PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 9, September 2017, Pages 3829–3841
http://dx.doi.org/10.1090/proc/13360
Article electronically published on May 24, 2017

SEPARABLE QUOTIENTS IN $C_c(X)$, $C_p(X)$, AND THEIR DUALS

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(Communicated by Thomas Schlumprecht)

Abstract. The quotient problem has a positive solution for the weak and strong duals of $C_c(X)$ (X an infinite Tichonov space), for Banach spaces $C_c(X)$, and even for barrelled $C_c(X)$, but not for barrelled spaces in general. The solution is unknown for general $C_c(X)$. A locally convex space is *properly separable* if it has a proper dense $\aleph_0$-dimensional subspace. For $C_c(X)$ quotients, *properly separable* coincides with *infinite-dimensional separable*. $C_c(X)$ has a properly separable *algebra* quotient if X has a compact denumerable set. Relaxing *compact* to *closed*, we obtain the converse as well; likewise for $C_p(X)$. And the weak dual of $C_p(X)$, which always has an $\aleph_0$-dimensional quotient, has no *properly* separable quotient when X is a P-space of a certain special form $X = X_\kappa$

1. Introduction

Here we assume topological vector spaces (tvs's) and their quotients are Hausdorff and infinite-dimensional. Banach's famous unsolved problem asks: *Does every Banach space admit a separable quotient?* [Popov's] ⟨Our⟩ negative [tvs] ⟨lcs⟩ solution found a [metrizable tvs] ⟨barrelled lcs⟩ without a separable quotient [16, 20]. (By *lcs* we mean *locally convex tvs over the real or complex scalar field.*)

The familiar Banach spaces admit separable quotients, as do all non-Banach Fréchet spaces [6, Satz 2] and (LF)-spaces [28, Theorem 3]. The non-Banach (LF)-space φ is an $\aleph_0$-dimensional space with the strongest locally convex topology whose only quotients are copies of φ itself. While separable, φ is not *properly* separable by Robertson's *ad hoc* definition [21] (see Abstract). Saxon's answer [26] to her quarter-century-old question proves φ is the *only* non-Banach (LF)-space without a *properly* separable quotient if and only if every Banach space has a separable quotient.

Her notion, unexpectedly characterized in weak barrelledness terms [26], intrigues the more: When/how may separable *vs.* properly separable quotients exist/differ in an lcs? Section 2 explores differences but finds no lcs without a separable quotient other than the (rather exotic) examples we found earlier [16]. The main (third) section concentrates on the function spaces $C_c(X)$ and $C_p(X)$, where

Received by the editors May 3, 2016 and, in revised form, June 1, 2016 and June 21, 2016.
2010 *Mathematics Subject Classification.* Primary 46A08, 46A30, 54C35.
Key words and phrases. (Properly) separables quotients, $C(X)$, weak barrelledness, P-spaces.
Thanks to Professor Aaron R. Todd for vital discussions/encouragement/prequels.
The first author's research was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, grant PROMETEO/2013/058, and by GACR grant 16-34860L and RVO 67985840 (Czech Republic).

the two notions of separable quotient coincide and beg the existential question (we fully answer) for separable *algebra* quotients. Robertson's notion keys new analytic characterizations (Theorems 23, 24) for P-spaces X of a certain form $X = X_\kappa$. The final section reviews open questions.

2. Weak barrelledness motivations, general lcs quotients

Remarkably, dense subspaces of GM-spaces are barrelled [5]. A Banach space has no separable quotient if and only if its dense subspaces are barrelled [30, 31]. An lcs E has no separable quotient if and only if its dense subspaces are non-S_σ (defined below) [16]. E has no *properly* separable quotient if and only if its dense subspaces are primitive [26].

S_σ*-spaces* are those lcs's covered by increasing sequences of closed proper subspaces. An lcs E is [*inductive*] $\langle$*primitive*$\rangle$ if ϕ is continuous whenever $\{E_n\}_n$ is an increasing covering sequence of subspaces and ϕ is a [seminorm] $\langle$linear form$\rangle$ on E such that each restriction $\phi|_{E_n}$ is continuous. These notions from the study of weak barrelledness relate as follows:

$$\begin{array}{ccccc} & & \text{barrelled} & & \\ & & \Downarrow & & \\ \text{non-}S_\sigma & \Rightarrow & \text{inductive} & \Rightarrow & \text{primitive.} \end{array}$$

Hence GM-spaces lack properly separable quotients. Quotients and countable-codimensional subspaces preserve each of the four properties. *Non-*S_σ and *primitive* are duality invariant properties, unlike *barrelled* and *inductive.* Under the Mackey topology, *inductive* $\Leftrightarrow$ *primitive.* Under metrizability, *non-*S_σ $\Leftrightarrow$ *primitive* [27, 29].

