4].

Theorem 2.4. *If A is a finitely generated p.i. algebra, then $exp(A) = e$ is an integer. Moreover, e is the largest integer such that there exist partitions λ with $m_\lambda \neq 0$ and λ_e indefinitely large, i.e., such that $\lambda \in \Lambda'_e$.*

The number e is called the eventual arm length of the cocharacter.

The third theorem is due to Berele from [2]

Theorem 2.5. *Under the hypothesis of Giambruno and Zaicev's theorem the multiplicities m_λ are bounded by a polynomial in $|\lambda|$ of degree $\binom{e}{2}$.*

Combining these theorems we can easily prove Theorem 1.1 under the hypothesis $k \geq exp(A)$. If A is generated by k elements and satisfies an identity of degree d, then by Theorem 2.2 and the remark following it, $exp(A) \leq \lfloor d/2\rfloor^2$. Letting $e = exp(A)$ Theorem 2.4 implies that $m_\lambda = 0$ unless $\lambda \in \Lambda'_e$. Combining (1) with Lemma 2.1 and Theorem 2.5 we get that rate of growth of A is bounded by

$$|\cup_{i\leq n}\Lambda'_e(i)| \times \max_{|\lambda|\leq n} m_\lambda \times \max_{\Lambda'_e} d_k(\lambda) \tag{2}$$

which is bounded by a polynomial of degree

$$e + \binom{e}{2} + ek - \binom{e+1}{2} = ek \leq k\lfloor d/2\rfloor^2.$$

This completes the proof if $k \geq exp(A)$. For the remaining case of $k < exp(A) = e$ we need to look more closely at the proof of Theorem 2.5 in [2]. Based on that proof we can prove this lemma.

Lemma 2.6. *If λ is a partition of height less than or equal to $e = exp(A)$, then $m_\lambda = L(ke - \binom{k+1}{2})$.*

Proof. In Theorem 2.10 of [2] we show that m_λ is bounded by e times the number of what we call $\mathcal{A}$-semistandard tableau of shape λ. An $\mathcal{A}$-semistandard tableau will be an e semistandard tableau with $O(1)$ entries from an additional finite alphabet. The number of such will be $O(d_e(\lambda))$, and since λ will have height at most k the number $d_h(\lambda)$ will be $L(ke - \binom{k+1}{2})$ by Lemma 2.1 $\square$

If A is a p.i. algebra on k generators and satisfies an identity of degree d, then $e = exp(A) \leq \lfloor d/2\rfloor^2$. Assuming that $k < e$ we consider (2). The first term will be replaced by the cardinality of $\cup_{i\leq n}\Lambda_k(i)$ which is $L(k)$; by Lemma 2.6 the second term is $L(ke - \binom{k+1}{2})$; and, by Lemma 2.1, the third term will be $L(\binom{k}{2})$ since λ is in Λ'_k. Hence the degree of the product will be $k + ek - \binom{k+1}{2} + \binom{k}{2}$ which equals $ek \le k\lfloor d/2 \rfloor^2$, completing the proof in this case. We note this corollary of the proof.

Corollary 2.7. *Let A be generated by k elements and assume that the cocharacter sequence of A has eventual arm length e. Then the GK-dimension of A is less than or equal to ke.*

Based on this corollary we can strengthen Conjecture 1.

Conjecture 2. *Let A be a characteristic zero p.i. algebra, generated by k elements and with eventual arm length e. Then the GK-dimension of A is less than or equal to $(k-1)e+1$.*

3. Proof of Theorem 1.2

In [3] Berele and Regev develop a method for computing $exp(A)$ based on what we called prime product algebras. In the case of algebras satisfying Capelli identities every prime product algebra is simply an algebra of block triangular matrices $UT(n_1, \dots, n_t)$, i.e., the algebra of matrices of the form

$$\begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_t \end{pmatrix}$$

where each A_i is in $M_{n_i}(F)$. If $A = UT(n_1, \dots, n_t)$ it is known, see Corollary 6.6.2 of [6], that $exp(A) = n_1^2 + \cdots + n_t^2$ and that the minimal degree of an identity satisfied by A is $2(n_1 + \cdots + n_t)$. In [3] we proved the following.

Theorem 3.1. *If A is any p.i. algebra, then $exp(A)$ is the minimum of $exp(\bar{A})$, where $\bar{A}$ runs over the prime product algebras satisfying all of the identities of A.*

Remark 3.2. Conjecture 2 is equivalent to the statement that if A is finitely generated and p.i., then the GK-dimension of the universal algebra in k generators $U_k(A)$ is equal to the maximum of the GK-dimensions $U_k(\bar{A})$ where, as above, $\bar{A}$ runs over the prime product algebras satisfying all of the identities of A.

In the case we are interested in A satisfies a Capelli identity and so the $\bar{A}$ in the theorem will run over block upper triangular matrices.

