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SHARP ESTIMATES OF RADIAL MINIMIZERS OF p–LAPLACE EQUATIONS

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Abstract. We study semi-stable, radially symmetric and decreasing solutions $u \in W^{1,p}(B_1)$ of $-\Delta_p u = g(u)$ in $B_1 \setminus \{0\}$, where B_1 is the unit ball of $\mathbb{R}^N$, $p > 1$, Δ_p is the $p-$Laplace operator and g is a general locally Lipschitz function. We establish sharp pointwise estimates for such solutions, which do not depend on the nonlinearity g. By applying these results, sharp pointwise estimates are obtained for the extremal solution and its derivatives (up to order three) of the equation $-\Delta_p u = \lambda f(u)$, posed in B_1, with Dirichlet data $u|_{\partial B_1} = 0$, where the nonlinearity f is an increasing C^1 function with $f(0) > 0$ and $\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty$.

1. Introduction and main results

This paper is concerned with the semi-stability of radially symmetric and decreasing solutions $u \in W^{1,p}(B_1)$ of

$$-\Delta_p u = g(u) \text{ in } B_1 \setminus \{0\}\,, \tag{1.1}$$

where $p > 1$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian of u, B_1 is the unit ball of $\mathbb{R}^N$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function.

By abuse of notation, we write $u(r)$ instead of $u(x)$, where $r = |x|$ and $x \in \mathbb{R}^N$. We denote by u_r the radial derivative of a radial function u.

Since $u \in W^{1,p}(B_1)$ is radial, by the Sobolev embedding in one dimension, we obtain $u \in L^\infty_{\text{loc}}\left(\overline{B_1} \setminus \{0\}\right)$. Hence, by standard regularity results it is deduced that $u \in C^{1,\beta}_{\text{loc}}\left(\overline{B_1} \setminus \{0\}\right)$ for some $\beta \in (0,1)$.

A radial solution $u \in W^{1,p}(B_1)$ of (1.1) such that $u_r(r) < 0$ for all $r \in (0,1)$ is called semi-stable if

$$\int_{B_1} (p-1)\,|u_r|^{p-2}\,|\xi_r|^2 - g'(u)\xi^2 \geq 0,$$

for every radially symmetric function $\xi \in C^1_c\left(B_1 \setminus \{0\}\right)$.

Note that the above expression is formally the second variation of the energy functional associated to (1.1):

$$E_\Omega(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \int_\Omega G(u)\, dx, \tag{1.2}$$

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where $G' = g$ and $\Omega \subset B_1$. Thus, if u is a radial local minimizer of (1.2) with $\Omega = B_1$ (i.e., for every $\delta \in (0,1)$ there exists $\varepsilon_\delta > 0$ such that $E_{B_1 \setminus \overline{B_\delta}}(u) \leq E_{B_1 \setminus \overline{B_\delta}}(u+\xi)$, for all radial functions $\xi \in C_c^1\left(B_1 \setminus \overline{B_\delta}\right))$ satisfying $\|\xi\|_{C^1} \leq \varepsilon_\delta)$, then u is a semi-stable solution of (1.1). Other general situations include semi-stable solutions: for instance, minimal solutions, extremal solutions, and also some solutions between a sub- and a supersolution (see [3, Remark 1.7] for more details). All the results obtained in this paper were obtained by the second author in [16] for the Laplace operator ($p = 2$).

Theorem 1.1 ([3]). *Let g be a locally Lipschitz function and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution in $B_1 \setminus \{0\}$ of (1.1) satisfying $u_r(r) < 0$ for all $r \in (0,1)$.*

Then:

a) *If $N < p + 4p/(p-1)$, then $u \in L^\infty(B_1)$. Moreover,*

$$\|u\|_{L^\infty(B_1)} \leq C_{N,p} \|u\|_{W^{1,p}(B_1)},$$

where $C_{N,p}$ is a constant depending only on N and p.

b) *If $N = p + 4p/(p-1)$, then $u \in L^q(B_1)$ for all $q < +\infty$. Moreover,*

$$|u(r)| \leq C_p \|u\|_{W^{1,p}(B_1)} (|\log r| + 1) \text{ in } B_1,$$

where C_p is a constant depending only on p.

