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STABLE EQUIVALENCES OF MORITA TYPE DO NOT PRESERVE TENSOR PRODUCTS AND TRIVIAL EXTENSIONS OF ALGEBRAS

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Abstract. It is well known that derived equivalences preserve tensor products and trivial extensions. We disprove both constructions for stable equivalences of Morita type.

1. Introduction

Let k be a field and let A be a finite dimensional k-algebra. We denote by $\mathrm{mod}A$ the category of all finite dimensional left A-modules, and by $\underline{\mathrm{mod}}A$ the stable module category of $\mathrm{mod}A$ modulo projective modules. Two finite dimensional k-algebras A and B are said to be *stably equivalent* if $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}B$ are equivalent as k-categories ([2]). The stable category $\underline{\mathrm{mod}}A$ is a natural quotient of the module category $\mathrm{mod}A$ by the ideal of maps that factor through projective modules, and in case that A is self-injective it is also a natural quotient (in the sense of triangulated categories) of the bounded derived module category $D^b(\mathrm{mod}A)$ ([10], [19]). Examples of stable equivalences naturally arise in the representation theory of groups and algebras (see [2], [1], [16], [5], [13], [14]).

However, unlike the classical Morita theory for module categories and the Morita theory for derived categories ([18]), it is not known how to describe stable equivalences in terms of generators of stable categories (cf. [12]). For this reason, much less is known for stable equivalences comparing to Morita and derived equivalences. In practice, one often uses *stable equivalences of Morita type*, which form a class of stable equivalences with properties needed in most applications and which are close to derived equivalences.

Definition 1.1 ([5]). Two finite dimensional algebras A and B are said to be stably equivalent of Morita type if there are two bimodules ${}_AM_B$ and ${}_BN_A$ which

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are projective as left modules and as right modules such that there are bimodule isomorphisms:

$$ {}_AM \otimes_B N_A \simeq {}_AA_A \oplus {}_AP_A, \quad {}_BN \otimes_A M_B \simeq {}_BB_B \oplus {}_BQ_B, $$

where ${}_AP_A$ and ${}_BQ_B$ are projective bimodules.

Clearly, in the above situation, the exact functors $N \otimes_A -$ and $M \otimes_B -$ induce mutually inverse equivalences between $\underline{\mathrm{mod}}A$ and $\underline{\mathrm{mod}}B$. In fact, any stable equivalence that is induced by an exact functor between the module categories of two self-injective algebras is isomorphic to a stable equivalence of Morita type ([21]). Under some mild condition, this even holds for general finite dimensional algebras ([7]). All derived equivalences between self-injective k-algebras induce stable equivalences of Morita type ([20]). On the other hand, there do exist stable equivalences of Morita type which are not induced by derived equivalences (see [5], [13] and Section 3).

Although we have a better understanding of stable equivalences of Morita type than on general stable equivalences, we still cannot answer some basic questions about them. For example, one of the most important open problems is the following fundamental conjecture of Auslander and Reiten:

Conjecture 1.2 ([2], [17]). *Two stably equivalent algebras have the same number of isomorphism classes of non-projective simple modules.*

This conjecture is largely open even for stable equivalences of Morita type. We refer the reader to [15], [11] for some equivalent descriptions of this conjecture in this situation.

In the present paper, we will study two basic questions on stable equivalences of Morita type. Before stating these questions, we first recall two classical results of Rickard on derived equivalences. To state Rickard's result, we need to recall the notion of trivial extensions.

Definition 1.3. Let A be a finite dimensional k-algebra. Let $D(A) = \mathrm{Hom}_k(A, k)$ be its k-dual. Denote $T(A) = A \oplus D(A)$ as k-vector spaces, and define the multiplication by

$$ (a, f)(b, g) = (ab, ag + fb) $$

for $a, b \in A$ and $f, g \in D(A)$. It is easy to see that this is a k-algebra and this algebra $T(A)$ is called the trivial extension of A and is denoted sometimes by $T(A) = A \ltimes D(A)$.

