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CHARACTERIZATIONS OF BLOCKS BY LOEWY LENGTHS OF THEIR CENTERS

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ABSTRACT. We study a block B of a finite group with respect to an algebraically closed field of prime characteristic through the Loewy length $\mathrm{ll}\mathbf{Z}B$ of the center $\mathbf{Z}B$. In this paper, we give some upper bounds for $\mathrm{ll}\mathbf{Z}B$ in terms of characters, subsections and defect groups associated to B. As a corollary to these results, we characterize some blocks by $\mathrm{ll}\mathbf{Z}B$.

1. INTRODUCTION

The present paper deals with the Loewy length of the center of a block of a finite group with respect to an algebraically closed field of prime characteristic.

Let G be a finite group, $\mathcal{O}$ a complete discrete valuation ring with quotient field K of characteristic 0 and $F = \mathcal{O}/\operatorname{rad}(\mathcal{O})$ the residue field of characteristic $p > 0$. We assume that K contains all $|G|$-th roots of unity and F is algebraically closed. For a block B of the group algebra FG with defect d, we denote by $\mathrm{ll}\mathbf{Z}B$ the Loewy length of the center $\mathbf{Z}B$. Okuyama [11] has proved that $\mathrm{ll}\mathbf{Z}B$ is bounded above by p^d with equality if and only if B is a nilpotent block with cyclic defect group. Our purpose is to develop this result.

The paper is organized as follows. In the next section we show fundamental properties of $\mathrm{ll}\mathbf{Z}B$ and compute them for some blocks with abelian defect groups. In the third section we give a new upper bound for $\mathrm{ll}\mathbf{Z}B$ using two parameters. This is a generalization of the preceding result and its proof is influenced by Okuyama [11]. Finally, we characterize some blocks by $\mathrm{ll}\mathbf{Z}B$. Our main theorems in the last section indicate that we can classify all blocks with $p^d - 3 \leq \mathrm{ll}\mathbf{Z}B \leq p^d$ into 9 types.

At the end of this section, we note some notation in this paper. Let b_D be a root of B, that is, a block of $F[DC_G(D)]$ such that $(b_D)^G = B$ where D is a defect group of B. We denote by $N_G(D, b_D)$ the inertial group of b_D in $N_G(D)$, by $I(B)$ the inertial quotient $N_G(D, b_D)/DC_G(D)$ and by $e(B) = |I(B)|$ the inertial index of B. For integers $m, n \geq 1$, Z_m and $Z_m \times Z_n$ denote a cyclic group of order m and a direct product of two cyclic groups, respectively. Moreover, we write $N \rtimes H$ to denote a non-trivial semi-direct product of N with H.

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2. Some fundamental properties

In this section, we denote by $k(B)$ and $l(B)$ the numbers of irreducible ordinary and Brauer characters associated to B, respectively. The next proposition is clear by the fact that $\mathbf{Z}B$ is local.

Proposition 2.1. *The following are equivalent:*

(1) *D is trivial.*
(2) $\mathrm{ll}\mathbf{Z}B = 1$.

Moreover, we give an upper bound using $k(B)$ and $l(B)$.

Proposition 2.2.

$$\mathrm{ll}\mathbf{Z}B \leq k(B) - l(B) + 1.$$

Proof. Let us denote by $\mathbf{S}B$ and $\mathbf{S}\mathbf{Z}B$ the socle of B and $\mathbf{Z}B$, respectively. Then $k(B) = \dim \mathbf{Z}B, l(B) = \dim \mathbf{S}B \cap \mathbf{Z}B$ and $\mathbf{S}B \cap \mathbf{Z}B \subseteq \mathbf{S}\mathbf{Z}B$ are known to hold. Thus we have $\mathrm{ll}\mathbf{Z}B - 1 \leq \dim \mathbf{Z}B - \dim \mathbf{S}\mathbf{Z}B \leq \dim \mathbf{Z}B - \dim \mathbf{S}B \cap \mathbf{Z}B = k(B) - l(B)$ as required. □

In the case D is cyclic, the Loewy length is given in Koshitani-Külshammer-Sambale [7].