c) *If $N > p + 4p/(p-1)$ and $q < q_0$, then $u \in L^q(B_1)$ and*

$$\|u\|_{L^q(B_1)} \leq C_{N,p,q} \|u\|_{W^{1,p}(B_1)},$$

where $C_{N,p,q}$ is a constant depending only on N, p, and q. Moreover,

$$|u(r)| \leq C_{N,p} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)} \left(|\log r|^{\frac{1}{p}} + 1\right) \text{ in } B_1,$$

where $C_{N,p}$ is a constant depending only on N and p.

d) *Assume that g is nonnegative. Then:*

d1) *We have*

$$\|\nabla u\|_{L^p(B_1)} \leq C_{N,p} \left\{ \left\|(u-u(1))^{p-1}\right\|_{L^1(B_1)}^{\frac{1}{p-1}} + \|g(u)\|_{L^1(B_1)}^{\frac{1}{p-1}} \right\},$$

for some constant $C_{N,p}$ depending only on N and p.

d2) *$u \in W^{1,q}(B_1)$ for every $q < q_1$, and*

$$\|u\|_{W^{1,q}(B_1)} \leq C \quad \text{if } q < q_1,$$

where C is a constant depending only on N, p, q, and on upper bounds for $\|u\|_{L^1(B_1)}$ and g.

d3) *If $N \geq p + 4p/(p-1)$, then*

$$|u_r(r)| \leq C_{N,p} \|u\|_{W^{1,p}(B_1)} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)} |\log r|^{\frac{1}{p}} \text{ in } B_{1/4},$$

where $C_{N,p}$ is a constant depending only on N and p.

The definition of q_k for $k = 0, 1$ is given by

$$\begin{cases} \frac{1}{q_k} := \frac{1}{p} - \frac{2}{Np}\sqrt{\frac{N-1}{p-1}} + \frac{k-1}{N} - \frac{2}{Np} & \text{for } N \geq p + 4p/(p-1), \\ q_k := +\infty & \text{for } N < p + 4p/(p-1). \end{cases}$$

In this paper we establish sharp pointwise estimates for semi-stable radially symmetric and decreasing solutions $u \in W^{1,p}(B_1)$ of (1.1) and its derivatives (up to order three). We improve the above theorem, answering affirmatively an open question raised in [3] about the removal of the factor $|\log r|^{\frac{1}{p}}$.

Theorem 1.2. *Let $N \geq p > 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of (1.1) satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then there exists a constant $C_{N,p}$ depending only on N and p such that:*

i) *If $p \leq N < p + 4p/(p-1)$, then*

$$|u(r)| \leq C_{N,p} \|u\|_{W^{1,p}\left(B_1 \setminus \overline{B_{1/2}}\right)}, \ \forall r \in (0,1].$$

ii) *If $N = p + 4p/(p-1)$, then*

$$|u(r)| \leq C_p \|u\|_{W^{1,p}\left(B_1 \setminus \overline{B_{1/2}}\right)} (|\log r| + 1), \ \forall r \in (0,1].$$

iii) *If $N > p + 4p/(p-1)$, then*

$$|u(r)| \leq C_{N,p} \|u\|_{W^{1,p}\left(B_1 \setminus \overline{B_{1/2}}\right)} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)}, \ \forall r \in (0,1].$$

Remark 1.3. For $N < p$, since $u \in W^{1,p}(B_1)$, the Sobolev embedding leads to $u \in L^\infty(B_1)$ and $\|u\|_{L^\infty(B_1)} \leq C_{N,p} \|u\|_{W^{1,p}(B_1)}$.

Theorem 1.4. *Let $N \geq p + 4p/(p-1)$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of (1.1) satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then there exists a constant $C'_{N,p}$ depending only on N and p such that:*

i) *If $g \geq 0$, then*

$$|u_r(r)| \leq C'_{N,p} \|\nabla u\|_{L^p\left(B_1 \setminus B_{1/2}\right)} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}, \ \forall r \in (0,1/2].$$

ii) *If $g \geq 0$ is nondecreasing, then*

$$|u_{rr}(r)| \leq C'_{N,p} \|\nabla u\|_{L^p\left(B_1 \setminus B_{1/2}\right)} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}+p-2\right)}, \ \forall r \in (0,1/2].$$

iii) *If $g \geq 0$ is nondecreasing and convex, then*

$$|u_{rrr}(r)| \leq C'_{N,p} \|\nabla u\|_{L^p\left(B_1 \setminus B_{1/2}\right)} r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}+2p-2\right)}, \ \forall r \in (0,1/2].$$