Easily, *properly* separable quotients and separable quotients coincide for metrizable spaces and for non-S_σ spaces, *e.g.*, for all $C_c(X)$ and $C_p(X)$. Our negative barrelled solution [16] now follows from the two paragraphs above; we merely needed a GM-space that is non-S_σ, and such spaces exist (that are even Baire) [5].

An lcs E is properly separable if and only if E is separable and $E' \neq E^*$, a corollary to the fact that (†) *finite*-codimensional subspaces of separable lcs's are separable [4]. Moreover, (††) *countable*- cannot replace *finite*- [4, 32].

The *separable quotient* analogs of (†) and (††) hold:

Theorem 1. *If an lcs E has a separable quotient, so do the finite-codimensional subspaces of E.*

Proof. Immediate from (†) and [23, Theorem 2(b)]. □

Example 2. A *countable*-codimensional subspace G of a barrelled space E does not necessarily admit a separable quotient when E does.

Proof. Let G be any non-S_σ GM-space and set $E = G \oplus \varphi$. □

The *properly separable quotient* story excludes Theorem 1:

Example 3. There is a Mackey space E with dense hyperplane H such that E admits a properly separable quotient and H does not.

Proof. Let (H, τ) be any S_σ GM-space, *e.g.*, φ. By [29, Theorem 3.2], H is a dense hyperplane of a non-primitive Mackey space (E, μ) with $(H, \mu)' = (H, \tau)'$. Thus all the dense subspaces of H are primitive. This section's first paragraph assures (i) E has a properly separable quotient, but (ii) H does not. □

3. $C_c(X)$, $C_p(X)$ AND THEIR DUALS

Throughout, X denotes an infinite completely regular Hausdorff topological space with Stone-Čech compactification βX. Let $C(X)$ $\left[\text{resp., } C^b(X)\right]$ denote the vector space of $\mathbb{R}$-valued continuous [resp., and bounded] functions on X. Let $C_c(X)$ denote $C(X)$ endowed with the compact-open topology. For $A \subset X$ and $\varepsilon > 0$, we put $[A, \varepsilon] = \{f \in C(X) : |f(x)| \leq \varepsilon \text{ for all } x \in A\}$. Sets of the form $[K, \varepsilon]$ with K a compact $\langle$resp., finite$\rangle$ subset of X and $\varepsilon > 0$ constitute a base of neighborhoods of 0 for $C_c(X)$ $\langle$resp., for the lcs denoted by $C_p(X)\rangle$. By $C_u^b(X)$ we denote the Banach space whose unit ball is $[X, 1]$.

For compact X, Rosenthal [22] implies the Banach space c is an algebra quotient of $C_c(X)$ if X has a denumerable compact subset. We prove the result and its converse for arbitrary X. We prove $C_c(X)$ and $C_p(X)$ have separable algebra quotients if and only if X has a denumerable closed subset. We prove $C_c(X)$ admits a separable quotient when X is non-pseudocompact or a P-space or of pointwise countable type.

It is unknown whether $C_c(X)$ or $C_p(X)$ always has a separable quotient. Their weak and strong duals do [16]. The dual $L(X)$ of $C_p(X)$ given any topology compatible with the dual pairing, *e.g.*, the weak dual $L_p(X)$ or the Mackey dual $L_m(X)$, even has an $\aleph_0$-dimensional quotient [9]. When $L_p(X)$ does <u>not</u> have a *properly* separable quotient, $C_p(X)$ <u>does</u>, and when $C_p(X)$ does, so does $C_c(X)$ [Corollaries 11, 20]. Example 3 (with the same proof) holds for $H = L_m(X_\kappa)$ if and only if X_κ is a P-space, as we show later. Or, we could combine Theorem 23 with [2] to obtain [X_κ is a (non-discrete) P-space] $\Leftrightarrow$ [every dense subspace of $L_m(X_\kappa)$ is primitive and not barrelled], adding new examples/characterizations to the previous section and recent work [9, 10, 24]. (Many other modern marriages of topology and analysis may be found in [13].)