Theorem 1.4 ([19], [20]). *Let A and B be two derived equivalent finite dimensional k-algebras and assume the same condition for C and D. Then*

(1) *the trivial extension algebras $T(A)$ and $T(B)$ are derived equivalent;*
(2) *the tensor product algebras $A \otimes_k C$ and $B \otimes_k D$ are derived equivalent.*

It is natural to ask whether the same is true for stable equivalences of Morita type. In fact, such questions are closely related to the Auslander-Reiten Conjecture 1.2.

Proposition 1.5. *Let k be an algebraically closed field of characteristic $p > 0$ and let C_p be the cyclic group of order p. Let A and B be two indecomposable, non-semisimple finite dimensional algebras which are stably equivalent of Morita type. Then the assertion that $A \otimes_k kC_p$ and $B \otimes_k kC_p$ are stably equivalent of Morita type implies the validity of the Auslander-Reiten Conjecture* 1.2 *for A and B.*

Proof. Compare with the proof of [21, Theorem 3.7]). We first observe that A and $A \otimes_k kC_p$ have the same number of non-isomorphic simple modules. Let C_A be the Cartan matrix of A. The Cartan matrix of $A \otimes_k kC_p$ is equal to pC_A, so its p-rank is zero. The statement follows from Theorem 4.1 of [15] which says that the invariance of the p-rank of the Cartan matrix under a stable equivalence of Morita type is equivalent to the Auslander-Reiten Conjecture 1.2.

One can also give a proof by computing the degree zero stable Hochschild homology (see [15]) of $A \otimes_k kC_p$ and of $B \otimes_k kC_p$. The details are left to the reader. □

In [21], Rickard raised the following question.

Question 1.6 ([21])**.** Let A and B be two indecomposable, non-semisimple self-injective k-algebras which are stably equivalent of Morita type and assume the same condition for C and D. Are $A \otimes_k C$ and $B \otimes_k D$ stably equivalent of Morita type?

There would be trivial counterexamples if we did not request that algebras be indecomposable, since A and $A \times k$ are stably equivalent of Morita type. If the stable equivalences are all induced by derived equivalences, then the answer is "yes" since the derived equivalence preserves tensor product. If they are not, Rickard mentions that the answer is probably "no" in general. However, as Rickard stated, the simplest possible counterexamples are already quite complicated.

Note that in case $p = 2$ and A is symmetric, the above construction is just the trivial extension, as the following more general proposition shows.

Proposition 1.7. *Let k be a field and A be a symmetric k-algebra. Then the tensor algebra $A \otimes_k k[x]/(x^2)$ is isomorphic to the trivial extension algebra $T(A) = A \ltimes D(A)$ of A.*

Proof. Since A is symmetric, we can fix an A-A-bimodule isomorphism $A \to D(A)$ (mapping $a \in A$ to $a' \in D(A)$). Define a map

$$\alpha : A \otimes_k k[x]/(x^2) \to T(A) \text{ by } \alpha(a \otimes \overline{1} + b \otimes \overline{x}) = (a, b').$$

It is straightforward that α is an algebra isomorphism. □

Remark 1.8. Note that following Definition 1.3 one can define the trivial extension algebra $T(A)$ of arbitrary finite dimensional k-algebra A. It is well known that $T(A)$ is always a symmetric k-algebra, that is, $T(A) \cong D(T(A))$ as $T(A)$-$T(A)$-bimodules.

In [11], König and the first two named authors proved the following result relating the Auslander-Reiten conjecture to trivial extensions.

Proposition 1.9 ([11, Corollary 8.2])**.** *Let A and B be two symmetric k-algebras over an algebraically closed field of characteristic $p > 0$. Suppose that A and B are stably equivalent of Morita type. Then the condition that $T(A)$ and $T(B)$ are stably equivalent of Morita type implies the validity of the Auslander-Reiten conjecture for A and B.*

This motivates the following question in [11].

Question 1.10 ([11, Question 8.3])**.** Let A and B be two indecomposable, non-simple finite dimensional algebras which are stably equivalent of Morita type. Are their trivial extensions algebras $T(A)$ and $T(B)$ stably equivalent of Morita type?