Remark 1.5. Observe that the estimates obtained in Theorems 1.2 and 1.4 are stated in terms of the $W^{1,p}$ norm of the annulus $B_1 \setminus \overline{B_{1/2}}$, while u is required to belong to $W^{1,p}(B_1)$. This requirement is essential to obtain our results, since we can easily find semi-stable radially decreasing solutions of (1.1) (for instance $u(r) = r^s$, with $s \ll 0$), not in the energy class $W^{1,p}(B_1)$, which the statements of Theorems 1.2 and 1.4 fail to satisfy.

Remark 1.6. To our knowledge there are no estimates of $|u_{rr}|$ or $|u_{rrr}|$ in the literature for these kinds of solutions.

As an application of some general results obtained in this paper for this class of solutions (for arbitrary $g \in C^1(\mathbb{R})$), we consider the following problem:

$$(1.3_{\lambda,p}) \qquad \begin{cases} -\Delta_p u &= \lambda f(u) & \text{in } B_1, \\ u &> 0 & \text{in } B_1, \\ u &= 0 & \text{on } \partial B_1, \end{cases}$$

where $\lambda > 0$ and f is an increasing C^1 function with $f(0) > 0$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty. \tag{1.4}$$

This problem is studied by Cabré and Sanchón in [4] for general smooth bounded domains Ω of $\mathbb{R}^N$. It is proved that there exists a positive parameter λ^* such that if $\lambda \in (0, \lambda^*)$, then $(1.3_{\lambda,p})$ admits a minimal (smallest) solution $u_\lambda \in C^1(\overline{\Omega})$ and if $\lambda \in (\lambda^*, +\infty)$, then $(1.3_{\lambda,p})$ admits no regular solution. In addition, for $\lambda \in (0, \lambda^*)$ the minimal solution u_λ is semi-stable (in a similar sense of the definition when $\Omega = B_1$). On the other hand, we may consider the increasing limit

$$u^* := \lim_{\lambda\uparrow\lambda^*} u_\lambda.$$

In the case $p = 2$ it is well known that u^* is a weak solution of $(1.3_{\lambda,p})$ for $\lambda = \lambda^*$. It is called the extremal solution. For general p, Ω and f, it is not known if u^* is a weak solution of $(1.3_{\lambda,p})$ for $\lambda = \lambda^*$. In the case $\Omega = B_1$, Cabré, Capella and Sanchón [3] proved that u^* is actually a semi-stable radially decreasing energy solution (i.e., $u^* \in W_0^{1,p}$) of $(1.3_{\lambda,p})$. Hence we can apply to the extremal solution the results obtained in this paper for these kinds of solutions.

We refer to [2, 5] for surveys on minimal and extremal solutions and to [1, 6–10, 12–15, 17] for other interesting results in the topic of extremal solutions.

Theorem 1.7. *Let $N \geq p > 1$. Suppose that f satisfies* (1.4)*. Let u^* be the extremal solution of* $(1.3_{\lambda^*,p})$*. We have that*

i) *If $p \leq N < p + 4p/(p-1)$, then $u^*(r) \leq C(1-r)$, $\forall r \in (0,1]$.*
ii) *If $N = p + 4p/(p-1)$, then $u^*(r) \leq C\,|\log r|$, $\forall r \in (0,1]$.*
iii) *If $N > p + 4p/(p-1)$, then*

$$u^*(r) \leq C\left(r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)} - 1\right), \ \forall r \in (0,1].$$

iv) *If $N \geq p + 4p/(p-1)$, then*

$$\left|\partial_r^{(k)} u^*(r)\right| \leq C r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}+(k-1)p-2\right)}, \ \forall r \in (0,1], \forall k \in \{1,2\}.$$

v) *If $N \geq p + 4p/(p-1)$ and f is convex, then*

$$|u^*_{rrr}(r)| \leq C r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}+2p-2\right)}, \ \forall r \in (0,1],$$

where $C = C_{N,p} \min_{t\in[1/2,1]} |u^*_r(t)|$, and $C_{N,p}$ is a constant depending only on N and p .