Now X is compact if and only if $C_u^b(X) = C_c(X)$. Always, $C_c(X)$ and $C_p(X)$ are non-S_σ [18, II.4.7] since $C^b(X)$ is dense and $C_u^b(X)$ is non-S_σ. Density derives from a well-known corollary to [11, 3.11(a),(c)]:

Lemma 4. *If $K \subset \mathcal{O} \subset X$ with K compact and $\mathcal{O}$ open, and $g \in C(K)$, then there exists $f \in C^b(X)$ such that* $\sup\{|f(x)| : x \in X\} = \sup\{|g(x)| : x \in K\}$, *$f|_K = g$, and f vanishes on $X \backslash \mathcal{O}$.*

The next lemma, likely known, has a simple proof.

Lemma 5. *The following five statements are equivalent.*

(1) *X admits a non-trivial convergent sequence.*
(2) *X admits a sequentially compact infinite set.*
(3) *X admits a compact denumerable set.*
(4) *X admits a countably compact denumerable set.*
(5) *X admits a compact metrizable infinite set.*

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (2) $\Rightarrow$ (1). Moreover, (1) $\Rightarrow$ (5), since any sequence $\{x_0, x_1, x_2, x_3, \dots\}$ of distinct points converging to x_0 in X is clearly homeomorphic to $\{0, 1, 1/2, 1/3, \dots\}$ in the (metrizable) unit interval $[0, 1]$.

Finally, note that $\neg$ (1) $\Rightarrow \neg$ (4): Let $S = \{y_1, y_2, \dots\}$ be an arbitrary sequence of distinct points in X. By hypothesis S does not converge to y_1, so there exists some neighborhood N_1 of y_1 which misses a subsequence $S_1 = \{x_{11}, x_{12}, \dots\}$ of S. We inductively find a neighborhood N_k of y_k which misses a subsequence $S_k = \{x_{k1}, x_{k2}, \dots\}$ of S_{k-1} for $k = 2, 3, \dots$. The diagonal sequence $T = \{x_{11}, x_{22}, \dots\}$ of distinct points in S has no cluster point in S, since N_k is a neighborhood of y_k that contains at most $k-1$ points of T $(k = 1, 2, \dots)$. Therefore S is not countably compact, denying (4). □

The proof of [22, Corollary 2.6] explicitly uses a form of

Lemma 6. *If X is countably compact, at least one of the following two cases holds:*

Case 1. *X admits a non-trivial convergent sequence.*
Case 2. *The derived set X^d of all cluster points is infinite and perfect.*

Proof. If X^d is finite, then its union with any denumerable set in X verifies (4), hence (1). If X^d is not perfect, then there exists x_0 in $X^d \backslash X^{dd}$. Let V be a closed neighborhood of x_0 that misses $X^d \backslash \{x_0\}$, let $x_1, x_2, \dots$ be distinct points in V, and define $S := \{x_0, x_1, x_2, \dots\}$. Since $S^d \subset V \cap X^d = \{x_0\}$, the set S is countably compact by hypothesis; *i.e.*, (4) holds. Then so does (1). □

In the simplest Case 1 examples, X is a convergent sequence of distinct points, making $C_c(X)$ isomorphic to the Banach space c of convergent scalar sequences, and $C_p(X)$ isomorphic to c with the topology induced by $\mathbb{R}^{\mathbb{N}}$, both separable.

We can now sketch a proof from [22] of the seminal

Theorem 7 (Rosenthal)**.** *When X is compact, the Banach space $C_c(X)$ has a (separable) quotient isomorphic to either c or the Hilbert space ℓ^2.*

Proof. By Lemma 6, there are only two cases possible.

Case 1. *X contains a sequence $\{x_n\}_n$ of distinct points converging to some point $x_0 \neq x_n$ $(n \in \mathbb{N})$.* The linear map $T : C_c(X) \to c$ defined by $f \mapsto (f(x_n))_{n \geq 1}$ is obviously continuous and is onto the Banach space c by Lemma 4. Hence the quotient $C_c(X)/T^{-1}(0)$ is isomorphic to c.