Proposition 2.3 (Corollary 2.8 in [7])**.** *If D is cyclic, then*

$$\mathrm{ll}\mathbf{Z}B = \frac{p^d - 1}{e(B)} + 1.$$

In the following, $\mathbf{J}A$ and $\mathrm{ll}A$ denote the Jacobson radical and the Loewy length of any algebra A over F, respectively. The remainder of this section is devoted to some blocks with $D \simeq Z_{p^m} \times Z_{p^n}$ for some $m, n \geq 1$. These results are applied in the last section. First of all, we show the next lemma.

Lemma 2.4. *If D is normal in G, then $\mathrm{ll}\mathbf{Z}B \leq \mathrm{ll}FD$. In particular, $\mathrm{ll}\mathbf{Z}B \leq p^m + p^n - 1$ in case of $D \simeq Z_{p^m} \times Z_{p^n}$.*

Proof. By a result of Külshammer [8], B is Morita equivalent to $F^\alpha[D \rtimes I(B)]$ for some 2-cocycle α of $D \rtimes I(B)$. Hence $\mathbf{Z}B \simeq \mathbf{Z}F^\alpha[D \rtimes I(B)]$ as algebras and $\mathrm{ll}\mathbf{Z}B = \mathrm{ll}\mathbf{Z}F^\alpha[D \rtimes I(B)] \leq \mathrm{ll}F^\alpha[D \rtimes I(B)]$. By Lemmas 1.2, 2.1 and Proposition 1.5 in Passman [12], $\mathbf{J}F^\alpha[D \rtimes I(B)] = \mathbf{J}FD \cdot F^\alpha[D \rtimes I(B)] = F^\alpha[D \rtimes I(B)] \cdot \mathbf{J}FD$ and thus $\mathrm{ll}F^\alpha[D \rtimes I(B)] = \mathrm{ll}FD$. Moreover, by Theorem (3) in Motose [9], we have $\mathrm{ll}FD = p^m + p^n - 1$. □

Now we consider the case $p = 2$.

Proposition 2.5. *If $D \simeq Z_{2^m} \times Z_{2^n}$ for some $m, n \geq 1$ and $d = m + n$, then one of the following holds:*

(1) *B is nilpotent; in this case, $\mathrm{ll}\mathbf{Z}B = \mathrm{ll}FD = 2^m + 2^n - 1$.*
(2) *$m = n$, and B is Morita equivalent to $F[D \rtimes Z_3]$; in this case, $\mathrm{ll}\mathbf{Z}B \leq \mathrm{ll}FD = 2^{m+1} - 1$. In particular, $\mathrm{ll}\mathbf{Z}B = 2$ in case of $m = n = 1$.*
(3) *$m = n = 1$, and B is Morita equivalent to the principal block of FA_5 where A_5 is the five degree alternating group; in this case, $\mathrm{ll}\mathbf{Z}B = 2$.*

Furthermore, if $2^d - 3 \leq \mathrm{ll}\mathbf{Z}B \leq 2^d - 1$, then $D \simeq Z_2 \times Z_2$ or $Z_4 \times Z_2$.

Proof. Without loss of generality, we may assume $m \geq n$. We first calculate the order of automorphism group $\operatorname{Aut}(D)$ of D as follows:

$$|\operatorname{Aut}(D)| = \begin{cases} 3 \cdot 2^{4m-3} & \text{if } m = n, \\ 2^{m+3n-2} & \text{if } m > n. \end{cases}$$

We remark that $e(B)$ divides the odd part of $|\operatorname{Aut}(D)|$.

Case 1 ($e(B) = 1$). By Broué-Puig [2] and Puig [13], B is nilpotent and Morita equivalent to FD. Therefore $\mathrm{ll}\mathbf{Z}B = \mathrm{ll}FD = 2^m + 2^n - 1$.

In the following we may assume $e(B) = 3$ and $m = n$.

Case 2 ($m = n = 1$ and $e(B) = 3$). By a result of Erdmann [4], B is Morita equivalent to FA_4 or the principal block of FA_5. In each case $\mathrm{ll}\mathbf{Z}B = 2$ by Proposition 2.2 since $k(B) - l(B) = 1$.

Case 3 ($m = n \geq 2$ and $e(B) = 3$). B is Morita equivalent to $F[D \rtimes Z_3]$ by Eaton-Kessar-Külshammer-Sambale [3]. Thus (2) follows by Lemma 2.4.