Remark 1.8. In [11] García-Azorero, Peral and Puel proved that if $f(u) = e^u$ and $N = p + 4p/(p-1)$, then

$$u^*(r) = -p \log r \text{ and } \lambda^* = p^{p-1}(N-p).$$

This shows that the pointwise estimates of Theorem 1.7 are optimal for $N = p + 4p/(p-1)$.

On the other hand, in [3] Cabré and Sanchón proved that if $N > p + 4p/(p-1)$ and $f(u) = (1+u)^m$, where

$$m := \frac{(p-1)N - 2\sqrt{(p-1)(N-1)} - p + 2}{N - 2\sqrt{\frac{N-1}{p-1}} - p - 2},$$

then

$$u^*(r) = r^{-\frac{1}{p}\left(N - 2\sqrt{\frac{N-1}{p-1}} - p - 2\right)} - 1$$

and

$$\lambda^* = \left(\frac{p}{m-(p-1)}\right)^{p-1}\left(N - \frac{mp}{m-(p-1)}\right).$$

This also shows the optimality of the pointwise estimates of Theorem 1.7 for the case $N \geq p + 4p/(p-1)$.

2. Proof of the main results

Lemma 2.1. *Let $N \geq p > 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of (1.1) satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then there exists a constant $K_{N,p}$ depending only on N and p such that*

$$\int_0^r |u_r(t)|^p\, t^{N-1}\, dt \leq K_{N,p}\, \|\nabla u\|^p_{L^p(B_1 \setminus B_{1/2})}\, r^{2\sqrt{\frac{N-1}{p-1}}+2},\ \forall r \in [0,1].$$

Proof. Let us use [3, Lem. 2.2] (see also the proof of [3, Lem. 2.3]) to assure that

$$(N-1)\int_{B_1} |u_r|^p\, \eta^2\, dx \leq (p-1)\int_{B_1} |u_r|^p\, |\nabla(|x|\,\eta)|^2\, dx, \tag{2.1}$$

for every radial Lipschitz function η vanishing on ∂B_1.

We now fix $r \in (0, 1/2)$ and consider the function

$$\eta_\epsilon(t) = \begin{cases} \dfrac{r^{-\sqrt{\frac{N-1}{p-1}}}}{\epsilon} & \text{if } 0 \leq t \leq \epsilon, \\[2ex] \dfrac{r^{-\sqrt{\frac{N-1}{p-1}}}}{t} & \text{if } \epsilon < t \leq r, \\[2ex] t^{-\sqrt{\frac{N-1}{p-1}}-1} & \text{if } r < t \leq 1/2, \\[2ex] 2^{\sqrt{\frac{N-1}{p-1}}+2}(1-t) & \text{if } 1/2 < t \leq 1. \end{cases}$$

Inequality (2.1) shows that

$$(N-p)\left(\frac{r^{-\sqrt{\frac{N-1}{p-1}}}}{\epsilon}\right)^2 \int_0^\epsilon |u_r(t)|^p\, t^{N-1} dt + (N-1) r^{-2\sqrt{\frac{N-1}{p-1}}-2} \int_\epsilon^r \left(\frac{r}{t}\right)^2 |u_r(t)|^p\, t^{N-1} dt$$

$$+ 2^{2\sqrt{\frac{N-1}{p-1}}+4} \int_{1/2}^1 \left((N-1)(1-t)^2 - (p-1)(1-2t)^2\right) |u_r(t)|^p\, t^{N-1}\, dt \leq 0.$$

Since $N \geq p$ and $r/t \geq 1$ for $0 < t \leq r$, letting $\epsilon \to 0$, we obtain

$$\int_0^r |u_r(t)|^p \, t^{N-1} \, dt \leq \left(\frac{(p-1)2^{2\sqrt{\frac{N-1}{p-1}}+4}}{N-1} \right) r^{2\sqrt{\frac{N-1}{p-1}}+2} \int_{1/2}^1 |u_r(t)|^p \, t^{N-1} \, dt,$$

and the lemma is proved for $0 < r \leq 1/2$.

