

## ZEROS OF POLYNOMIALS OF DERIVATIVES OF ZETA FUNCTIONS

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**ABSTRACT.** Let  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  be a polynomial whose coefficients are the ring of all general Dirichlet series which converge absolutely in the half-plane  $\Re(s) > 1/2$ . In the present paper, we show that the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  has infinitely many zeros in the vertical strip  $D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$  if  $L(s)$  is hybridly universal and  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  is a polynomial such that at least one of the degrees of  $X_1, \dots, X_l$  is greater than zero. As a corollary, we prove that the function  $(d^k/ds^k)P_s(L(s))$  with  $k \in \mathbb{N}$  has infinitely many zeros in the strip  $D$  when  $L(s)$  is hybridly universal and  $P_s \in \mathcal{D}_s[X]$  is a polynomial with degree greater than zero. The upper bounds for the numbers of zeros of  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  and  $(d^k/ds^k)P_s(L(s))$  are studied as well.

### 1. INTRODUCTION AND THE MAIN RESULTS

**1.1. Main results.** For a complex variable  $s = \sigma + it$ , let  $D(s)$  be a general Dirichlet series of the form

$$(1.1) \quad D(s) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad a_n \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}.$$

Let us denote by  $\mathcal{D}_s$  the ring of all general Dirichlet series defined as (1.1) which are absolutely convergent in the half-plane  $\Re(s) > 1/2$ . Let  $\mathbb{N}$  be the set of integers greater than 0, and  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  be a polynomial whose coefficients are in  $\mathcal{D}_s$ . Then we have the following results which are proved in Section 2. Note that the definition of the hybrid universality is written in the next subsection. Some remarks and examples related to the main results are given in Section 3.

**Theorem 1.1.** *Let  $L(s)$  be hybridly universal and  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  be a polynomial such that at least one of the degrees of  $X_1, \dots, X_l$  is greater than zero. Then, for any  $1/2 < \sigma_1 < \sigma_2 < 1$ , there exists a constant  $C_1 > 0$  such that for sufficiently large  $T$ , the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  has more than  $C_1 T$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2$ ,  $0 < t < T$ .*

**Corollary 1.2.** *Suppose that a function  $L(s)$  is hybridly universal, and  $P_s \in \mathcal{D}_s[X]$  is a polynomial with degree greater than zero. Then, for any  $1/2 < \sigma_1 < \sigma_2 < 1$ , there exists a constant  $C_1 > 0$  such that for sufficiently large  $T$ , the function*

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$(d^k/ds^k)P_s(L(s))$  with  $k \in \mathbb{N}$  has more than  $C_1T$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2, 0 < t < T$ .

**Proposition 1.3.** *For each  $0 \leq j \leq l$ , let  $L^{(j)}(s)$  be a function defined as a Dirichlet series for  $\sigma > 1$  which can be continued analytically to a meromorphic function on  $\Re(s) > 1/2$  with a finite number of poles and all of them lie on the line  $\Re(s) = 1$ . Moreover for each  $0 \leq j \leq l$ , suppose that  $L^{(j)}(s)$  is a function of finite order and for any fixed  $1/2 < \Re(s) < 1$ , the square mean-value satisfies*

$$(1.2) \quad \int_0^T |L^{(j)}(\sigma + it)|^2 dt \ll T, \quad \text{as } T \rightarrow \infty.$$

Then, for any  $\sigma_0 > 1/2$  and any polynomial  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$ , there exists a constant  $C_2 > 0$  such that for sufficiently large  $T$ , the number of zeros  $\rho$  with multiplicities of  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  with  $\Re(\rho) > \sigma_0$  and  $0 < \Im(\rho) < T$  is less than  $C_2T$ .

**Corollary 1.4.** *Let  $L^{(j)}(s)$  satisfy all assumptions in Proposition 1.3. Then, for any  $\sigma_0 > 1/2$  and any polynomial  $P_s \in \mathcal{D}_s[X_0]$ , there exists a constant  $C_2 > 0$  such that for sufficiently large  $T$ , the number of zeros  $\rho$  with multiplicities of  $(d^k/ds^k)P_s(L(s))$  with  $\Re(\rho) > \sigma_0$  and  $0 < \Im(\rho) < T$  is less than  $C_2T$ .*

It should be mentioned that Theorem 1.1 and Proposition 1.3 are generalizations of Main Theorems 1 and 2 in [18], respectively (see Theorem B).

