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BIHARMONIC HYPERSURFACES IN A SPHERE

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ABSTRACT. In this short paper we will survey some recent developments in the geometric theory of biharmonic submanifolds, with an emphasis on the newly discovered Liouville type theorems and applications of known Liouville type theorems in the research of the nonexistence of biharmonic submanifolds. A new Liouville type theorem for superharmonic functions on complete manifolds is proved and its applications in a kind of nonexistence of biharmonic hypersurfaces in a sphere is provided.

1. INTRODUCTION

In their celebrated 1964 paper Eells and Sampson (cf. [15]) made great progress in the theory of harmonic maps. Since then many works on harmonic maps are done and they have been used in various fields in differential geometry. However there are nonexistence results for harmonic maps. Therefore a generalization of harmonic maps is an important subject.

In their paper Eells and Sampson suggested considering the (intrinsic) bi-energy of a map ϕ defined by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 d\mu_g, \tag{1.1}$$

where $\tau(\phi)$ is the tension field of ϕ and $d\mu_g$ is the volume element on (M, g). Stationary points of the bi-energy functional are called biharmonic maps. Jiang (cf. [21], [22], [23]) is the first mathematician who seriously considered the bi-energy functional, and he computed the first and second variations of E_2. The stationary points of the functional E_2 satisfy the E-L equation (cf. [21])

$$-\Delta\tau(\phi) = \sum_{i=1}^{m} R^N(d\phi(e_i), \tau(\phi))d\phi(e_i), \tag{1.2}$$

where Δ is the Laplacian, R^N is the Riemann curvature tensor of the ambient manifold N and $\{e_i, i = 1, \ldots, m\}$ is a local orthonormal frame field of M. If further ϕ is an isometry, ϕ satisfies the equation

$$-\Delta\vec{H} = \sum_{i=1}^{m} R^N(e_i, \vec{H})e_i, \tag{1.3}$$

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where $\vec{H}$ is the mean curvature vector field. By decomposing $-\Delta\vec{H} - \sum_{i=1}^{m} R^N(e_i, \vec{H})e_i$ into its tangential and normal parts, we see that a submanifold is biharmonic if and only if it satisfies (cf. [9], [10])

$$\Delta^{\perp}\vec{H} - \sum_{i=1}^{m} B(A_{\vec{H}}e_i, e_i) + \sum_{i=1}^{m}(R^N(e_i, \vec{H})e_i)^{\perp} = 0, \tag{1.4}$$

$$m\nabla|\vec{H}|^2 + 4\sum_{i=1}^{m} A_{\nabla^{\perp}_{e_i}\vec{H}}e_i - 4\sum_{i=1}^{m}(R^N(e_i, \vec{H})e_i)^{T} = 0, \tag{1.5}$$

where $(R^N(e_i, \vec{H})e_i)^{\perp}, (R^N(e_i, \vec{H})e_i)^{T}$ denote the normal and tangential parts of $R^N(e_i, \vec{H})e_i$ respectively, B is the second fundamental form, A is the shape operator and $\Delta^{\perp}$ is the Laplacian of the normal connection $\nabla^{\perp}$. In particular, if N is a space form of constant sectional curvature c, then a submanifold is biharmonic if and only if

$$\Delta^{\perp}\vec{H} - \sum_{i=1}^{m} B(A_{\vec{H}}e_i, e_i) + cm\vec{H} = 0, \tag{1.6}$$

$$m\nabla|\vec{H}|^2 + 4\sum_{i=1}^{m} A_{\nabla^{\perp}_{e_i}\vec{H}}e_i = 0. \tag{1.7}$$

