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TOPOLOGICAL INVARIANT MEANS ON ALMOST PERIODIC FUNCTIONALS: SOLUTION TO PROBLEMS BY DALES–LAU–STRAUSS AND DAWS

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ABSTRACT. Let $\mathcal{G}$ be a locally compact group, and denote by $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ and $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ the spaces of weakly almost periodic, respectively, almost periodic functionals on the measure algebra $\mathrm{M}(\mathcal{G})$. Problem 3 in [H.G. Dales, A.T.-M. Lau, D. Strauss, *Second duals of measure algebras*, Dissertationes Math. (Rozprawy Mat.) 481 (2012), 1–121] asks if $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ and $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ admit topological invariant means, and if yes, whether they are unique. The questions regarding existence had already been raised in [M. Daws, *Characterising weakly almost periodic functionals on the measure algebra*, Studia Math. 204 (2011), no. 3, 213–234]. We answer all these problems in the affirmative.

1. The problems

Let $\mathcal{A}$ be a Banach algebra. For any $f \in \mathcal{A}^*$, consider the canonical map $\lambda_f : \mathcal{A} \to \mathcal{A}^*$, $\lambda(f) := fa$, where $\langle fa, b\rangle = \langle f, ab\rangle$ $(a, b \in \mathcal{A})$. The spaces $\mathrm{WAP}(\mathcal{A})$ and $\mathrm{AP}(\mathcal{A})$ of all weakly almost periodic, respectively, almost periodic functionals on $\mathcal{A}$ are defined as the sets of all $f \in \mathcal{A}^*$ such that λ_f is weakly compact, respectively, compact; trivially, $\mathrm{AP}(\mathcal{A}) \subseteq \mathrm{WAP}(\mathcal{A})$. Denote by $\Box$ and $\Diamond$ the left and right Arens products on $\mathcal{A}^{**}$, as well as the corresponding canonical module actions of $\mathcal{A}^{**}$ on $\mathcal{A}^*$. We recall [2, Proposition 3.11] that $f \in \mathcal{A}^*$ belongs to $\mathrm{WAP}(\mathcal{A})$ if and only if

$$\langle m\Box n, f\rangle = \langle m\Diamond n, f\rangle \text{ for all } m, n \in \mathcal{A}^{**}. \tag{1}$$

Now assume $\mathcal{A}$ has a bounded approximate identity (BAI). Then, by [3, Proposition 1.13], $\mathrm{WAP}(\mathcal{A})$ and $\mathrm{AP}(\mathcal{A})$ are essential (i.e., neo-unital) Banach $\mathcal{A}$-bimodules, whence $\mathrm{WAP}(\mathcal{A}) = \mathcal{A}\ \mathrm{WAP}(\mathcal{A}) = \mathrm{WAP}(\mathcal{A})\ \mathcal{A}$, and analogously for $\mathrm{AP}(\mathcal{A})$. Note that if $\mathcal{A}$ has a BAI of norm 1, then $\mathcal{A}$ can be identified isometrically as an ideal in its multiplier algebra $\mathcal{M}(\mathcal{A})$, and $\mathrm{WAP}(\mathcal{A})$ and $\mathrm{AP}(\mathcal{A})$ become Banach $\mathcal{M}(\mathcal{A})$-bimodules (extending the actions of $\mathcal{A}$); see [6, pp. 40–41].

For instance, consider the group algebra $\mathcal{A} = L_1(\mathcal{G})$ for a locally compact group $\mathcal{G}$. Then, by the above, $\mathrm{WAP}(L_1(\mathcal{G}))$ and $\mathrm{AP}(L_1(\mathcal{G}))$ are essential Banach $L_1(\mathcal{G})$-bimodules, and Banach $\mathrm{M}(\mathcal{G})$-bimodules, where $\mathrm{M}(\mathcal{G})$ is the measure algebra of $\mathcal{G}$ (noting that $\mathcal{M}(\mathcal{A}) = \mathrm{M}(\mathcal{G})$, by Wendel's theorem); see also [3, Theorem 5.14] for a different proof of the latter statement. We note that, as is well known, $\mathrm{WAP}(L_1(\mathcal{G})) = \mathrm{WAP}(\mathcal{G})$ and $\mathrm{AP}(L_1(\mathcal{G})) = \mathrm{AP}(\mathcal{G})$, the C^*-algebras of weakly almost periodic, respectively, almost periodic functions on $\mathcal{G}$; see, e.g., [2, (7.5)].

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In this paper, we shall be mainly concerned with the spaces $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ and $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$, which are C^*-subalgebras of $\mathrm{M}(\mathcal{G})^*$; see [4, Corollaries 1 and 2]. We denote by $*$ the convolution product in $\mathrm{M}(\mathcal{G})$, as well as the corresponding left and right action of $\mathrm{M}(\mathcal{G})$ on $\mathrm{M}(\mathcal{G})^*$, and analogously for $L_1(\mathcal{G})$.