**1.2. Hybrid universality.** In 1975, Voronin [27] showed the following universality theorem for the Riemann zeta function  $\zeta(s) := \sum_{n=1}^\infty n^{-s}$ . In order to state it, we need some notation. Let  $D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$ . Denote by  $\mu(A)$  the Lebesgue measure of the set  $A$ , and, for  $T > 0$ , write

$$\nu_T\{\dots\} := T^{-1}\mu\{\tau \in [0, T] : \dots\}$$

where the dots stand for a condition satisfied by  $\tau$ . Then the modern version of the Voronin theorem can be written as follows.

**Theorem A.** *Let  $K \subset D$  be a compact set with connected complement, and  $f(s)$  be a nonvanishing continuous function on  $K$  which is analytic in the interior of  $K$ . Then for any  $\varepsilon > 0$ , it holds that*

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Roughly speaking, this theorem implies that any nonvanishing analytic function can be uniformly approximated by the Riemann zeta function. In the current era, it is known that many zeta and  $L$ -functions, for example, Dirichlet  $L$ -functions, Dedekind zeta functions, Lerch zeta functions and  $L$ -functions associated with newforms, have the universality property (see for instance [12], [16] and [24]).

Combining the Kronecker-Weyl theorem on diophantine approximations and Voronin’s universality theorem, we can define the hybrid universality. Denote the distance to the nearest integer by  $\|\cdot\|$ . The precise definition is as follows.

**Definition 1.5.** *Hybrid universality* for the function  $L(s)$  is the following property: Let  $K$  and  $f(s)$  be the same as in Theorem A. Suppose that  $\{\alpha_j\}_{1 \leq j \leq k}$  are real numbers linearly independent over  $\mathbb{Q}$ . Then for any  $\varepsilon > 0$  and any real numbers  $\{\theta_j\}_{1 \leq j \leq k}$ , it holds that

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |L(s + i\tau) - f(s)| < \varepsilon, \quad \|\tau\alpha_j - \theta_j\| < \varepsilon, \quad 1 \leq j \leq k \right\} > 0.$$

As mentioned in [18], the first result on hybrid universality was proved in weaker form by Gonek [4] and improved by Kaczorowski and Kulas [9]. They showed that Dirichlet  $L$ -functions  $L(s, \chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$  satisfy the inequality in the above definition for  $\alpha_n = \log p_n$ , where  $p_n$  denotes the  $n$ -th prime number. Afterwards, Pańkowski [21] proved the hybrid universality in the most general form for an axiomatically defined wide class of  $L$ -functions having Euler product. By using the hybrid universality, Nakamura and Pańkowski showed the following theorem in [18, Main Theorem 1].

**Theorem B.** *Suppose that a function  $L(s)$  is hybridly universal and  $P_s \in \mathcal{D}_s[X]$  is not a monomial but a polynomial with degree greater than zero. Then, the function  $P_s(L(s))$  has infinitely many zeros in  $D$ . More precisely, for any  $1/2 < \sigma_1 < \sigma_2 < 1$ , there exists a constant  $C > 0$  such that for sufficiently large  $T$ , the function  $P_s(L(s))$  has more than  $CT$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2$ ,  $0 < t < T$ .*

Moreover, they gave an upper bound for the number of zeros of certain polynomials of  $L$ -functions in [18, Main Theorem 2]. We can say that Theorem 1.1 and Proposition 1.3 are multidimensional cases of Main Theorems 1 and 2 in [18], respectively. It should be mentioned that they also showed that a lot of zeta or  $L$ -functions have infinitely many complex zeros off the critical line in [18, Section 3] (see also Section 3.2).

**1.3. Zeros of the derivatives of the Riemann zeta function.** Zeros of the derivatives of the Riemann zeta function have been investigated by many mathematicians. In 1935, Speiser [23] showed that the Riemann Hypothesis is equivalent to  $\zeta'(s)$  having no zeros in the vertical strip  $0 < \Re(s) < 1/2$ . Let  $N_k(T)$ ,  $k \geq 1$ , denote the number of zeros  $\beta_k + i\gamma_k$  of the  $k$ -th derivative of the Riemann zeta function  $\zeta^{(k)}(s)$  such that  $0 < \gamma_k < T$ . Berndt [1, p. 577] proved that

$$N_k(T) = \frac{T \log T}{2\pi} - \frac{1 + \log 4\pi}{2\pi} T + O(\log T), \quad T \rightarrow \infty.$$

Levinson and Montgomery [14] obtained several results on the zeros of  $\zeta^{(k)}(s)$ . For example, they showed

$$2\pi \sum_{0 < \gamma_k \leq T} (\beta_k - 1/2) = kT \log \log(T/2\pi) - 2\pi k \text{Li}(T/2\pi) \\ + (\log 2 - 2k \log \log 2)(T/2) + O(\log T),$$

where  $\text{Li}(x) := \int_2^x dy/\log y$  in [14, Theorem 10]. They also proved in [14, Theorem 7] that if  $\zeta^{(j)}$ ,  $j \in \mathbb{N} \cup \{0\}$  has only a finite number of nonreal zeros in  $\Re(s) < 1/2$ , then  $\zeta^{(j+k)}$  has the same property for  $k \geq 1$ .