Lemma 3.3. *Let A be finitely generated and satisfy an identity of degree d, and let $h = \lfloor d/2 \rfloor$. If A satisfies an identity which is not an identity of $h \times h$ matrices, then $exp(A) \le h^2 - 2h + 2$.*

Proof. If $\bar{A} = UT(n_1, \dots, n_t)$ satisfies all of the identities of A, then since it satisfies an identity of degree d,

$$2(n_1 + \cdots + n_t) \le d$$

and so $n_1 + \cdots + n_t \le h$. Let n_1 be the largest n_i. Since A, and therefore $\bar{A}$ satisfy an identity which is not an identity for $M_h(F)$, it is not the case that $n_1 = h$ and the rest of the n_i are zero. Then

$$exp(\bar{A}) = n_1^2 + \cdots + n_t^2 \le n_1^2 + (n_2 + \cdots + n_t)^2.$$

We maximize this sum by setting $n_1 = h - 1$, $n_2 = 1$ and the remaining $n_i = 0$ yielding an exponential rate of growth $(n-1)^2 + 1$. $\square$

Hence, by Corollary 2.7 if A is generated by k elements, then the GK-dimension of A will be less than or equal to $k(h^2 - 2h + 2)$. It is straightforward to compare this to $(k-1)h^2 + 1$.

Theorem 3.4. *Assume that the algebra A is generated by k elements and satisfies an identity of degree d and an identity (perhaps the same one) which is not an identity for $h \times h$ matrices, $h = \lfloor d/2 \rfloor$. If $2k \geq h + 1$, then the GK-dimension of A is less than $(k-1)h^2 + 1$.*

Proof.

$$\begin{aligned}(k-1)h^2 + 1 - k(h^2 - 2h + 2)& \\ &= -h^2 + 1 + 2kh - 2k \\ &= -(h+1)(h-1) + 2k(h-1) \\ &= (h-1)(2k - h - 1) \geq 0.\end{aligned}$$

□

Note that $2k \geq \lfloor d/2 \rfloor + 1$ is equivalent to $k \geq \lfloor d/4 \rfloor + 1$.

The remaining case to consider in Theorem 1.2 is that in which A satisfies only identities of $M_h(F)$. In this case we need the following two theorems, the first is due to Procesi in [8] and the second is due to Giambruno and Zaicev, Theorem 2 from [5].

Theorem 3.5. *The generic algebra generated by k $h \times h$ matrices has GK-dimension $(k-1)h^2 + 1$.*

Theorem 3.6. *The T-ideal generated by the even standard identity S_{2h} is the intersection $T_1 \cap T_2$ where T_1 is the T-ideal of identities of $M_h(F)$ and T_2 is the T-ideal of identities for a certain finite dimesional algebra B with $exp(B) < h^2$.*

The T-ideal generated by the odd standard identity S_{2h+1} is the intersection $T_3 \cap T_4 \cap T_5$ where T_3 and T_4 are gotten by multiplying T_1, above, from the right or left, respectively, by the ideal of all polynomials with 0 constant term, and T_5 is the T-ideal of identities for a certain finite dimensional algebra C with $exp(C) < h^2$.

Lemma 3.7. *Let A be the generic algebra on k generators satisfying the standard identity S_d and assume that $k \geq \lfloor d/4 \rfloor + 1$. Then A has GK-dimension $(k-1)h^2 + 1$, where $h = \lfloor d/2 \rfloor$.*

Proof. We first consider the case in which $d = 2h$ is even. Referring to Giambruno and Zaicev's theorem, let U_1 be the generic algebra in k variables for $M_h(F)$ and U_2 be the generic algebra for B. Note that the GK-dimension of A will be the larger of the GK-dimensions of U_1 and U_2. Since B satisfies S_{2h} and so $exp(B) < h^2$, by Theorem 3.4, the GK-dimension of U_2 is less than that of U_1. Applying Procesi's theorem the lemma follows in this case.

The case of odd d follows similarly, using the fact that the growth function for the univeral algebra for T_3 or T_4 is at most k times the growth function of T_1. □

Although not germane to the present work we mention this corollary of our methods. Based on Giambruno and Zaicev's paper a similar result is also true for the Capelli identity C_{k^2+1}.

Corollary 3.8. *Let $m_\lambda(S_{2h})$ be the multiplicity in the cocharacter sequence of the generic algebra of the standard identity, and let $m_\lambda(M_h(F))$ be the multiplicity in the cocharacter sequence of $M_h(F)$ and let $2k \geq h+1$. Then if λ_k is large enough $m_\lambda(M_h(F)) = m_\lambda(S_{2h})$.*

Proof of Theorem 1.2. Let A be a generic p.i. algebra satisfying an identity of degree d, let $h = \lfloor d/2 \rfloor$ and let the number of generators be k with $2k > h+1$. If A satisfies an identity which is not an identity for $h \times h$ matrices, then by Theorem 3.4 the GK-dimension is less than $(k-1)h^2+1$.

If A satisfies no such identity, then the hypotheses of Theorem 1.2 imply that A satisfies a standard identity and we are done by Lemma 3.7. □

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