Furthermore, if M is a hypersurface in N and H is the mean curvature of M, then we have (cf. [27])

$$\Delta H - H|A|^2 + HRic^N(\xi, \xi) = 0. \tag{1.8}$$

A submanifold satisfying equation (1.3) is called a biharmonic submanifold. In [22] Jiang proved that every biharmonic surface in $\mathbb{R}^3$ is minimal. In the study of his finite type submanifolds Chen proposed to consider submanifolds with harmonic mean curvature, which is just biharmonic submanifolds whose target manifold is a Euclidean space. In [11] Chen found Jiang's nonexistence result independently (in an earlier version of [11]) and conjectured that every biharmonic submanifold in a Euclidean space is minimal, which is nowadays called Chen's conjecture. Several partial answers of Chen's conjecture were obtained by Dimitric, who was Chen's Ph.D. student, in his Ph.D. thesis (cf. [14]). In particular, he proved that every biharmonic curve in a Euclidean space is an open part of a straight line. Another striking result is proved by Hasanis and Vlachos (cf. [19]) that every biharmonic hypersurface in $\mathbb{R}^4$ is minimal. See also [13] for a more mathematical proof. These results make Chen's conjecture popular, particularly in the past decade. Inspired by the success of Chen's conjecture, Caddeo, Montaldo and Oniciuc (cf. [6]) proposed the generalized Chen's conjecture.

Generalized Chen's Conjecture (GCC). *Every biharmonic submanifold in a non-positively curved manifold is minimal.*

Until now there have been many affirmative partial answers to the GCC. In particular, there are affirmative partial answers to the GCC in the case where ambient space is hyperbolic space. Caddeo, Montaldo and Oniciuc (cf. [7]) showed that every biharmonic surface in $\mathbb{H}^3(-1)$ is minimal. Balmuş, Montaldo and Oniciuc (cf. [4]) showed that every biharmonic hypersurface in $\mathbb{H}^4(-1)$ is minimal. Now we know the GCC is false, by a counterexample constructed by Ou and Tang (cf. [28]).

See also [29] for further examples. Hence the second author (cf. [31]) proposed the following global version of the GCC:

Generalized Chen's Conjecture: global version. *Every complete biharmonic submanifold in a nonpositively curved manifold is minimal.*

Under the completeness assumption, things should be easier. For example we could use cut-off functions and the integral by parts argument. With this method, Nakauchi and Urakawa (cf. [34]) first obtained an affirmative partial answer to the GCC under the assumption of completeness and the integrability of the mean curvature vector field. After their studies, several generalizations of their results were obtained (cf. [24], [30], [35]). In particular, in [24] the first author proved among other things that every complete biharmonic submanifold in a nonpositively curved manifold with an $L^p(0 < p < +\infty)$ integrable mean curvature vector field is minimal. He used certain Liouville type theorems on complete manifolds, and it was made explicit and broadened in a subsequent paper (cf. [25, Theorem 1.9]). Here we remark that the application of cut-off functions to biharmonic submanifolds was first used by Wheeler [39] and independently used by Nakauchi and Urakawa [34].

The study of the GCC is a good prototype of applying Liouville type theorems and it is also very helpful in finding new Liouville type theorems of functions on complete manifolds, as we showed in the last paragraph and will show in the following one. Akutagawa and the second author (cf. [1]) showed that every properly immersed biharmonic submanifold in a Euclidean space is minimal, where they used essentially a new Liouville type theorem inspired by Omori-Yau's generalized maximal principle. This argument was later developed by the authors to obtain a new Liouville type theorem for submanifolds in nonpositively curved target manifolds with certain curvature decay order at infinity (cf. [26], [31]).