On the other hand, Launrinčikas [11] showed the following. For any  $\sigma_1$  and  $\sigma_2$  such that  $1/2 < \sigma_1 < \sigma_2 < 1$ , there exists a  $C = C(\sigma_1, \sigma_2) > 0$  such that for sufficiently large  $T$  one can find more than  $CT$  zeros of the function  $\zeta'(s)$  lying in the domain  $\sigma_1 < \Re(s) < \sigma_2$  and  $0 < \Im(s) < T$ . In [15, Theorem 2], Meyrath proved that  $\zeta^{(k)}(s)$ ,  $k \geq 1$  has the strong universality, namely,  $\zeta^{(k)}(s)$  can approximate any analytic function (with zeros) in the sense of the universality (see for example [24, p. 30] for the definition of the strong universality). Therefore, we have the following.

**Theorem C.** For any  $\sigma_1$  and  $\sigma_2$  with  $1/2 < \sigma_1 < \sigma_2 < 1$ , there exists a constant  $C$  such that for sufficiently large  $T$  one can find more than  $CT$  zeros of the function  $\zeta^{(k)}(s)$  with  $k \in \mathbb{N}$  lying in  $\sigma_1 < \Re(s) < \sigma_2$  and  $0 < \Im(s) < T$ .

Furthermore, Launrinčikas [13, p. 200] showed that  $\sum_{k=1}^l a_k \zeta^{(k)}(s)$ , where  $a_k \in \mathbb{C} \setminus \{0\}$  has the strong universality (see also [13, Theorem 3.1]). Hence  $\sum_{k=1}^l a_k \zeta^{(k)}(s)$  with  $a_k \in \mathbb{C} \setminus \{0\}$  has more than  $CT$  zeros in  $\sigma_1 < \Re(s) < \sigma_2$  and  $0 < \Im(s) < T$ . Obviously, Theorem 1.1 is a generalization of this result and Theorem C.

2. PROOFS

**2.1. Preliminary.** We quote the first lemma from [12, Lemma 6.6.1] or [24, p. 194]. We can find the second lemma in [17, Lemma 1] and [19, Lemma 2.1].

**Lemma 2.1.** Let  $c_0 \neq 0, c_1, \dots, c_m$  be complex numbers. Then there exists a polynomial  $p_m(s) = b_0 + b_1s + \dots + b_ms^m$  such that

$$c_n = \frac{d^n}{ds^n} \left[ \exp(p_m(s)) \right]_{s=0}, \quad 0 \leq n \leq m.$$

**Lemma 2.2.** Let  $u, v \in \mathbb{N}$  and  $D_u(s) = \sum_{n=1}^\infty a_{u,n} e^{-\lambda_{u,n}s}$  be a general Dirichlet series for  $1 \leq u \leq v$ . Then, for every  $\varepsilon > 0$  and every compact set  $K$  lying in the half-plane of absolute convergence, there exist  $\delta > 0, M \in \mathbb{N}$  and a finite set  $I \in \{1, 2, \dots, v\} \times \mathbb{N}$  such that the numbers  $\lambda_{j,k}$ , where  $(j, k) \in I$ , are linearly independent over  $\mathbb{Q}$  and, moreover, if

$$\max_{(j,k) \in I} \left\| \frac{\tau \lambda_{j,k}}{2\pi M} \right\| < \delta,$$

then it holds that

$$\max_{1 \leq u \leq v} \max_{s \in K} |D_u(s + i\tau) - D_u(s)| < \varepsilon.$$