In the present paper, we will answer Question 1.6 and Question 1.10 in the negative for general finite dimensional algebras. More precisely, in Section 2, we prove that if two algebras are stably equivalent (even of Morita type), then their corresponding triangular matrix algebras are usually not stably equivalent of Morita type. Since the triangular matrix algebras are special cases of tensor algebras, we get a negative answer to Question 1.6. In Section 3, starting from the group algebra of the dihedral group of order 8 in characteristic 2, we first use a method in [14] to construct two algebras Λ and Γ which are stably equivalent of Morita type, and then form their trivial extensions $T(\Lambda)$ and $T(\Gamma)$; although Λ and Γ have isomorphic (stable-) centers, the (stable-) centers of $T(\Lambda)$ and $T(\Gamma)$ are non-isomorphic, this shows that $T(\Lambda)$ and $T(\Gamma)$ are not stably equivalent of Morita type and we get a negative answer to Question 1.10.

Using a GAP [9] computer program and the ideas of the present paper Bouc and the last named author proved in [4] that Rickard's original question has a negative answer in general. However, the proof there heavily depends on a computation by GAP.

2. Triangular matrix algebras

In this section, we answer Question 1.6 in the negative for general finite dimensional algebras.

Recall that for a finite dimensional k-algebra A, the stable category $\overline{\mathrm{mod}}A$ of $\mathrm{mod}A$ modulo injective modules can be defined similarly. There is an equivalence τ from $\underline{\mathrm{mod}}A$ to $\overline{\mathrm{mod}}A$, which is called the Auslander-Reiten translation. If $F : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}B$ is a stable equivalence, then there is an induced stable equivalence (modulo injectives) $\tau_B F \tau_A^{-1} : \overline{\mathrm{mod}}A \to \overline{\mathrm{mod}}B$.

Given a finite dimensional k-algebra A, we denote by $T_2(A)$ the lower triangular matrix algebra $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$. Note that there is an algebra isomorphism between the tensor algebra $A \otimes_k T_2(k)$ and $T_2(A)$ given by the map $a \otimes \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \mapsto \begin{pmatrix} au & 0 \\ av & aw \end{pmatrix}$. We refer to [2] for the description of $T_2(A)$-modules in terms of A-modules.

Theorem 2.1. *Let A and B be two self-injective algebras with no semisimple summands. If $\Lambda := T_2(A)$ and $\Gamma := T_2(B)$ are stably equivalent, then A and B are Morita equivalent.*

Proof. First we observe that although A and B are self-injective algebras, Λ and Γ are not self-injective any more. Suppose now that there is a stable equivalence $F : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Gamma$. Let $H = \tau_\Gamma F \tau_\Lambda^{-1} : \overline{\mathrm{mod}}\Lambda \to \overline{\mathrm{mod}}\Gamma$ be the induced stable equivalence modulo injectives. By [1, Corollary 3.2], H induces a one-to-one correspondence between the isomorphism classes of indecomposable non-simple non-injective projective modules in $\mathrm{mod}\Lambda$ and those in $\mathrm{mod}\Gamma$. Under our assumption, there are no simple projective modules over Λ and Γ. Therefore H induces a one-to-one correspondence between the isomorphism classes of indecomposable non-injective projective modules in $\mathrm{mod}\Lambda$ and those in $\mathrm{mod}\Gamma$.

Each Λ-module can be described as a triple (X, Y, f), where X and Y in $\mathrm{mod}A$, and f is an A-module homomorphism from X to Y. A homomorphism from

(X, Y, f) to (X', Y', f') is precisely a pair (α, β) in $\mathrm{Hom}_A(X, X') \times \mathrm{Hom}_A(Y, Y')$ such that $\beta f = f'\alpha$. From this description we see that the indecomposable projective Λ-modules are isomorphic to modules of the form $(P, P, 1_P)$ and $(0, P, 0)$ where P is an indecomposable projective A-module. Dually, the indecomposable injective Λ-modules are isomorphic to modules of the form $(P, P, 1_P)$ and $(P, 0, 0)$ where P is an indecomposable projective A-module. By the previous discussion, we see that under the stable equivalence H, each indecomposable non-injective projective Λ-module $(0, P, 0)$ corresponds to some indecomposable non-injective projective Γ-module $(0, Q, 0)$, and this gives a bijection between the isomorphism classes of indecomposable non-injective projective modules in $\mathrm{mod}\Lambda$ and those in $\mathrm{mod}\Gamma$. Observe that we have the following easy fact: for any two A-modules X and X', we have $\overline{\mathrm{Hom}}_\Lambda((0, X, 0), (0, X', 0)) \cong \mathrm{Hom}_\Lambda((0, X, 0), (0, X', 0)) \cong \mathrm{Hom}_A(X, X')$. Without loss of generality we may assume that both A and B are basic algebras. Then we have that