If $r \in (1/2, 1]$, then, applying the above inequality for $r = 1/2$, we obtain

$$\begin{aligned} \int_0^r |u_r(t)|^p \, t^{N-1} \, dt &\leq \int_0^{1/2} |u_r(t)|^p \, t^{N-1} \, dt + \int_{1/2}^1 |u_r(t)|^p \, t^{N-1} \, dt \\ &\leq \left[\left(\frac{(p-1)2^{2\sqrt{\frac{N-1}{p-1}}+4}}{N-1} \right) \left(\frac{1}{2} \right)^{2\sqrt{\frac{N-1}{p-1}}+2} + 1 \right] \int_{1/2}^1 |u_r(t)|^p \, t^{N-1} \, dt \\ &\leq (2r)^{2\sqrt{\frac{N-1}{p-1}}+2} \left(\frac{4(p-1)}{N-1} + 1 \right) \int_{1/2}^1 |u_r(t)|^p \, t^{N-1} \, dt, \end{aligned}$$

and the proof is completed for $1/2 < r \leq 1$. □

Proposition 2.2. *Let $N \geq p > 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of (1.1) satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then there exists a constant $K'_{N,p}$ depending only on N and p such that*

$$\left| u(r) - u\left(\frac{r}{2}\right) \right| \leq K'_{N,p} \, \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} \, r^{-\frac{1}{p}\left(N - 2\sqrt{\frac{N-1}{p-1}} - p - 2\right)}, \ \forall r \in (0,1]. \tag{2.2}$$

Proof. Fix $r \in (0,1]$. Applying Hölder's inequality and Lemma 2.1 we deduce

$$\begin{aligned} \left| u(r) - u\left(\frac{r}{2}\right) \right| &= \int_{r/2}^r |u_r(t)| \, t^{\frac{N-1}{p}} t^{-\frac{N-1}{p}} \, dt \\ &\leq \left(\int_{r/2}^r |u_r(t)|^p \, t^{N-1} \, dt \right)^{\frac{1}{p}} \left(\int_{r/2}^r t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}} \\ &\leq K_{N,p}^{\frac{1}{p}} \, \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} \, r^{\frac{2}{p}\sqrt{\frac{N-1}{p-1}} + \frac{2}{p}} \left(r^{-\frac{N-p}{p-1}} \int_{1/2}^1 t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}}, \end{aligned}$$

and (2.2) is proved with $K'_{N,p} = K_{N,p}^{\frac{1}{p}} \left(\int_{1/2}^1 t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}}$. □

Proof of Theorem 1.2. Let $0 < r \leq 1$. Then, there exist $m \in \mathbb{N}$ and $1/2 < r_1 \leq 1$ such that $r = r_1/2^{m-1}$. Since u is radial we have $|u(r_1)| \leq \|u\|_{L^\infty(B_1 \setminus \overline{B_{1/2}})} \leq \gamma_{N,p} \, \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})}$, where $\gamma_{N,p}$ depends only on N and p. From this and Proposition 2.2, it follows that

$$\begin{aligned} |u(r)| &\leq |u(r) - u(r_1)| + |u(r_1)| = \sum_{i=1}^{m-1} \left| u\left(\frac{r_1}{2^{i-1}}\right) - u\left(\frac{r_1}{2^i}\right) \right| + |u(r_1)| \\ &\leq \left(K'_{N,p} \sum_{i=1}^{m-1} \left(\frac{r_1}{2^{i-1}} \right)^{-\frac{1}{p}\left(N - 2\sqrt{\frac{N-1}{p-1}} - p - 2\right)} + \gamma_{N,p} \right) \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})}. \end{aligned} \tag{2.3}$$

Let $\delta_{N,p} = -\frac{1}{p}\left(N - 2\sqrt{\frac{N-1}{p-1}} - p - 2\right)$. We have

$$\sum_{i=1}^{m-1}\left(\frac{r_1}{2^{i-1}}\right)^{\delta_{N,p}} \leq \alpha_{N,p}\begin{cases} r^{\delta_{N,p}} & \text{if } \delta_{N,p} < 0, \\ 1 & \text{if } \delta_{N,p} > 0, \\ |\log r| & \text{if } \delta_{N,p} = 0, \end{cases} \tag{2.4}$$

where $\alpha_{N,p}$ is a constant depending only on N and p.