*Proof.* For the reader’s convenience, we write the proof here. Take  $\varepsilon > 0$  and fix a compact set  $K$ . Then there exists a positive integer  $N$  such that

$$\max_{1 \leq u \leq v} \max_{s \in K} \sum_{n > N} |a_{u,n}| e^{-\lambda_{u,n} \Re(s)} < \varepsilon$$

since the Dirichlet series  $D_u(s)$  converges absolutely on the compact set  $K$ . Now take the set  $I$  of indices such that  $\{\lambda_{j,k} : (j, k) \in I\}$  is the basis of a vector space over  $\mathbb{Q}$  generated by all numbers  $\lambda_{u,n}$  with  $n \leq N$ . Moreover, let  $M$  be a positive integer such that all these numbers can be expressed as a linear combination of elements  $\lambda_{j,k}/M$ , where  $(j, k) \in I$ , with integer coefficients. Then, for  $s \in K$  and  $1 \leq u \leq v$ , it holds that

$$\begin{aligned} |P_u(s + i\tau) - P_u(s)| &\leq \sum_{n \leq N} |a_{u,n}| \left| e^{-\lambda_{u,n}(s+i\tau)} - e^{-\lambda_{u,n}s} \right| + 2 \sum_{n > N} |a_{u,n}| e^{-\lambda_{u,n} \Re(s)} \\ &\ll \max_{n \leq N} |e^{-i\tau \lambda_{u,n}} - 1| + 2\varepsilon \ll \max_{(j,k) \in I} \left\| \frac{\tau \lambda_{j,k}}{2\pi M} \right\| + 2\varepsilon. \end{aligned}$$

By taking suitable  $\delta > 0$ , we complete the proof. □

**Lemma 2.3.** *Let  $\mathcal{K}$ ,  $\mathcal{K}'$  and  $\mathcal{K}''$  be closed disks whose centers are  $a \in \mathbb{C}$  and radiuses are  $r$ ,  $r'$ ,  $r''$  which satisfy  $0 < r < r' < r''$ . Suppose that  $f(s)$  and  $g(s)$  are analytic in the interior of  $\mathcal{K}''$  and satisfy  $\max_{s \in \mathcal{K}'} |f(s) - g(s)| < \varepsilon$ . Then one has*

$$(2.1) \quad \max_{s \in \mathcal{K}} |f^{(k)}(s) - g^{(k)}(s)| < \frac{k!2^k \varepsilon}{(r' - r)^k}.$$

*Proof.* Put  $r_{\#} := (r' - r)/2$ . Note that  $s + r_{\#}e^{i\theta} \in \mathcal{K}'$  for all  $0 \leq \theta < 2\pi$  when  $s \in \mathcal{K}$ . From Cauchy's integral formula, we have

$$\begin{aligned} |f^{(k)}(s) - g^{(k)}(s)| &= \left| \frac{k!}{2\pi i} \int_{|z-s|=r_{\#}} \frac{f(z) - g(z)}{(z-s)^{k+1}} dz \right| \\ &= \left| \frac{k!}{2\pi i} \int_0^{2\pi} \frac{f(s + r_{\#}e^{i\theta}) - g(s + r_{\#}e^{i\theta})}{(r_{\#}e^{i\theta})^{k+1}} r_{\#}e^{i\theta} d\theta \right| < \frac{k! \varepsilon}{2\pi} \int_0^{2\pi} \frac{d\theta}{r_{\#}^k}, \end{aligned}$$

for any  $s \in \mathcal{K}$ . This inequality implies (2.1).  $\square$

## 2.2. Proof of main results.

*Proof of Theorem 1.1.* For any  $\varepsilon > 0$ , any closed disk  $K' \subset D$  and any function  $f(s)$  which is nonvanishing and continuous on  $K''$  and analytic in the interior of  $K''$ , where  $K'' \supset K'$ , there exists  $\tau \in \mathbb{R}$  such that

$$\max_{1 \leq u \leq v} \max_{s \in K'} |D_u(s + i\tau) - D_u(s)| < \varepsilon \quad \text{and} \quad \max_{s \in K'} |L(s + i\tau) - f(s)| < \varepsilon$$

from Lemma 2.2 and the definition of the hybrid universality. According to Lemma 2.3, for  $K \subset K'$ , there exists  $\tau \in \mathbb{R}$  such that

$$(2.2) \quad \begin{cases} \max_{1 \leq u \leq v} \max_{s \in K} |D_u(s + i\tau) - D_u(s)| < \varepsilon, \\ \max_{0 \leq j \leq l} \max_{s \in K} |L^{(j)}(s + i\tau) - f^{(j)}(s)| < \varepsilon. \end{cases}$$