Case 2. *X contains a perfect infinite set.* Via the Khinchin inequality (*cf.* [8]) one finds ℓ^2 is isomorphic to a subspace of $L^1[0,1]$, and in Case 2, $L^1[0,1]$ is isomorphic to a subspace of $L^1(X, \mathfrak{B}_X, \mu)$, with $\mathfrak{B}_X$ the Borel sets in X and μ a non-negative, finite, regular Borel measure on X. In turn, the latter space is isomorphic to a subspace of the strong dual $C_c(X)'_\beta$ of $C_c(X)$. Therefore the reflexive ℓ^2 is a subspace of $C_c(X)'_\beta$ that is w^*-closed [22, Corollary 1.6, Proposition 1.2]. This yields a quotient of $C_c(X)$ isomorphic to ℓ^2. □

Rosenthal recalled on p. 180 of [22] that ℓ^∞ may be identified with $C_u^b(\beta\mathbb{N})$, clearly aware of Corollaries 8, 9 below. Whether he actually observed Corollaries 10, 11 is less clear.

Corollary 8 (Rosenthal)**.** *$C_u^b(X) \approx C_c(\beta X)$ has a separable quotient.*

Proof. The Stone-Čech theorem [33], then Theorem 7. □

Corollary 9 (Rosenthal)**.** *Some quotient of ℓ^∞ is isomorphic to ℓ^2.*

Proof. $\ell^\infty = C_u^b(\mathbb{N}) \approx C_c(\beta\mathbb{N})$ and $(\beta\mathbb{N})^d = \beta\mathbb{N} \backslash \mathbb{N}$ is infinite and perfect, so Case 2 of Theorem 7 applies. □

Corollary 10. *If X has an infinite compact subset Y, then $C_c(X)$ has a quotient isomorphic to c or ℓ^2.*

Proof. The restriction map $f \mapsto f|_Y$ from $C_c(X)$ into $C_c(Y)$ is clearly linear and continuous. By Lemma 4, it is open. And quotient-taking is transitive. □

Corollary 11. *If $C_p(X)$ has a separable quotient, then so does $C_c(X)$.*

Proof. Either $C_c(X) = C_p(X)$, or Corollary 10 applies. □

Recall that an lcs E is a *GM-space* [5] if every linear map $t : E \to F$, where F is any metrizable lcs and t has closed graph, is necessarily continuous. Immediately from Mahowald's theorem, every GM-space is barrelled. No $C_u^b(X)$ is a GM-space, twice-proved: (i) no metrizable lcs F is a GM-space, since there is always a strictly finer metrizable locally convex topology on F; (ii) GM-spaces lack properly separable quotients. Moreover,

Theorem 12. *Neither $C_c(X)$ nor $C_p(X)$ is a GM-space.*

Proof. Barrelled $C_c(X)$ spaces admit (properly) separable quotients [16]. And if $C_p(X)$ is barrelled, then $C_p(X) = C_c(X)$ [2]. □

Corollary 13. *Always, there exists a discontinuous linear map with closed graph from $C_c(X)$ into some metrizable lcs.*

From Lemma 4, the closed ideals of $C_p(X)$ are precisely the spaces

$$\Im_A = \{f \in C(X) : f(x) = 0 \text{ for all } x \in A\}$$

where A ranges over the closed subsets of X. These are also the closed ideals of $C_c(X)$ [7, Theorem 4.10.6]. An *algebra quotient* of $C_c(X)$ or $C_p(X)$ is one by a closed ideal, thus preserving vector multiplication. In Rosenthal's Case 1 the quotient is an algebra quotient, since the kernel of T is $\Im_A$ with $A = \{x_0, x_1, x_2, \dots\}$.

Recall that X is *pseudocompact* if $C(X) = C^b(X)$. Algebra quotients add to a list [15, Theorem 1.1] of characterizations found jointly with Todd.

Theorem 14. *When X is non-pseudocompact, $C_c(X)$ and $C_p(X)$ admit separable quotients. In fact, the following seven statements are equivalent.*

(1) *X is not pseudocompact.*
(2) *$C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$.*
(3) *$C_p(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$.*
(4) *$C_c(X)$ admits a quotient isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(5) *$C_p(X)$ admits a quotient isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(6) *$C_c(X)$ admits an algebra quotient isomorphic to $\mathbb{R}^{\mathbb{N}}$.*
(7) *$C_p(X)$ admits an algebra quotient isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

Moreover, if one (and thus each) of (1-7) *holds, then $C_u^b(X)$ admits quotients isomorphic to ℓ^∞ and to ℓ^2.*

Proof. In [14] we showed that (1) ⇔ (2), and since the topology on $\mathbb{R}^{\mathbb{N}}$ is minimal, (2) ⇒ (3). In every lcs, each copy of $\mathbb{R}^{\mathbb{N}}$ is complemented [19, 2.6.5(iii)], so (2) ⇒ (4) and (3) ⇒ (5).