$$H((0, A, 0)) \cong (0, B, 0) \quad \text{and} \quad \overline{\mathrm{End}}_\Lambda((0, A, 0)) \cong \overline{\mathrm{End}}_\Gamma((0, B, 0)).$$

Therefore we have the following algebra isomorphisms:

$$\begin{aligned} \mathrm{End}_A(A) \cong \mathrm{End}_\Lambda((0, A, 0)) &\cong \overline{\mathrm{End}}_\Lambda((0, A, 0)) \\ &\cong \overline{\mathrm{End}}_\Gamma((0, B, 0)) \cong \mathrm{End}_\Gamma((0, B, 0)) \cong \mathrm{End}_B(B). \end{aligned}$$

It follows that the algebras A and B are isomorphic. □

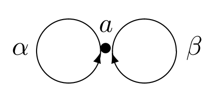
Remark 2.2. The above result shows that Question 1.6 has a negative answer for general finite dimensional algebras. Indeed, we can easily find two self-injective algebras A and B which are derived equivalent but not Morita equivalent. Clearly A and B are stably equivalent of Morita type, but $T_2(A) \simeq A \otimes_k T_2(k)$ and $T_2(B) \simeq B \otimes_k T_2(k)$ cannot be stably equivalent of Morita type by Theorem 2.1.

Remark 2.3. From the proof of Theorem 2.1, we obtain that the stable category of the triangular matrix algebra $T_2(A)$ determines the original algebra A in the following way: it is the (stable) endomorphism algebra of the sum of indecomposable non-projective injective modules over triangular matrix algebra.

3. Trivial extensions

In this section, we answer Question 1.10 in the negative for general finite dimensional algebras.

Let k be an algebraically closed field of characteristic 2. Then it is well known (see, for example, [8]) that the group algebra $A = kD_8$ of the dihedral group of order 8 is given by the following quiver:

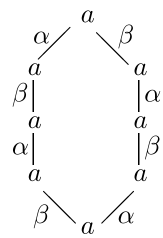


with relations

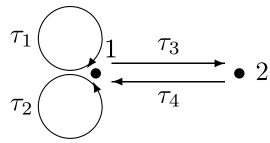
$$\alpha^2 = \beta^2 = 0, (\alpha\beta)^2 = (\beta\alpha)^2.$$

This is a local symmetric algebra with basis (for simplicity, we write α for its class

$\overline{\alpha}$ in A, etc.) $1 = e_a, \alpha, \beta, \alpha\beta, \beta\alpha, \alpha\beta\alpha, \beta\alpha\beta, \alpha\beta\alpha\beta = \beta\alpha\beta\alpha$. The Loewy diagram of the regular module ${}_AA$ looks like



The Cartan matrix C_A of A is given by $C_A = (8)$, and the center $Z(A)$ of A is a radical square zero local algebra with basis $1, \alpha\beta\alpha, \beta\alpha\beta, \alpha\beta\alpha\beta, \alpha\beta + \beta\alpha$. Let S be the unique simple A-module (which is also the trivial module k of the group algebra A) and let Λ be the endomorphism algebra $\mathrm{End}_A(A \oplus S)^{op}$. One can compute that Λ is given by the following quiver:



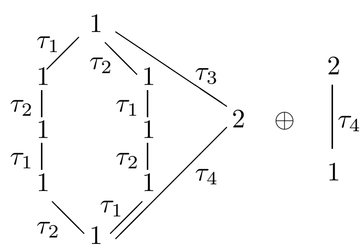
with relations

$$\tau_1{}^2 = \tau_2{}^2 = \tau_3\tau_4 = \tau_2\tau_4 = \tau_1\tau_4 = \tau_3\tau_1 = \tau_3\tau_2 = 0, (\tau_1\tau_2)^2 = (\tau_2\tau_1)^2 = \tau_4\tau_3.$$

This is an 11-dimensional algebra with basis (remember that we write τ_1 for its equivalences class $\overline{\tau_1}$ in Λ, etc.)

$$e_1, e_2, \tau_1, \tau_2, \tau_3, \tau_4, \tau_2\tau_1, \tau_1\tau_2, \tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2, \tau_2\tau_1\tau_2\tau_1 = \tau_1\tau_2\tau_1\tau_2 = \tau_4\tau_3.$$

The regular module ${}_\Lambda\Lambda$ has the following decomposition:



The Cartan matrix C_Λ of Λ is given by $C_\Lambda = \begin{pmatrix} 8 & 1 \\ 1 & 1 \end{pmatrix}$, and the center $Z(\Lambda)$ of Λ is a 5-dimensional algebra with basis $1, \tau_2\tau_1 + \tau_1\tau_2, \tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2, \tau_2\tau_1\tau_2\tau_1 = \tau_1\tau_2\tau_1\tau_2 = \tau_4\tau_3$. Since $\mathrm{char} k = 2$, it is easy to verify that $Z(\Lambda)$ is also a radical square zero local algebra. Next we want to compute the center $Z(T(\Lambda))$ of the trivial extension $T(\Lambda) = \Lambda \ltimes D(\Lambda)$. According to a result of Bessenrodt, Holm and the third named author (see [3, Proposition 3.2]), $Z(T(\Lambda)) = Z(\Lambda) \ltimes \mathrm{Ann}_{D(\Lambda)}(K(\Lambda))$, where $K(\Lambda)$ is the k-subspace of Λ spanned by all commutators $\lambda\mu - \mu\lambda$ $(\lambda, \mu \in \Lambda)$ and where $\mathrm{Ann}_{D(\Lambda)}(K(\Lambda)) = \{f \in D(\Lambda) | f(K(\Lambda)) = 0\}$. By a straightforward calculation we have the following (write ${e_1}^*$ as the dual basis element corresponding to e_1, etc.):

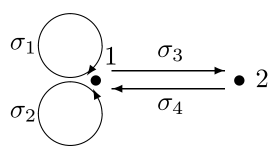
$$K(\Lambda) = \langle \tau_2\tau_1 + \tau_1\tau_2, \tau_3, \tau_4, \tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2, \tau_4\tau_3\rangle,$$
$$\mathrm{Ann}_{D(\Lambda)}(K(\Lambda)) = \langle {e_1}^*, {e_2}^*, {\tau_1}^*, {\tau_2}^*, (\tau_2\tau_1)^* + (\tau_1\tau_2)^*\rangle.$$

Again char $k = 2$ forces that $Z(T(\Lambda))$ is a (10-dimensional) radical square zero local algebra.

Now we come back to the group algebra A of the dihedral group D_8. According to [6], each direct summand of the module

$$\mathrm{rad}A/\mathrm{rad}^4A \quad = \quad \begin{array}{c} a \\ \beta \big| \\ a \\ \alpha \big| \\ a \end{array} \quad \oplus \quad \begin{array}{c} a \\ \big| \alpha \\ a \\ \big| \beta \\ a \end{array}$$

is an endotrivial module. Recall that for a group algebra kG of a finite group G, a kG-module X is called endotrivial if $D(X) \otimes_k X \simeq k \oplus \{\text{projective}\}$ as kG-modules (where k is the trivial module). It follows easily that $X \otimes_k -$ induces a stable self-equivalence of Morita type of $\underline{\mathrm{mod}}kG$ (here the defining kG-kG-bimodule is given by $X \otimes_k kG$ where the left kG-module structure is defined by diagonal G-action and the right kG-module structure is defined by multiplication on the right factor). Now let X be one of any direct summand of $\mathrm{rad}A/\mathrm{rad}^4A$. Then X induces a stable self-equivalence of Morita type over $\underline{\mathrm{mod}}A$ such that the trivial module S corresponds to X. Since X is not of the form $\Omega^i(S)$, this stable self-equivalence is not induced from a derived equivalence (cf. [13, Remark 3.10] or [22, Theorem 2.11]). Let Γ be the endomorphism algebra $\mathrm{End}_A(A \oplus X)^{op}$. Then by the construction in [14, Theorem 1.1], there is a stable equivalence of Morita type between Λ and Γ. One can compute that Γ has the same quiver as Λ (but here we use new notation to name the arrows)