The last part of the proposition is clear by the first part. $\square$

Finally, we study the case that $p = 3$ and $D \simeq Z_{3^n} \times Z_3$ for some $n \geq 1$.

Proposition 2.6. *If $D \simeq Z_{3^n} \times Z_3$ for some $n \geq 1$ and $d = n + 1$, then $\mathrm{ll}\mathbf{Z}B \leq 3^n + 2$. In particular, $\mathrm{ll}\mathbf{Z}B \leq p^d - 4$.*

Proof. We first obtain

$$|\operatorname{Aut}(D)| = \begin{cases} 16 \cdot 3 & \text{if } n = 1, \\ 4 \cdot 3^{n+1} & \text{if } n \geq 2. \end{cases}$$

Case 1 ($e(B) \leq 4$). If $e(B) = 1$, then $\mathrm{ll}\mathbf{Z}B = 3^n + 2$ by the same way as in Case 1 in the proof of Proposition 2.5. If $2 \leq e(B) \leq 4$, then B is perfectly isometric to its Brauer correspondent $\tilde{B}$ in $N_G(D)$ by Usami [16] and Puig-Usami [14], [15]. Hence $\mathrm{ll}\mathbf{Z}B = \mathrm{ll}\mathbf{Z}\tilde{B} \leq 3^n + 2$ by Lemma 2.4.

Since $e(B)$ divides the $3'$-part of $|\operatorname{Aut}(D)|$, we may assume $n = 1$ in the following.

Case 2 ($n = 1$ and $5 \leq e(B)$). $I(B)$ is isomorphic to one of the following groups:

Z_8, D_8(dihedral group of order 8)$, Q_8$(quaternion group of order 8),

SD_{16}(semi-dihedral group of order 16).

We first suppose $I(B)$ is isomorphic to D_8 or SD_{16}. By the results of Kiyota [6] and Watanabe [17], $k(B) - l(B)$ is at most 4 and thus $\mathrm{ll}\mathbf{Z}B \leq 5$ by Proposition 2.2. Finally, suppose $I(B)$ is isomorphic to Z_8 or Q_8. Kiyota [6] has not determined the invariants $k(B)$ and $l(B)$ in general. However, we can compute $k(B) - l(B)$ as follows. Since $I(B)$ acts on $D \backslash \{1\}$ regularly, the conjugacy classes of B-subsections are $(1, B)$ and (u, b_u) for some $u \in D \backslash \{1\}$ where b_u is a Brauer correspondent of B in $C_G(u)$. Moreover $I(b_u) \simeq C_{I(B)}(u)$ is trivial and thus b_u is nilpotent, $k(B) - l(B) = l(b_u) = 1$. Hence $\mathrm{ll}\mathbf{Z}B = 2$ as claimed.

The last part of the proposition is clear. $\square$

3. New upper bound

We first mention the result of Okuyama [11], the motivation of this paper.

Theorem 3.1 (Okuyama [11])**.** *Let d be the defect of B. Then*

$$\mathrm{ll}\mathbf{Z}B \leq p^d. \tag{3.1}$$

Equality occurs in (3.1) *if and only if B is a nilpotent block with cyclic defect group.*

We improve Theorem 3.1 in this section. Here we use a set $S(B)$ of representatives for the G-conjugacy classes of B-subsections. Namely, for each $(u, b) \in S(B)$, u is a p-element in G and b is a Brauer correspondent of B in $C_G(u)$. In the following, $|u|$ denotes the order of u and $\bar{b}$ denotes the unique block of $F[C_G(u)/\langle u\rangle]$ dominated by b. Now we give an upper bound of $\mathrm{ll}\mathbf{Z}B$ by using $S(B)$. The proof below is influenced by Okuyama [11].

Theorem 3.2.

$$\mathrm{ll}\mathbf{Z}B \leq \max\{(|u|-1)\mathrm{ll}\mathbf{Z}\bar{b} \mid (u,b) \in S(B)\} + 1. \tag{3.2}$$

Proof. We denote by λ the first part of the right side of (3.2). Note that $\mathbf{JZ}B = \mathbf{JZ}FG \cdot 1_B$ where 1_B is the block idempotent of B. We follow three steps.

Step 1. For each $(u, b) \in S(B)$, $(u-1)(\mathbf{JZ}b)^\lambda = 0$.