When the target manifold is a Euclidean sphere, in contrast to the nonpositive curvature case, there are natural examples of nonminimal biharmonic submanifolds. For example, $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ and $\mathbb{S}^{n-p}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{p-1}(\frac{1}{\sqrt{2}})(n-p \neq p-1)$ in $\mathbb{S}^n$ given by Jiang (cf. [21]). But similar to the nonpositive curvature case, it is conjectured (cf. [3]) that biharmonic submanifolds in a sphere is a constant mean curvature submanifold by Balmuş, Montaldo and Oniciuc:

BMO conjecture. *Every biharmonic submanifold in a sphere has constant mean curvature.*

There are affirmative partial answers to the BMO conjecture if M is one of the following: (i) A compact hypersurface with nowhere zero mean curvature vector field and $|B|^2 \geq m$ by J. H. Chen (cf. [12]), (ii) a compact hypersurface with nowhere zero mean curvature vector field and $|B|^2 \leq m$ by Balmuş, Montaldo and Oniciuc (cf. [4]), (iii) a compact submanifold with $|\vec{H}| \geq 1$ by Balmuş and Oniciuc (cf. [2]; see also [30]), where $|B|^2$ is the squared norm of the second fundamental form. There are many studies for biharmonic submanifolds in the sphere (cf. [5], [6], [7], [16], [36], etc.). Recently the second author considered the complete noncompact case, and he proved the BMO conjecture under the assumption of $H \geq 1$, together with certain integrality condition (cf. [32]), where he used an argument originated from [24], [30] and [34]. For the $H \in (0, 1]$ case, he proved the BMO conjecture under the assumption

$$\int_M H^{-p} d\mu_g < +\infty, \tag{1.9}$$

where he used a Liouville type theorem for the drift Laplacian due to Petersen and Wylie (cf. [37]) to get this result. We observe that we could use an old Liouville type theorem of Yau to get a stronger result. Actually we could get the following Liouville type theorem for superharmonic functions on a complete manifold, and as a corollary we can get a stronger result for biharmonic hypersurfaces. We have

Theorem 1.1. *Let (M,g) be a complete noncompact manifold and $u \in (0,C]$ $(C>0)$ a superharmonic function on M. If*

$$\int_M (\log^{(k)} \frac{Ce^{(k)}}{u})^p d\mu_g < +\infty \tag{1.10}$$

for some $p>0$ and $k \in \mathbb{N}$, then u is a constant. Here $\log^{(k)} = \log(\log^{(k-1)})$ and $e^{(k)} = e^{e^{(k-1)}}$, where $\log^{(1)} = \log$ and $e^{(1)} = e$.

Applying this Liouville type theorem, we obtain the following theorem.

Theorem 1.2. *Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a complete biharmonic hypersurface. Assume that the mean curvature H satisfies $0 < H \leq 1$. We also assume that $|B|^2 \leq Ric^N(\xi,\xi)$, where B is the second fundamental form of M in N, Ric^N is the Ricci curvature of N, and ξ is the unit normal vector field on M. If*

$$\int_M (\log^{(k)} \frac{e^{(k)}}{H})^p d\mu_g < +\infty \tag{1.11}$$

for some $p>0$ and $k \in \mathbb{N}$, then H is constant.

Remark 1.3. It is easy to see that condition (1.11) is weaker than condition (1.9).

As a corollary of Theorem 1.2 we have

Corollary 1.4. *Let $\phi : (M^m, g) \to (\mathbb{S}^{m+1}, h)$ be a complete biharmonic hypersurface with $H>0$. If $|B|^2 \leq m$ and*

$$\int_M (\log^{(k)} \frac{e^{(k)}}{H})^p d\mu_g < +\infty \tag{1.12}$$

for some $0<p<\infty$ and $k \in \mathbb{N}$, then H is constant.

Recently Hornung and Moser defined p-biharmonic ($p>1$) maps, and isometric p-biharmonic maps are called p-biharmonic submanifolds. In the appendix we will discuss the nonexistence results of p-biharmonic submanifolds.

The rest of this paper is organized as follows: Our main result is proved in section 2. In the appendix, we apply our main result to p-biharmonic hypersurfaces.

2. Proofs

In this section, we will use notation $\log^{(k)} = \log(\log^{(k-1)})$ and $e^{(k)} = e^{e^{(k-1)}}$, where $\log^{(1)} = \log$ and $e^{(1)} = e$. We will need the following Liouville type theorem due to Yau (cf. [38], Theorem 1).