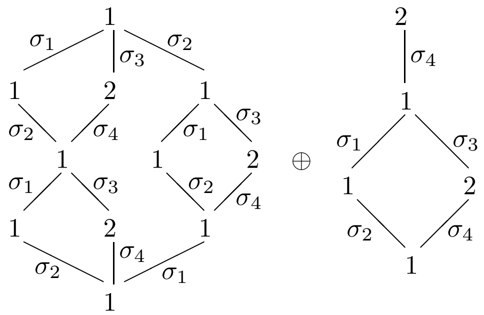
with relations

$${\sigma_1}^2 = {\sigma_2}^2 = \sigma_3\sigma_1 = \sigma_2\sigma_4 = \sigma_3\sigma_2\sigma_1\sigma_2 = 0, \sigma_2\sigma_1 = \sigma_4\sigma_3,$$
$$(\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^2 = (\sigma_4\sigma_3)^2 = \sigma_1\sigma_4\sigma_3\sigma_2.$$

This is a 16-dimensional algebra with basis

$$e_1, e_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_2\sigma_1 = \sigma_4\sigma_3, \sigma_1\sigma_2, \sigma_3\sigma_2, \sigma_3\sigma_4, \sigma_1\sigma_4, \sigma_1\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2,$$
$$\sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_4\sigma_3, \sigma_4\sigma_3\sigma_4 = \sigma_2\sigma_1\sigma_4, (\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^2 = (\sigma_4\sigma_3)^2.$$

The regular module ${}_\Gamma\Gamma$ has the following decomposition:



The Cartan matrix of Γ is given by $C_\Gamma = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}$. From this we can deduce that Λ and Γ are not derived equivalent, since their Cartan matrices are not congruent over the integers. The fact that the Cartan matrices of two derived equivalent algebras are congruent over the integers is one of few known invariants to distinguish between derived equivalence and stable equivalence of Morita type. Our next aim is to show that the trivial extensions $T(\Lambda)$ and $T(\Gamma)$ are also not stably equivalent of Morita type. We will verify this fact by proving that their stable centers are not isomorphic as algebras.

Let us first recall the definition of the stable center. For an algebra A, we can identify any A-A-bimodule with a left A^e-module where $A^e = A \otimes_k A^{op}$. In particular, the algebra A itself is naturally an A^e-module, and the endomorphism algebra $\mathrm{End}_{A^e}(A, A)$ is canonically isomorphic to the center $Z(A)$ of A (by $f \mapsto f(1)$). Set $Z^{pr}(A)$ to be the ideal of $Z(A)$ consisting of A^e-homomorphisms from A to A which factor through a projective A^e-module and we call it the *projective center* of A. The *stable center* of A is defined to be the quotient algebra $Z^{st}(A) = Z(A)/Z^{pr}(A)$. It is well known that a stable equivalence of Morita type preserves the stable centers of algebras (see [5]).

Theorem 3.1. *Let k be an algebraically closed field of characteristic 2, let D_8 be the dihedral group of order 8 and let $A = kD_8$. Denote by S the trivial kD_8-module. Then $radA/rad^4A = X \oplus Y$ for $X \neq 0 \neq Y$. Let $\Lambda = End_A(A \oplus S)^{op}$ and $\Gamma = End_A(A \oplus X)^{op}$. Then the stable centers of $T(\Lambda)$ and $T(\Gamma)$ are not isomorphic as algebras. In particular, $T(\Lambda)$ and $T(\Gamma)$ are not stably equivalent of Morita type.*