Then:

- If $p \leq N < p + 4p/(p-1)$, then $\delta_{N,p} > 0$. From (2.3) and (2.4) we obtain
$$|u(r)| \leq \left(K'_{N,p}\alpha_{N,p} + \gamma_{N,p}\right)\|u\|_{W^{1,p}\left(B_1\setminus\overline{B_{1/2}}\right)}.$$
- If $N = p + 4p/(p-1)$, then $\delta_{N,p} = 0$. From (2.3) and (2.4) we obtain
$$\begin{aligned}|u(r)| &\leq \left(K'_{N,p}\alpha_{N,p}|\log r| + \gamma_{N,p}\right)\|u\|_{W^{1,p}\left(B_1\setminus\overline{B_{1/2}}\right)} \\ &\leq \left(K'_{N,p}\alpha_{N,p} + \gamma_{N,p}\right)\|u\|_{W^{1,p}\left(B_1\setminus\overline{B_{1/2}}\right)}(|\log r| + 1).\end{aligned}$$
- If $N > p + 4p/(p-1)$, we have $\delta_{N,p} < 0$ and $r^{\delta_{N,p}} \geq 1$. From (2.3) and (2.4) we obtain
$$|u(r)| \leq \left(K'_{N,p}\alpha_{N,p} + \gamma_{N,p}\right)r^{\delta_{N,p}}\|u\|_{W^{1,p}\left(B_1\setminus\overline{B_{1/2}}\right)},$$

which completes the proof. □

Lemma 2.3. *Let $N \geq 1$, $p > 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative and nondecreasing locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of* (1.1) *such that $u_r < 0$ for all $r \in (0,1)$. Then*

$$g(u(r)) \leq N\frac{|u_r(r)|^{p-1}}{r}, \quad \forall r \in (0,1]. \tag{2.5}$$

Moreover, if g is convex, then

$$g'(u(r)) \leq M_{N,p}\frac{|u_r(r)|^{p-2}}{r^2}, \quad \forall r \in (0,1], \tag{2.6}$$

where $M_{N,p}$ is a constant depending only on N and p.

Proof. Consider the function

$$\Psi(r) := Nr^{1-1/N}\left|u_r\left(r^{1/N}\right)\right|^{p-1}, \; r \in (0,1].$$

It is easy to check that $\Psi'(r) = g\left(u\left(r^{1/N}\right)\right)$, $r \in (0,1]$. As g is nonnegative and nondecreasing we have that Ψ is a nonnegative nondecreasing concave function. It follows immediately that

$$0 \leq \Psi'(r) \leq \Psi(r)/r, \; r \in (0,1],$$

and we obtain (2.5).

To obtain ii), we first observe that from (1.1) we obtain

$$u_{rr} = -\frac{1}{p-1}\left(\frac{g(u)}{|u_r|^{p-2}} + \frac{N-1}{r}u_r\right), \quad \forall r \in (0,1].$$

Therefore, using the nonnegativeness of g and (2.5) we deduce that

$$|u_{rr}| \leq \frac{1}{p-1}\left(\frac{g(u)}{|u_r|^{p-2}} + \frac{N-1}{r}|u_r|\right) \leq \left(\frac{2N-1}{p-1}\right)\frac{|u_r|}{r}, \quad \forall r \in (0,1]. \tag{2.7}$$

For fixed $\alpha \in \mathbb{R}$ an easy computation shows that

$$\begin{aligned} \partial_r\left(r^\alpha |u_r|^{p-2}\right) &= \alpha r^{\alpha-1}|u_r|^{p-2} - (p-2)r^\alpha u_{rr}|u_r|^{p-3} \\ &\geq r^{\alpha-1}|u_r|^{p-2}\left(\alpha - \frac{|p-2|(2N-1)}{p-1}\right), \quad \forall r \in (0,1]. \end{aligned}$$

Thus $r^\alpha |u_r|^{p-2}$ is nondecreasing for $\alpha = \frac{|p-2|(2N-1)}{p-1}$. Using this, the monotocity of $g'(u(r))$ and the semi-stability of u, we deduce that

$$\begin{aligned} g'(u(r))\int_0^r s^{N-1}\xi(s)^2 ds &\leq \int_0^r s^{N-1} g'(u(s))\xi(s)^2 ds \\ &\leq (p-1)\int_0^r |u_r(s)|^{p-2} s^\alpha s^{N-1-\alpha}\xi'(s)^2 ds \\ &\leq (p-1)|u_r(r)|^{p-2} r^\alpha \int_0^r s^{N-1-\alpha}\xi'(s)^2 ds, \end{aligned}$$

for every $r \in (0,1)$ and every $\xi \in C^1$ with compact support in $(0,r)$.