First suppose

$$(2.3) \quad P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s)) = D(s)L^{(k)}(s),$$

where  $1 \leq k \leq l$  and  $D(s)$  is a general Dirichlet series converges absolutely when  $\Re(s) > 1/2$ . Now let  $1/2 < \sigma_1 < \sigma_2 < 1$ ,  $K \subset \{s \in \mathbb{C} : \sigma_1 < \Re(s) < \sigma_2\}$  be a closed disk whose center is  $\alpha \in \mathbb{C}$  and  $f(s) := se^{\eta s}$  with  $\eta \neq 0$ . Obviously, the function  $se^{\eta s}$  is nonvanishing and continuous on  $K$  and analytic in the interior of  $K$ . In addition, one has

$$f^{(k)}(s) = \frac{d^k}{ds^k} se^{\eta s} = k\eta^{k-1}e^{\eta s} + \eta^k se^{\eta s} = \eta^{k-1}e^{\eta s}(k + \eta s)$$

by the principle of mathematical induction. Put  $\eta = -k/\alpha$ . Then the function

$$(2.4) \quad A(s, \alpha, k) := \frac{d^k}{ds^k} se^{-ks/\alpha} = (-1)^{k-1} k^{k-1} \alpha^{-k} e^{-ks/\alpha} (k\alpha - ks)$$

has only one zero at  $s = \alpha$  when  $s \in K$ . From (2.2), we have

$$\max_{|s-\alpha|=r} |D(s+i\tau)L^{(k)}(s+i\tau) - D(s)f^{(k)}(s)| < \min_{|s-\alpha|=r} |D(s)f^{(k)}(s)|$$

if  $D(s)$  does not vanish on the circle  $|s - \alpha| = r$ . By applying Rouché's theorem, we can show that whenever the inequality above holds, the function  $D(s+i\tau)L^{(k)}(s+i\tau)$  has a zero in the interior of  $K$ . Moreover, the measure of such  $\tau \in [0, T]$  is greater than  $cT$  for some constant  $c > 0$  according to the hybrid universality. Hence, for

some positive constant  $C_1$ , the function  $D(s)L^{(k)}(s)$  has more than  $C_1T$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2, 0 < t < T$  when  $1 \leq k \leq l$ .

Assume that  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  is a monomial such that at least one of the degrees of  $X_1, \dots, X_l$  is greater than zero. Then a factor of the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  is expressed as the form (2.3). Therefore, the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  satisfies the condition above has more than  $C_1T$  nontrivial zeros in  $\sigma_1 < \sigma < \sigma_2, 0 < t < T$ .

Next suppose  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  is a nonmonomial polynomial. Then we can see that at least one of the degrees of  $X_0, \dots, X_l$  is greater than zero. It should be noted that we show the case that not at least one of the degrees of  $X_1, \dots, X_l$  is greater than zero but at least one of the degrees of  $X_0, X_1, \dots, X_l$  is greater than zero since the latter case needs less assumptions and contains [18, Main Theorem 1] or Theorem B. In this case, the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  can be written by

$$(2.5) \quad \begin{aligned} &P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s)) \\ &= \sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} D_{d_0 d_1 \dots d_l}(s) (L^{(0)}(s))^{d_0} (L^{(1)}(s))^{d_1} \dots (L^{(l)}(s))^{d_l} \end{aligned}$$

where  $D_{d_0 d_1 \dots d_l}(s)$  are general Dirichlet series converging absolutely when  $\Re(s) > 1/2$  and at least two of  $D_{d_0 d_1 \dots d_l}(s)$  with  $0 \leq d_j \leq n_j$  for  $0 \leq j \leq l$  are not identically vanishing. Let  $K \subset \{s \in \mathbb{C} : 1/2 < \sigma_1 < \Re(s) < \sigma_2 < 1\}$  be a closed disk whose center is  $\alpha \in \mathbb{C}$  and let at least two of  $D_{d_0 d_1 \dots d_l}(\alpha)$  with  $0 \leq d_j \leq n_j$  for  $0 \leq j \leq l$  be not zero. This is possible from the assumption that  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  is a nonmonomial polynomial. Then we can choose nonzero complex numbers  $\theta_0, \theta_1, \dots, \theta_l$  which satisfy

$$(2.6) \quad 0 = \sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} D_{d_0 d_1 \dots d_l}(\alpha) \theta_0^{d_0} \theta_1^{d_1} \dots \theta_l^{d_l}.$$