Let $\tau : FC_G(u) \to F[C_G(u)/\langle u\rangle]$ be the natural epimorphism. Then $\tau((\mathbf{JZ}b)^{\mathrm{ll}\mathbf{Z}\bar{b}}) \subseteq (\mathbf{JZ}\bar{b})^{\mathrm{ll}\mathbf{Z}\bar{b}} = 0$ and thus $(\mathbf{JZ}b)^{\mathrm{ll}\mathbf{Z}\bar{b}} \subseteq \operatorname{Ker}\tau = (u-1)FC_G(u)$. Thereby $(\mathbf{JZ}b)^\lambda \subseteq (\mathbf{JZ}b)^{(|u|-1)\mathrm{ll}\mathbf{Z}\bar{b}} \subseteq \{(u-1)FC_G(u)\}^{|u|-1} = (u-1)^{|u|-1}FC_G(u)$. Hence the claim follows.

Step 2. Take an element $a = \sum a_g g$ in $(\mathbf{JZ}B)^\lambda$. Then $a_{xy} = a_y$ for all p-elements x in G and p'-elements y in $C_G(x)$.

Let us denote by $\mathrm{Br}_{\langle x\rangle} : \mathbf{Z}FG \to \mathbf{Z}FC_G(x)$ the Brauer homomorphism. If $\mathrm{Br}_{\langle x\rangle}(1_B) = 0$, then $\mathrm{Br}_{\langle x\rangle}(a) = 0$ and hence $a_{xy} = a_y = 0$ as required. So we may assume $\mathrm{Br}_{\langle x\rangle}(1_B) \neq 0$. Then there exists a B-subsection $(u, b) \in S(B)$ and $t \in G$ such that $x = t^{-1}ut$. Since $a \in \mathbf{Z}FG$, $a_{xy} = a_{t^{-1}uty} = a_{utyt^{-1}}$ and $a_y = a_{tyt^{-1}}$. Therefore we need only prove the claim above for u and p'-element v in $C_G(u)$. Since $\mathrm{Br}_{\langle u\rangle}$ maps nilpotent elements to nilpotent elements, we have $\mathrm{Br}_{\langle u\rangle}(\mathbf{JZ}FG) \subseteq \mathbf{JZ}FC_G(u)$ and thus $\mathrm{Br}_{\langle u\rangle}((\mathbf{JZ}B)^\lambda) \subseteq \sum(\mathbf{JZ}b)^\lambda$ where $\mathrm{Br}_{\langle u\rangle}(1_B)1_b \neq 0$. Hence it follows from Step 1 that $(u-1)\mathrm{Br}_{\langle u\rangle}(a) = 0$. This implies $a_{uv} = a_v$ as asserted.

Step 3 (Completion of the proof). We denote by $Z_{p'}$ the F-subspace of $\mathbf{Z}FG$ spanned by all p'-section sums. It suffices to prove that $(\mathbf{JZ}B)^\lambda \subseteq Z_{p'}$ since $\mathbf{JZ}FG \cdot Z_{p'} = 0$ (see Brauer [1] or Okuyama [10]). Take an element $a = \sum a_g g \in (\mathbf{JZ}B)^\lambda$. We want to show that $a_g = a_h$ for all $g, h \in G$ if the p'-parts of them are G-conjugate. However, it is an immediate consequence of the claim in Step 2 since $a \in \mathbf{Z}FG$. Thus the theorem is completely proved.

□

In addition we give an upper bound of the right side of (3.2) in terms of the defect groups of B.

Corollary 3.3. *Let p^d and p^ε be the order and the exponent of a defect group D of B, respectively. Then*

$$\max\{(|u|-1)\mathrm{ll}\mathbf{Z}\bar{b} \mid (u,b) \in S(B)\} \leq p^d - p^{d-\varepsilon}. \tag{3.3}$$

If equality occurs in (3.3)*, then* $D \simeq Z_{p^\varepsilon} \times Z_{p^{d-\varepsilon}}$.