Taking $\xi(s) = \zeta(\frac{s}{r})$ for $s \in [0,r]$, where $\zeta \in C^1$ is any function with compact support in $(0,1)$, we obtain (2.6). □

Proof of Theorem 1.4.

i) We first observe that $\partial_r\left(r^{N-1}|u_r|^{p-1}\right) = r^{N-1}g(u)$. Hence $r^{N-1}|u_r|^{p-1}$ is a positive nondecreasing function and so is $\left(r^{N-1}|u_r|^{p-1}\right)^{\frac{p}{p-1}}$. Thus, for $0 < r \leq 1/2$, we have

$$\begin{aligned} \int_0^{2r} |u_r(t)|^p t^{N-1}\, dt &\geq \int_r^{2r} |u_r(t)|^p t^{N-1}\, dt \\ &= \int_r^{2r} \left(t^{N-1}|u_r|^{p-1}\right)^{\frac{p}{p-1}} t^{-\frac{N-1}{p-1}}\, dt \\ &\geq r^{\frac{p(N-1)}{p-1}}|u_r(r)|^p \int_r^{2r} t^{-\frac{N-1}{p-1}}\, dt \\ &= r^N |u_r(r)|^p \int_1^2 t^{-\frac{N-1}{p-1}}\, dt. \end{aligned}$$

From this and Lemma 2.1 we obtain i).

ii) From (2.7) and i) we obtain ii).

iii) From (1.1) we obtain

$$u_{rrr} = -\frac{1}{p-1}\left(\frac{g'(u)u_r}{|u_r|^{p-2}} - (p-2)\frac{u_r u_{rr} g(u)}{|u_r|^p} - \frac{N-1}{r^2}u_r + \frac{N-1}{r}u_{rr}\right),$$

for every $r \in (0,1]$. Therefore from (2.5), (2.6) and (2.7), we obtain

$$
\begin{aligned}
|u_{rrr}| &\leq \frac{1}{p-1}\left(\frac{g'(u)\,|u_r|}{|u_r|^{p-2}} + |p-2|\,\frac{|u_r|\,|u_{rr}|\,g(u)}{|u_r|^p}\right.\\
&\qquad\left. + \frac{N-1}{r^2}\,|u_r| + \frac{N-1}{r}\,|u_{rr}|\right)\\
&\leq \frac{1}{p-1}\left(M_{N,p} + \frac{|p-2|\,N(2N-1)}{p-1}\right.\\
&\qquad\left. + (N-1) + \frac{(N-1)(2N-1)}{p-1}\right)\frac{|u_r|}{r^2},\ \forall r \in (0,1].
\end{aligned}
\tag{2.8}
$$

Then iii) follows from i).

□

Lemma 2.4. *Let $N \geq 1$, $p > 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz nonnegative and nondecreasing function and u be a radial solution of* (1.1) *satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then:*

i) *$r^{N-1}\,|u_r|^{p-1}$ is nondecreasing for $r \in (0,1]$.*
ii) *$r^{-1}\,|u_r|^{p-1}$ is nonincreasing for $r \in (0,1]$.*
iii) *$\max_{t\in[1/2,1]} |u_r(t)| \leq 2^{\frac{N}{p-1}} \min_{t\in[1/2,1]} |u_r(t)|$.*
iv) *$\|\nabla u\|_{L^p\left(B_1\setminus B_{1/2}\right)} \leq q_{N,p} \min_{t\in[1/2,1]} |u_r(t)|$ for a certain constant $q_{N,p}$ depending only on N and p.*

Proof.