Theorem 2.1 (Yau)**.** *Suppose f is bounded from below by a constant and is Lebesgue integrable with $0 < \int_M f \leq \infty$. Assume that $\Delta \log v = f, v>0$; then $\int_M v^p = \infty$ for $p>0$, unless v is a constant function. If f is zero almost everywhere, the same conclusion holds.*

Proof of Theorem 1.1. (I) The case $k = 1$: A direct computation shows that

$$\begin{aligned}\Delta \log\log \frac{Ce}{u} &= \frac{\log \frac{Ce}{u} \Delta \log \frac{Ce}{u} - |\nabla \log \frac{Ce}{u}|^2}{(\log \frac{Ce}{u})^2} \\ &= \frac{-\log \frac{Ce}{u} \Delta \log u - |\nabla \log u|^2}{(\log \frac{Ce}{u})^2}.\end{aligned}$$

Since $\Delta \log u = \frac{1}{u}\Delta u - |\nabla \log u|^2$ we finally get

$$\Delta \log\log \frac{Ce}{u} = \frac{-\frac{1}{u}\Delta u}{\log \frac{Ce}{u}} + \frac{\log \frac{C}{u} |\nabla \log \frac{Ce}{u}|^2}{(\log \frac{Ce}{u})^2}. \tag{2.1}$$

Let us apply Theorem 2.1 to get the conclusion. Let $v = \log \frac{Ce}{u} = 1 + \log \frac{C}{u}$ and $f = \frac{-\frac{1}{u}\Delta u}{\log \frac{Ce}{u}} + \frac{\log \frac{C}{u} |\nabla \log \frac{Ce}{u}|^2}{(\log \frac{Ce}{u})^2}$. Then $\Delta \log v = f$ by equation (2.1), and it is easy to see that $f \geq 0$. Therefore we must have that f is zero almost everywhere or $0 < \int_M f \leq \infty$. By Theorem 2.1 we conclude that v is a constant.

(II) The case $k \geq 2$: A direct computation shows that

$$\Delta \log^{(k+1)} f = \frac{\Delta \log^{(k)} f \cdot \log^{(k)} f - |\nabla \log^{(k)} f|^2}{(\log^{(k)} f)^2}$$

and

$$|\nabla \log^{(k+1)} f|^2 = \frac{|\nabla \log^{(k)} f|^2}{(\log^{(k)} f)^2}.$$

By an elementary argument we have

$$\Delta \log^{(k+1)} f = \frac{\Delta \log f \cdot \prod_{i=1}^k \log^{(i)} f - |\nabla \log f|^2 \Big(\sum_{i=2}^k \prod_{j=i}^k \log^{(j)} f + 1 \Big)}{\prod_{i=1}^k (\log^{(i)} f)^2}. \tag{2.2}$$

Let $f = \frac{Ce^{(k)}}{u}$. We obtain

$$\begin{aligned}&\Delta \log^{(k+1)} \frac{Ce^{(k)}}{u} \\ &= \frac{-\Delta u \cdot u \cdot \prod_{i=1}^k \log^{(i)} f + |\nabla u|^2 \Big\{ \prod_{i=1}^k \log^{(i)} f - \Big(\sum_{i=2}^k \prod_{j=i}^k \log^{(j)} f + 1 \Big) \Big\}}{u^2 \prod_{i=1}^k (\log^{(i)} f)^2}.\end{aligned} \tag{2.3}$$

Since $f = \frac{Ce^{(k)}}{u}$ and $u \leq C$, we have

$$\begin{aligned}&\prod_{i=1}^k \log^{(i)} f - \Big(\sum_{i=2}^k \prod_{j=i}^k \log^{(j)} f + 1 \Big) \\ &\geq (\log f - k) \prod_{i=2}^k \log^{(i)} f \\ &\geq (\log e^{(k)} - k) \prod_{i=2}^k \log^{(i)} e^{(k)} \geq 0,\end{aligned}$$

where in the first inequality we used $\log^{(i)} f \geq \log^{(i)} e^{(k)} \geq 1$ for any $1 \leq i \leq k$.