As a consequence, we have

$$\mathrm{ll}\mathbf{Z}B \leq p^d - p^{d-\varepsilon} + 1. \tag{3.4}$$

Proof. We may assume D is not trivial. Fix $(u, b) \in S(B)$ associated to the left side of (3.3). We let D' be a defect group of b of order $p^{d'}$ and put $|u| = p^{\varepsilon'}$. Then D' is contained in D up to G-conjugacy since $b^G = B$, $\varepsilon' \leq \varepsilon$ and we may assume that a defect group of $\bar{b}$ is $\bar{D}' = D'/\langle u\rangle$ (see Chapter V, Lemma 4.5, in Feit [5]). Hence we obtain from (3.1) that $(|u| - 1)\mathrm{ll}\mathbf{Z}\bar{b} \leq (p^{\varepsilon'} - 1)p^{d'-\varepsilon'} \leq (p^{\varepsilon'} - 1)p^{d-\varepsilon'} = p^d - p^{d-\varepsilon'} \leq p^d - p^{d-\varepsilon}$ as claimed. We next suppose equality holds in (3.3). Then we have $d = d', \varepsilon = \varepsilon'$ and $\bar{D}$ is cyclic. Since $\langle u\rangle$ is contained in the center of D', D' is abelian. Therefore we deduce $D \simeq D' = \langle u\rangle \times H$ where $H \simeq \bar{D}'$. □

4. Characterizations of blocks

In this last section, we determine all blocks with $p^d - 3 \leq \mathrm{ll}\mathbf{Z}B \leq p^d$. We remark that the notation given in Corollary 3.3 will be used throughout this section. The case of $\mathrm{ll}\mathbf{Z}B = p^d$ is due to Theorem 3.1. We next study the case of $\mathrm{ll}\mathbf{Z}B = p^d - 1$.

Theorem 4.1. *Let D be a defect group of B of order p^d. If $\mathrm{ll}\mathbf{Z}B = p^d - 1$, then one of the following holds:*

1. *$D \simeq Z_3$ and $I(B) \simeq Z_2$.*
2. *B is nilpotent and $D \simeq Z_2 \times Z_2$.*

Proof. In the case D is cyclic, (1) follows by Proposition 2.3. So we may assume that $\varepsilon < d$. Then, since $\mathrm{ll}\mathbf{Z}B = p^d - 1 \leq p^d - p^{d-\varepsilon} + 1 < p^d$, we have $D \simeq Z_2 \times Z_{2^{d-1}}$ by Corollary 3.3. Furthermore, we deduce $d = 2$ and (2) holds by Proposition 2.5. □

The next problem is the case of $\mathrm{ll}\mathbf{Z}B = p^d - 2$.

Theorem 4.2. *Let D be a defect group of B of order p^d. If $\mathrm{ll}\mathbf{Z}B = p^d - 2$, then one of the following holds:*

1. *$D \simeq Z_5$ and $I(B) \simeq Z_2$.*
2. *$D \simeq Z_2 \times Z_2$ and B is Morita equivalent to FA_4.*
3. *$D \simeq Z_2 \times Z_2$ and B is Morita equivalent to the principal block of FA_5, where A_4 and A_5 are four and five degree alternating groups, respectively.*

Proof. For the same reason as in the proof of Theorem 4.1, we may assume $\varepsilon < d$ and $\mathrm{ll}\mathbf{Z}B = p^d - 2 \leq \mathrm{ll}\mathbf{Z}\bar{b}\ (p^{\varepsilon'} - 1) + 1 \leq p^d - p^{d-\varepsilon} + 1 \leq p^d - 1$.

Case 1 ($\mathrm{ll}\mathbf{Z}B = p^d - p^{d-\varepsilon} + 1$). By Corollary 3.3, $D \simeq Z_3 \times Z_{3^{d-1}}$. However, $\mathrm{ll}\mathbf{Z}B \neq p^d - 2$, in this case by Proposition 2.6.

Case 2 ($\mathrm{ll}\mathbf{Z}\bar{b}\ (p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 1$). We have $D \simeq Z_2 \times Z_{2^{d-1}}$ and thus (2) or (3) holds by Proposition 2.5.