For  $0 \leq j \leq l$ , we can take  $\eta_0, \eta_1, \dots, \eta_l \in \mathbb{C}$  satisfying

$$\theta_j = \frac{d^j}{ds^j} [f(s)]_{s=\alpha}, \quad f(s) := \exp(\eta_0 + \eta_1(s - \alpha) + \dots + \eta_l(s - \alpha)^l)$$

from Lemma 2.1. Hence  $A(s, \alpha; f)$  written by

$$(2.7) \quad A(s, \alpha; f) := \sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} D_{d_0 d_1 \dots d_l}(s) (f^{(0)}(s))^{d_0} (f^{(1)}(s))^{d_1} \dots (f^{(l)}(s))^{d_l}$$

has a zero at the point  $s = \alpha$  from (2.6) and the definition of  $f(s)$ . For simplicity, we put

$$(2.8) \quad Z(s + i\tau) := P_{s+i\tau}(L(s + i\tau), L^{(1)}(s + i\tau), \dots, L^{(l)}(s + i\tau)).$$

By using (2.2), we have

$$\begin{aligned} &\max_{|s-\alpha|=r} \left| Z(s + i\tau) - \sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} D_{d_0 d_1 \dots d_l}(s) (f^{(0)}(s))^{d_0} (f^{(1)}(s))^{d_1} \dots (f^{(l)}(s))^{d_l} \right| \\ &< \min_{|s-\alpha|=r} \left| \sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} D_{d_0 d_1 \dots d_l}(s) (f^{(0)}(s))^{d_0} (f^{(1)}(s))^{d_1} \dots (f^{(l)}(s))^{d_l} \right| \end{aligned}$$

when the function  $A(s, \alpha; f)$  does not vanish on the circle  $|s - \alpha| = r$ . Therefore, the function  $Z(s + i\tau)$  defined by (2.8) also has a zero in the interior of  $K$  by Rouché's theorem. Thus  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  has more than  $C_1 T$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2$ ,  $0 < t < T$  by the hybrid universality when  $P_s \in \mathcal{D}_s[X_0, X_1, \dots, X_l]$  is a nonmonomial polynomial.  $\square$

*Proof of Proposition 1.3.* The function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  can be continued analytically to a meromorphic function on  $\Re(s) > 1/2$  with a finite number of poles and all of them lie on the line  $\Re(s) = 1$  according to (2.5) and the assumptions of  $L^{(j)}(s)$ . Moreover,  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  is a function of finite order from (2.5) and the assumption that  $L^{(j)}(s)$  are functions of finite order for all  $0 \leq j \leq l$ .

Put  $n := (l + 1)^2 n_0 n_1 \cdots n_l$ . Due to Hölder's inequality, one has

$$\begin{aligned} & \int_0^T \left| D_{d_0 d_1 \dots d_l}(\sigma + it) (L^{(0)}(\sigma + it))^{d_0} \cdots (L^{(l)}(\sigma + it))^{d_l} \right|^{2/n} dt \\ & \ll \int_0^T |L^{(0)}(\sigma + it) \cdots L^{(l)}(\sigma + it)|^{2/(l+1)} dt \\ & \leq \left( \int_0^T |L^{(0)}(\sigma + it)|^2 dt \right)^{1/(l+1)} \cdots \left( \int_0^T |L^{(l)}(\sigma + it)|^2 dt \right)^{1/(l+1)}. \end{aligned}$$

Hence, by the assumption (1.2), it holds that

$$\sum_{0 \leq d_j \leq n_j, 0 \leq j \leq l} \int_0^T \left| D_{d_0 d_1 \dots d_l}(\sigma + it) (L^{(0)}(\sigma + it))^{d_0} \cdots (L^{(l)}(\sigma + it))^{d_l} \right|^{2/n} dt \ll T.$$

Therefore, we obtain Proposition 1.3 by modifying the proof of [18, Main Theorem 2] (see also the proofs of [24, Theorems 7.1 and 7.6]).  $\square$

*Proof of Corollaries 1.2 and 1.4.* Let  $D(s)$  be a general Dirichlet series converging absolutely in the half-plane  $\Re(s) > 1/2$ . Then it is well known that the general Dirichlet series of  $D^{(k)}(s)$  also converges absolutely when  $\Re(s) > 1/2$  for any  $1 \leq k \leq l$  (see for instance [7, Theorem 4]). Hence  $(d^k/ds^k)P_s(L(s))$  can be expressed as (2.5). Thus Corollaries 1.2 and 1.4 are deduced by Theorem 1.1 and Proposition 1.3, respectively.  $\square$