By Theorem 2.1, we conclude that u is constant. $\square$

Proof of Theorem 1.2. Recall that

$$\Delta H - H|A|^2 + HRic^N(\xi,\xi) = 0.$$

Hence

$$\Delta H = H|A|^2 - HRic^N(\xi,\xi) = H|B|^2 - HRic^N(\xi,\xi) \leq 0,$$

where we used $|A|^2 = |B|^2$. Thus H is a superharmonic function on M. Let $H = u$ in Theorem 1.1 with $C = 1$. Thus we have completed the proof of Theorem 1.2. □

Proof of Theorem 1.4. Since $N = \mathbb{S}^{m+1}$, $Ric^N(\xi,\xi) = m$, and hence by assumption $|B|^2 \leq m$. Now since $mH^2 \leq |B|^2$, $H \leq 1$ is automatically satisfied.

Therefore we prove that H is a constant function on M. □

3. Appendix

We can apply our method to p-biharmonic submanifolds ($p > 1$) (cf. [20]). If an isometric immersion $\phi : (M, g) \to (N, h)$ satisfies

$$-\Delta(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^{m} R^N(e_i, |\vec{H}|^{p-2}\vec{H})e_i = 0, \tag{3.1}$$

then M is called a p-biharmonic submanifold. By decomposing (3.1) into its tangential and normal parts, we see that a submanifold is p-biharmonic if and only if it satisfies (cf. [17])

(3.2)
$$\Delta^{\perp}\left(|\vec{H}|^{p-2}\vec{H}\right) - \sum_{i=1}^{m} B(A_{|\vec{H}|^{p-2}\vec{H}}e_i, e_i) + \sum_{i=1}^{m}\left(R^N(|\vec{H}|^{p-2}\vec{H}, e_i,)e_i\right)^{\perp} = 0,$$

$$\mathrm{Tr}_g(\nabla A_{|\vec{H}|^{p-2}\vec{H}}) + \mathrm{Tr}_g[A_{\nabla^{\perp}|\vec{H}|^{p-2}\vec{H}}(\cdot)] - \sum_{i=1}^{m}\left(R^N(|\vec{H}|^{p-2}\vec{H}, e_i,)e_i\right)^{T} = 0. \tag{3.3}$$

For p-biharmonic submanifolds, it is easy to see that we can get similar results as in the results of biharmonic submanifolds in many cases (cf. [8], [17], [18], [26]). In fact, we have the following theorem.

Theorem 3.1. *Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a complete p-biharmonic hypersurface. Assume that the mean curvature H satisfies $0 < H \leq 1$. We also assume that $|B|^2 \leq Ric^N(\xi,\xi)$, where B is the second fundamental form of M in N, Ric^N is the Ricci curvature of N, and ξ is the unit normal vector field on M. If*

$$\begin{cases} (i)\ \int_M (\log^{(k)} \dfrac{e^{(k)}}{H^{p-1}})^q d\mu_g < +\infty \\ or \\ (ii)\ p \geq 2 \ and\ \int_M (\log^{(k)} \dfrac{e^{(k)}}{H})^q d\mu_g < +\infty, \end{cases}$$

for some $q > 0$ and $k \in \mathbb{N}$, then H is constant.

Proof. The proof is similar to the proof of Theorem 1.2.

The case (i): Let $u = H^{p-1}$. Since $\Delta H^{p-1} \leq 0$, we can apply Theorem 1.1.

The case (ii): Let $u = H$. Since $p \geq 2$, we have $\Delta H \leq 0$. Applying Theorem 1.1, we get the result. □

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