Case 3 ($\mathrm{ll}\mathbf{Z}B = \mathrm{ll}\mathbf{Z}\bar{b}\ (p^{\varepsilon'} - 1) + 1$ and $p^d - p^{d-\varepsilon} + 1 = p^d - 1$). We obtain $p = 2, d - \varepsilon = 1$ and $\mathrm{ll}\mathbf{Z}\bar{b} = \frac{2^d - 3}{2^{\varepsilon'} - 1}$. Since $\mathrm{ll}\mathbf{Z}\bar{b} \leq |\bar{D}'| = 2^{d'-\varepsilon'} \leq 2^{d-\varepsilon'}, d - \varepsilon' = 1$ (remark that $0 < d - \varepsilon \leq d - \varepsilon'$), and so $\mathrm{ll}\mathbf{Z}\bar{b} = 1$ or 2. Thus we have $\varepsilon' = 1$ and $d = 2$. In this case, (2) or (3) holds by Proposition 2.5.

□

Finally, we consider the case of $\mathrm{ll}\mathbf{Z}B = p^d - 3$.

Theorem 4.3. *Let D be a defect group of B of order p^d. If $\mathrm{ll}\mathbf{Z}B = p^d - 3$, then one of the following holds:*

(1) *$D \simeq Z_5$ and $I(B) \simeq Z_4$.*
(2) *$D \simeq Z_7$ and $I(B) \simeq Z_2$.*
(3) *B is nilpotent and $D \simeq Z_4 \times Z_2$.*

Proof. We may assume D is not cyclic, $\varepsilon < d$ and $\mathrm{ll}\mathbf{Z}B = p^d - 3 \leq \mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1 \leq p^d - p^{d-\varepsilon} + 1 \leq p^d - 1$.

Case 1 ($\mathrm{ll}\mathbf{Z}B = p^d - p^{d-\varepsilon} + 1$). By Corollary 3.3, we have $D \simeq Z_4 \times Z_{2^{d-2}}$ and hence we obtain $d = 3$ by using Proposition 2.5.

Case 2 ($\mathrm{ll}\mathbf{Z}B = \mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1, p^d - p^{d-\varepsilon} + 1 = p^d - 2$). Clearly, $p = 3, d - \varepsilon = 1$ and $\mathrm{ll}\mathbf{Z}\bar{b} = \frac{3^d - 4}{3^{\varepsilon'} - 1}$. However, this case cannot occur since $\mathrm{ll}\mathbf{Z}\bar{b}$ is not an integer.

Case 3 ($\mathrm{ll}\mathbf{Z}B = \mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1, p^d - p^{d-\varepsilon} + 1 = p^d - 1$). We first obtain $p = 2, d - \varepsilon = 1$ and $\mathrm{ll}\mathbf{Z}\bar{b} = \frac{2^d - 4}{2^{\varepsilon'} - 1}$. Since $2^{\varepsilon'} - 1$ is odd, we have $\varepsilon' = 1$. Hence $2^d - 4 = \mathrm{ll}\mathbf{Z}\bar{b} \leq |\bar{D}'| \leq 2^{d-1}$ and thus $d = 3$ (remark that $\mathrm{ll}\mathbf{Z}B \geq 2$). Moreover, since we have $\bar{D}' \simeq Z_4$ and $d = d'$, D' is abelian by the same reason as in Corollary 3.3 and thus $D \simeq D' = Z_4 \times Z_2$.

Case 4 ($\mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 2$). In this case, $D \simeq Z_3 \times Z_{3^{d-1}}$. However, $\mathrm{ll}\mathbf{Z}B \neq p^d - 3$ by Proposition 2.6.

Case 5 ($\mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1 = p^d - 2, p^d - p^{d-\varepsilon} + 1 = p^d - 1$). We have $p = 2, d - \varepsilon = 1$ and $\mathrm{ll}\mathbf{Z}\bar{b} = \frac{2^d - 3}{2^{\varepsilon'} - 1}$. Since $\mathrm{ll}\mathbf{Z}\bar{b} \leq |\bar{D}'| = 2^{d' - \varepsilon'} \leq 2^{d - \varepsilon'}$, we deduce that $d - \varepsilon' = 1$ and $\mathrm{ll}\mathbf{Z}\bar{b} = 1$ or 2. Thus we obtain $d = 2$, but this case cannot occur.

Case 6 ($\mathrm{ll}\mathbf{Z}\bar{b}\,(p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 1$). We have $D \simeq Z_2 \times Z_{2^{d-1}}$ by Corollary 3.3 and hence $d = 3$ in this case using Proposition 2.5.

□

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