Proof. The Cartan matrices of Λ and Γ are given by $C_\Lambda = \begin{pmatrix} 8 & 1 \\ 1 & 1 \end{pmatrix}$ and $C_\Gamma = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}$, respectively. It follows easily that the Cartan matrices of $T(\Lambda)$ and $T(\Gamma)$ are given by $C_{T(\Lambda)} = \begin{pmatrix} 16 & 2 \\ 2 & 2 \end{pmatrix}$ and $C_{T(\Gamma)} = \begin{pmatrix} 16 & 6 \\ 6 & 4 \end{pmatrix}$, respectively. By [15, Proposition 2.3 and Corollary 2.7], the dimension of the projective center of a symmetric algebra over an algebraically closed field k of characteristic $p \geq 0$ is equal to the p-rank of the Cartan matrix. Since now $p = 2$, both 2-ranks of $C_{T(\Lambda)}$ and $C_{T(\Gamma)}$ are zero, and therefore the stable centers of $T(\Lambda)$ and $T(\Gamma)$ are the same as the centers of $T(\Lambda)$ and $T(\Gamma)$, respectively.

We have seen that the center $Z(T(\Lambda))$ is a 10-dimensional radical square zero local algebra. Similarly we can compute the center $Z(T(\Gamma))$ using the formula $Z(T(\Gamma)) = Z(\Gamma) \ltimes \mathrm{Ann}_{D(\Gamma)}(K(\Gamma))$. The center $Z(\Gamma)$ of Γ is a 5-dimensional algebra with basis $1, \sigma_2\sigma_1 + \sigma_1\sigma_2 + \sigma_3\sigma_4, \sigma_1\sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2, (\sigma_2\sigma_1)^2$. Since $\mathrm{char} k = 2$, it is easy to verify that $Z(\Gamma)$ is also a radical square zero local algebra. We also have the following:

$$K(\Gamma) = \langle \sigma_3, \sigma_4, \sigma_1\sigma_4, \sigma_3\sigma_2, \sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_4\sigma_3, \sigma_4\sigma_3\sigma_4, \sigma_2\sigma_1 + \sigma_1\sigma_2, \sigma_3\sigma_4 + \sigma_4\sigma_3,$$

$$\sigma_2\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1 = \sigma_1\sigma_4\sigma_3, (\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^2 = (\sigma_4\sigma_3)^2 \rangle,$$

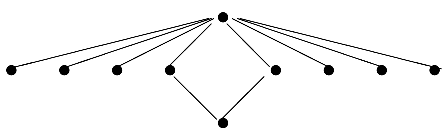
$$\mathrm{Ann}_{D(\Gamma)}(K(\Gamma)) = \langle {e_1}^*, {e_2}^*, {\sigma_1}^*, {\sigma_2}^*, (\sigma_2\sigma_1)^* + (\sigma_1\sigma_2)^* + (\sigma_3\sigma_4)^* \rangle.$$

We perform the following multiplication in $Z(T(\Gamma))$:

$$(\sigma_2\sigma_1 + \sigma_1\sigma_2 + \sigma_3\sigma_4)((\sigma_2\sigma_1)^* + (\sigma_1\sigma_2)^* + (\sigma_3\sigma_4)^*) = 2{e_1}^* + {e_2}^* = {e_2}^*.$$

Since char $k = 2$, the above multiplication is not equal to zero and therefore $Z(T(\Gamma))$ is not radical square zero. So $Z(T(\Lambda))$ and $Z(T(\Gamma))$ are not isomorphic as algebras. □

Remark 3.2. Suppose that k is of characteristic 2. Then the center $Z(T(\Gamma))$ is a 10-dimensional local algebra such that the regular module has the following Loewy structure:



Parallel edges correspond to multiplication with the same element. Here, in the square in the center one direction corresponds to multiplication with $(\sigma_2\sigma_1 + \sigma_1\sigma_2 + \sigma_3\sigma_4)$, whereas the other direction of the square in the center corresponds to multiplication with $(\sigma_2\sigma_1)^* + (\sigma_1\sigma_2)^* + (\sigma_3\sigma_4)^*$. The product of these two corresponds to multiplication with e_2^*.

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