i) Since $u_r < 0$ we have $\partial_r\left(r^{N-1}\,|u_r|^{p-1}\right) = r^{N-1}g(u) \geq 0$.
ii) From (2.5) of Lemma 2.3, we have that

$$
N r^{N-2}\,|u_r|^{p-1} \geq \partial_r\left(r^{N-1}\,|u_r|^{p-1}\right) = N r^{N-2}\,|u_r|^{p-1} + r^N \partial_r\left(r^{-1}\,|u_r|^{p-1}\right),
$$

and ii) follows immediately.
iii) Take $r_1, r_2 \in [1/2,1]$ such that $|u_r(r_1)| = \min_{t\in[1/2,1]} |u_r(t)|$ and $|u_r(r_2)| = \max_{t\in[1/2,1]} |u_r(t)|$.
-If $r_2 \leq r_1$, we deduce from i) that $|u_r(r_2)|^{p-1} \leq (r_1/r_2)^{N-1}\,|u_r(r_1)|^{p-1} \leq 2^N\,|u_r(r_1)|^{p-1}$.
-If $r_2 > r_1$, we deduce from ii) that $|u_r(r_2)|^{p-1} \leq (r_2/r_1)\,|u_r(r_1)|^{p-1} \leq 2\,|u_r(r_1)|^{p-1} \leq 2^N\,|u_r(r_1)|^{p-1}$.
iv) We see at once that

$$
\|\nabla u\|_{L^p\left(B_1\setminus B_{1/2}\right)} \leq \left|B_1 \setminus B_{1/2}\right|^{1/p} \max_{t\in[1/2,1]} |u_r(t)|,
$$

and iv) follows from iii).

□

Proof of Theorem 1.7. As we have mentioned, it is well known that u^* is a semi-stable radially decreasing $W^{1,p}(B_1)$ solution of (1.1) for $g(s) = \lambda^* f(s)$. Hence, we can apply to u^* the results obtained in Lemma 2.1 and Lemma 2.4.

Let $0 < s \leq 1$. From statement ii) of Lemma 2.4 and applying Hölder's inequality, we deduce that

$$\begin{aligned} |u_r^*(s)|^{p-1} &\leq 2\int_{s/2}^{s} t^{-1}\,|u_r^*(t)|^{p-1}\,dt = 2\int_{s/2}^{s} t^{\frac{(p-1)(N-1)}{p}}\,|u_r^*(t)|^{p-1}\,t^{\frac{N-pN-1}{p}}\,dt \\ &\leq 2\left(\int_{s/2}^{s} t^{N-1}\,|u_r^*(t)|^{p}\,dt\right)^{\frac{p-1}{p}}\left(\int_{s/2}^{s} t^{N-pN-1}\,dt\right)^{\frac{1}{p}}. \end{aligned}$$

From this, Lemma 2.1 and statement iv) of Lemma 2.4, we have

$$|u_r^*(s)| \leq C_{N,p}\min_{t\in[1/2,1]}|u_r^*(t)|\,s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)},\ \forall s\in(0,1], \tag{2.9}$$

where $C_{N,p}$ depends only on N and p.

Then as $u^*(1) = 0$, we use (2.9) to obtain

$$|u^*(r)| = \int_r^1 |u_r^*(s)|\,ds \leq C_{N,p}\min_{t\in[1/2,1]}|u_r^*(t)|\int_r^1 s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}\,ds, \tag{2.10}$$

for all $0 < r \leq 1$. Next, we estimate the integral $\int_r^1 s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}\,ds$.

- If $p \leq N < p + 4p/(p-1)$, we have $-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right) > -1$. Then

$$\begin{aligned} \int_r^1 s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}\,ds &= \frac{1-r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)}}{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)} \\ &\leq \left(1+\frac{1}{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)}\right)(1-r). \end{aligned}$$

- If $N = p + 4p/(p-1)$, we have $-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right) = -1$. Then

$$\int_r^1 s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}\,ds = -\log r.$$

- If $N > p + 4p/(p-1)$, we have $-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right) < -1$. Then

$$\int_r^1 s^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-2\right)}\,ds = \frac{r^{-\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)}-1}{\frac{1}{p}\left(N-2\sqrt{\frac{N-1}{p-1}}-p-2\right)}.$$

From this and (2.10), we conclude i), ii), and iii).

Finally, the proof of iv) and v) follows from (2.7), (2.8), and (2.9). □

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