### 3. REMARKS AND EXAMPLES

**3.1. Remarks.** We have the following remarks.

(1) In general, polynomials  $P_s(L(s))$  are not universal even if  $L(s)$  is hybridly universal. As an example, consider the nonuniversality of the following function:

$$F(s) = (1 - 9^{9(s-3/4)})\zeta(s) + 2$$

in the close disk  $\mathcal{K}$  whose center is  $3/4$  and the radius is  $\pi/\log 9^9 < r < 1/4$ . Note that  $\pi/\log(9^9) = 0.158867 \cdots < 1/4$ . Then every shifted function  $F(s + i\tau) - 2$  has at least one zero inside  $\mathcal{K}$ , and hence it cannot approximate uniformly functions which do not vanish inside  $\mathcal{K}$ . In particular, the function  $F(s + i\tau) - 2$  cannot approximate the constant function 1 (we can find this argument in [10, p. 114]). Therefore, the function  $F(s)$  cannot approximate the constant function 3 in the sense of universality. Despite the fact above, we can show that  $F(s)$  has zeros in the strip  $1/2 < \sigma < 1$  by Theorem 1.1.

(2) In [5, Theorem 4], Gonek, Lester and Milinovich proved that a positive proportion of the  $a$ -points of  $\zeta(s)$  are simple in fixed strips to the right of the line  $\Re(s) = 1/2$ . As a generalization of this result, we consider simple zeros of the function  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$ . Suppose that  $A(s, \alpha, k)$  or  $A(s, \alpha; f)$ , defined by (2.4) or (2.7) respectively, has only a simple zero in the disk  $K$  whose center is  $\alpha$  and the radius is  $r$ . Then  $Z(s+i\tau)$  written by (2.8) also has a simple zero in the disk  $K$  from Rouché's theorem. Hence there is a positive constant  $c_1$  such that at least  $c_1T$  of zeros of  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  in the region  $(\sigma_1, \sigma_2) \times (0, T)$  are simple in this case. It should be mentioned that  $A(s, \alpha; f)$  does not have simple zeros when  $P_s(L(s), L^{(1)}(s), \dots, L^{(l)}(s))$  can be expressed as  $(\zeta(s) - 1)^2, (\zeta^2(s) - \zeta(2s))^3, (\zeta'(s) + \zeta''(s) + \zeta(s+1))^5$  and so on.

(3) The Riemann zeta function  $\zeta(s)$  satisfies all assumptions in Proposition 1.3. This is proved as follows. By the following partial summation:

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s \int_N^\infty \frac{[u] - u}{u^{s+1}} du,$$

where  $[u]$  denotes the maximal integer less than or equal to  $u$  (see for example [24, (1.3)]), we can see that  $\zeta^{(j)}(s)$  is a function of finite order and for any  $0 \leq j \leq l$  and any fixed  $1/2 < \Re(s) < 1$ . Moreover, it is known that if  $\eta$  and  $\theta$  are fixed and  $\eta, \theta > -1/2, \eta + \theta > 1$ , then it holds that

$$\int_1^\infty \zeta^{(u)}(\eta + it) \zeta^{(v)}(\theta - it) dt \sim \zeta^{(u+v)}(\eta + \theta)T,$$

where  $u$  and  $v$  are nonnegative integers (see for example [8, (2)]). Therefore, the derivatives of the Riemann zeta function  $\zeta^{(j)}(s)$  fulfill (1.2) for all  $0 \leq j \leq l$ .

(4) There exist some functions composed by  $\zeta(s)$  which do not vanish in the strip  $D$ . For example, the function

$$\exp(\zeta(s)) = \sum_{n=0}^\infty \frac{\zeta^n(s)}{n!} = 1 + \zeta(s) + \frac{\zeta^2(s)}{2!} + \frac{\zeta^3(s)}{3!} + \dots$$

does not have zeros in the vertical strip  $D$ . Hence we cannot remove the assumption that  $P_s$  is a polynomial (with finite degree) in Theorem 1.1. Furthermore, we introduce functions  $G(s), F_\pm(s) \in \mathcal{D}_s[\zeta(s), s]$  which do not vanish when  $s \in D$  as in Examples (4) and (5).

**3.2. Examples.** We have the following examples which we can apply to the main results.

(1) By Corollaries 1.2 and 1.4, the derivatives of zeta functions appearing in [18, Section 3], for example, the zeta functions associated to symmetric matrices treated by Ibukiyama and Saito, certain spectral zeta functions and the Euler-Zagier multiple zeta functions, have more than  $C_1T$  and less than  $C_2T$  nontrivial zeros in some rectangle since these zeta functions can be expressed by the form (2.5).