A left (respectively, right) Banach $\mathcal{A}$-submodule X of $\mathcal{A}^*$ is called left-introverted (respectively, right-introverted) if $m \Box f \in X$ (respectively, $f \Diamond m \in X$) whenever $m \in \mathcal{A}^{**}$ and $f \in X$; a Banach $\mathcal{A}$-subbiomodule X of $\mathcal{A}^*$ is called introverted if it is both left- and right-introverted.

Now, assume that the Banach algebra $\mathcal{A}$ is a Lau algebra, i.e., $\mathcal{A}^*$ is a von Neumann algebra whose unit 1 lies in the spectrum of $\mathcal{A}$. Denote by $P_1(\mathcal{A})$ the set of all $p \in \mathcal{A}$ such that $\|p\| = \langle p, 1\rangle = 1$. We recall the following [3, Definition 1.20]:

Definition 1.1. Let $\mathcal{A}$ be a Lau algebra, and X an introverted C^*-subalgebra of $\mathcal{A}^*$. A topological invariant mean on X is an element $m \in P_1(\mathcal{A}^{**})$ such that

$$\langle m, xp\rangle = \langle m, px\rangle = \langle m, x\rangle \text{ for all } x \in X,\ p \in P_1(\mathcal{A}).$$

Let $\mathcal{G}$ be a locally compact group. We are interested in the cases where $\mathcal{A} = L_1(\mathcal{G})$ and $X = \mathrm{WAP}(L_1(\mathcal{G}))$ (or $\mathrm{AP}(L_1(\mathcal{G}))$), as well as $\mathcal{A} = \mathrm{M}(\mathcal{G})$ and $X = \mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ (or $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$); see [3, Proposition 1.14] concerning the property of introversion. Given any locally compact group $\mathcal{G}$, it is well known that $\mathrm{WAP}(L_1(\mathcal{G}))$ and $\mathrm{AP}(L_1(\mathcal{G}))$ each admit a topological invariant mean which is unique when restricted to $\mathrm{WAP}(L_1(\mathcal{G}))$, respectively, $\mathrm{AP}(L_1(\mathcal{G}))$; cf., e.g., [7, p. 361, Remark (b)]. Note that in view of Definition 1.1 above, we need to consider the restriction to $\mathrm{WAP}(L_1(\mathcal{G}))$, respectively, $\mathrm{AP}(L_1(\mathcal{G}))$, to obtain uniqueness; for instance, there are $2^{\mathfrak{c}}$ many topological invariant means on $\ell_\infty(\mathbb{Z})$ whose restrictions to $\mathrm{WAP}(\ell_1(\mathbb{Z})) = \mathrm{WAP}(\mathbb{Z})$ coincide; cf. [1, Theorem 1]. In the following, we will, however, simply refer to *the* topological invariant mean on $\mathrm{WAP}(L_1(\mathcal{G}))$, respectively, $\mathrm{AP}(L_1(\mathcal{G}))$.

In [3], Dales–Lau–Strauss list several open problems, including, as no. 3, the following: "Let $\mathcal{G}$ be a locally compact group. Do $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ or $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ always have a topological invariant mean. If so, is it unique?" (Of course, uniqueness refers to the restricted functional.) The questions on existence were already formulated by Daws: "It would be interesting to know if $\mathrm{C}(K_{\mathrm{AP}}) = \mathrm{AP}(\mathrm{M}(\mathcal{G}))$ always carries an invariant probability measure", and "it seems natural to ask about [an] invariant measure on K_{WAP}, but we have made no progress in this direction" [5, pp. 222 and 233]; here, K_{AP} and K_{WAP} denote the spectra of the C^*-algebras $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ and $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$, respectively. In [3, Proposition 5.16], existence is established in the case where $\mathcal{G}$ is amenable. In the following, we shall prove that all of these problems have an affirmative answer for any locally compact group $\mathcal{G}$.

2. The proofs

We start by showing uniqueness of a topological invariant mean in the general setting of Lau algebras.

Theorem 2.1. *Let $\mathcal{A}$ be a Lau algebra. Then* $\mathrm{WAP}(\mathcal{A})$ *and* $\mathrm{AP}(\mathcal{A})$ *can each have at most one topological invariant mean (when restricted to* $\mathrm{WAP}(\mathcal{A})$*, respectively,* $\mathrm{AP}(\mathcal{A})$*).*

Proof. Let m_1, m_2 be topological invariant means on $\mathrm{WAP}(\mathcal{A})$, and denote by 1 the unit of the von Neumann algebra $\mathcal{A}^*$. Using (1), we have for all $f \in \mathrm{WAP}(\mathcal{A})$,

$$\langle m_1, m_2 \Box f\rangle = \langle m_1 \Box m_2, f\rangle = \langle m_1 \Diamond m_2, f\rangle = \langle m_2, f \Diamond m_1\rangle.$$

Since m_2 is right invariant, we have $\langle m_2 \Box f, a\rangle = \langle m_2, fa\rangle = \langle a, 1\rangle\langle m_2, f\rangle$ for all $a \in \mathcal{A}$, whence $m_2 \Box f = \langle m_2, f\rangle 1$. Similarly, as m_1 is left invariant, we obtain that $f \Diamond m_1 = \langle m_1, f\rangle 1$. By the above, and noting that $\langle m_1, 1\rangle = \langle m_2, 1\rangle = 1$, we thus have for all $f \in \mathrm{WAP}(\mathcal{A})$:

$$\langle m_2, f\rangle = \langle m_1, m_2 \Box f\rangle = \langle m_2, f \Diamond m_1\rangle = \langle m_1, f\rangle,$$

as claimed. The proof in the case of $\mathrm{AP}(\mathcal{A})$ is analogous. □

To prove existence, we will use the following folklore result (see [5, Lemma 3.3] for a quick proof):

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $T : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then T^* maps $\mathrm{WAP}(\mathcal{B})$ to $\mathrm{WAP}(\mathcal{A})$, and $\mathrm{AP}(\mathcal{B})$ to $\mathrm{AP}(\mathcal{A})$.*

We now come to the main result.

Theorem 2.3. *Let $\mathcal{G}$ be a locally compact group. Then $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ and $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ each have a topological invariant mean that is unique (when restricted to $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$, respectively, $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$).*

Proof. Uniqueness follows from Theorem 2.1. We will prove existence for $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$, the argument for $\mathrm{AP}(\mathrm{M}(\mathcal{G}))$ being analogous.

Denote by $m_0 \in P_1(L_1(\mathcal{G})^{**})$ the topological invariant mean on $\mathrm{WAP}(L_1(\mathcal{G})) = \mathrm{WAP}(\mathcal{G})$; so m_0 is invariant under the left and right (convolution) action of $P_1(L_1(\mathcal{G}))$ on $\mathrm{WAP}(L_1(\mathcal{G}))$. Denote by $\iota : L_1(\mathcal{G}) \to \mathrm{M}(\mathcal{G})$ the canonical embedding. By Lemma 2.2, the restriction $R := \iota^* : \mathrm{M}(\mathcal{G})^* \to L_\infty(\mathcal{G})$ to $L_1(\mathcal{G})$ maps $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$ to $\mathrm{WAP}(L_1(\mathcal{G}))$. Define $m := m_0 \circ R \in \mathrm{M}(\mathcal{G})^{**}$. Denote by 1 the function constant 1. Obviously, $\|m\| \leq \|m_0\| = 1$ and $\langle m, 1\rangle = \langle m_0, 1\rangle = 1$, whence $\|m\| = 1 = \langle m, 1\rangle$. Thus, $m \in P_1(\mathrm{M}(\mathcal{G})^{**})$.

To show that m is left invariant on $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$, let $\mu \in P_1(\mathrm{M}(\mathcal{G}))$ and $h \in \mathrm{WAP}(\mathrm{M}(\mathcal{G}))$. Note first that $R(\mu * h) = \mu * R(h)$, since we have for all $g \in L_1(\mathcal{G})$:

$$\langle R(\mu * h), g\rangle = \langle \mu * h, \iota(g)\rangle = \langle h, \iota(g) * \mu\rangle = \langle h, \iota(g * \mu)\rangle = \langle R(h), g * \mu\rangle = \langle \mu * R(h), g\rangle,$$

using that $L_1(\mathcal{G})$ is an ideal in $\mathrm{M}(\mathcal{G})$. Write (e_α) for the standard BAI of $L_1(\mathcal{G})$, which lies in $P_1(L_1(\mathcal{G}))$. Note that $e_\alpha * f \to f$ (in norm) for all $f \in \mathrm{WAP}(L_1(\mathcal{G}))$, as $\mathrm{WAP}(L_1(\mathcal{G}))$ is an essential left Banach $L_1(\mathcal{G})$-module; of course, this also follows directly from the fact that $\mathrm{WAP}(L_1(\mathcal{G})) = \mathrm{WAP}(\mathcal{G}) \subseteq \mathrm{UC}(\mathcal{G})$, the space of bounded (left and right) uniformly continuous functions on $\mathcal{G}$. Now we obtain, using that m_0 is left invariant under the $P_1(L_1(\mathcal{G}))$-action on $\mathrm{WAP}(L_1(\mathcal{G}))$:

$$\begin{aligned}
\langle m, \mu * h\rangle &= \langle m_0, R(\mu * h)\rangle \\
&= \langle m_0, \mu * R(h)\rangle \\
&= \lim_\alpha \langle m_0, \mu * e_\alpha * R(h)\rangle \text{ since } R(h) \in \mathrm{WAP}(L_1(\mathcal{G})) \\
&= \langle m_0, R(h)\rangle \text{ since } \mu * e_\alpha \in P_1(L_1(\mathcal{G})) \text{ and } R(h) \in \mathrm{WAP}(L_1(\mathcal{G})) \\
&= \langle m, h\rangle.
\end{aligned}$$

This proves left invariance of m on $\mathrm{WAP}(\mathrm{M}(\mathcal{G}))$. Right invariance is shown analogously, using the right module actions. □

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