(2) It is known that the second derivative of the logarithm of the Riemann zeta function  $(\log \zeta(s))'' = (\zeta'(s)/\zeta(s))' = (\zeta(s))^{-2}(\zeta(s)\zeta''(s) - \zeta'(s)^2)$  appears in the pair correlation for the Riemann zeta function (see for example Bogomolny and Keating [2] and Rodgers [22]). Recently, Stopple [25, Theorem 3] showed that for

positive  $\delta$  the number of zeros of

$$\mu(s) := \zeta(s)\zeta''(s) - \zeta'(s)^2$$

in the region  $|\Im(s)| \leq T$ ,  $\Re(s) > 5/6 + \delta$  is less than  $C(\delta)T$ , where  $C(\delta) > 0$ . By using Theorem 1.1, Proposition 1.3 and the fact mentioned in Remark 3, the third remark in Section 3.1, we can prove that the number of zeros of  $\mu(s)$  in the region  $|\Im(s)| \leq T$ ,  $\Re(s) > 5/6$  is more than  $C_1T$  and less than  $C_2T$ , where  $C_2 \geq C_1 > 0$ .

(3) Let  $m \in \mathbb{N}$  and  $H(P)$ , where  $P \in \mathbb{P}^m(\mathbb{Q})$ , be the height of the projective space  $\mathbb{P}^m(\mathbb{Q})$ . Then the height zeta function  $\mathcal{Z}_m(s)$  is expressed as

$$\mathcal{Z}_m(s) := \sum_{P \in \mathbb{P}^m(\mathbb{Q})} H(P)^{-s} = \frac{1}{\zeta(s)} \sum_{n=0}^{\lfloor m/2 \rfloor} \binom{m+1}{2n+1} 2^{m-2n} \zeta(s-m+2n)$$

(see for example [26, Section 3.4.3]). It is well known that for  $\Re(s) > 1$ , one has  $(\zeta(s))^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ . Note that the Möbius function  $\mu(n)$  is defined by  $\mu(n) = 0$  if  $n$  is divisible by  $p^2$  for some prime number  $p$ , and otherwise  $\mu(n) = (-1)^{\omega(n)}$ , where  $\omega(n)$  is the number of distinct prime factors of  $n$  (see for instance [3, p. 156]). Therefore, the derivatives of zeta function  $\mathcal{Z}_m(s)$  have more than  $C_1T$  and less than  $C_2T$  nontrivial zeros in the rectangle  $\sigma_1 < \sigma < \sigma_2$ ,  $0 < t < T$ , where  $m - 1/2 < \sigma_1 < \sigma_2 < m$  when  $m \geq 2$  from Corollaries 1.2 and 1.4.

We can find some other zeta functions which can be written as (nonmonomial) polynomials in [6, D-16, D-19, D-64, D-70 and D-71].

(4) In [20, Theorem 1.3], it is proved that the function

$$G(s) := \zeta(s) + Cs \in \mathcal{D}_s[\zeta(s), s]$$

does not vanish in the half-plane  $\sigma > 1/18$  when  $|C| > 10$  and  $-19/2 \leq \Re(C) \leq 17/2$ . By Corollaries 1.2 and 1.4, the derivatives of the function  $(d^k/ds^k)(\zeta(s) + Cs)$  with  $k \in \mathbb{N}$  has more than  $C_1T$  and less than  $C_2T$  complex zeros in the rectangle  $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$ ,  $0 < t < T$  for some positive constants  $C_1 \leq C_2$ .

(5) All complex zeros of the function

$$F_{\pm}(s) := s - 1 \pm 2\pi \frac{\zeta(s-1)}{\zeta(s+1)} = \frac{\Gamma(s)}{\Gamma(s-1)} \pm 2\pi \frac{\zeta(s-1)}{\zeta(s+1)} \in \mathcal{D}_s[\zeta(s), s]$$

are on the critical line  $\Re(s) = 1/2$  (see [3, p. 262]). Hence the functions  $F_{\pm}(s)$  satisfy an analogue of the Riemann hypothesis. Note that  $1/\Gamma(s)$  is analytic on the whole complex plane and recall that  $(\zeta(s))^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ , where  $\mu(n)$  is the Möbius function. Therefore, the functions  $(d^k/ds^k)F_{\pm}(s)$  with  $k \in \mathbb{N}$  have more than  $C_1T$  and less than  $C_2T$  complex zeros in the rectangle  $3/2 < \sigma_3 < \sigma < \sigma_4 < 2$ ,  $0 < t < T$ .

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