Now let M be any closed subspace of $C_c(X)$ [resp., of $C_p(X)$]. If X is pseudocompact, then M is a closed subspace of $C_u^b(X)$. The topology of the Banach space $C_u^b(X)/M$ is finer than that of $C_c(X)/M$ [resp., of $C_p(X)/M$], so, by the open mapping theorem, the latter cannot be the non-Banach Fréchet space $\mathbb{R}^{\mathbb{N}}$. This shows (the contrapositive of) (4) $\Rightarrow$ (1) [resp., (5) $\Rightarrow$ (1)]. Thus (1-5) are equivalent.

[(1) $\Rightarrow$ (6),(7)]. By definition, the non-pseudocompact X admits a disjoint sequence $\{U_n\}_n$ of non-empty open sets that is locally finite; *i.e.*, each point in X has a neighborhood which meets only finitely many of the U_n. Choose x_n in U_n for each $n \in \mathbb{N}$ and define the linear map $T : C(X) \to \mathbb{R}^{\mathbb{N}}$ so that, for all $f \in C(X)$,

$$T(f) = (f(x_n))_n .$$

T is continuous on $C_p(X)$, and thus also on $C_c(X)$. By Lemma 4, for each n there exists $f_n \in [X, 1]$ with $f_n(x_n) = 1$ and $f_n(X \backslash U_n) = \{0\}$. For an arbitrary scalar sequence $(a_n)_n$, the pointwise sum $\sum_n a_n f_n$ is in $C(X)$ by local finiteness. If K is compact in X, it is countably compact and meets U_k only for those k in some finite set $\sigma \subset \mathbb{N}$. If $\varepsilon > 0$ and

$$W = \left\{(a_n)_n \in \mathbb{R}^{\mathbb{N}} : |a_k| \leq \varepsilon \text{ for each } k \in \sigma\right\},$$

then for each $(a_n)_n \in W$ we have $\sum_n a_n f_n \in [K, \varepsilon]$ with $T(\sum_n a_n f_n) = (a_n)_n$. Hence $T([K, \varepsilon]) \supset W$, so that T is an open map both from $C_c(X)$ and $C_p(X)$, and $T^{-1}(0) = \Im_A$, where $A = \{x_1, x_2, \dots\}$ is obviously closed. Thus (6) and (7) hold.

Trivially, (7) and (6) imply (5) and (4), respectively, completing the proof that (1-7) are equivalent.

The $C_u^b(X)$ case remains. If (1-7) hold, then T exists as above, and the restriction $T|_{C^b(X)}$ is clearly continuous and open from $C_u^b(X)$ onto the Banach space ℓ^∞. Thus ℓ^∞ is a quotient of $C_u^b(X)$, as is ℓ^2 by Corollary 9 and transitivity. $\square$

In [15] we proved that $C_c(X)$ contains a copy of a dense subspace of $\mathbb{R}^{\mathbb{N}}$ if and only if X is not Warner bounded. (X is *Warner bounded* if for every disjoint sequence $(U_n)_n$ of non-empty open sets in X there exists a compact $K \subset X$ such that $U_n \cap K \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.)

Lemma 15. *Let A be a closed infinite subset of X. Then $C_p(X)/\Im_A$ is isomorphic to a dense subspace of $C_p(A)$, itself a dense subspace of the product space $\mathbb{R}^A$. If A is also compact, then $C_c(X)/\Im_A$ is isomorphic to the Banach space $C_c(A)$.*

Proof. Let q denote the quotient map. One may use Lemma 4 to see that: (i) in both cases, the map $q(f) \mapsto f|_A$ is an isomorphism from the quotient onto its image in $C(A)$; (ii) the image is a dense subspace of $\mathbb{R}^A$ since some f in $C(X)$ achieves arbitrarily prescribed values on any given finite subset of A; (iii) the map is onto $C(A)$ when A is compact. $\square$

Rosenthal's Banach algebra quotient (Case 1) generalizes, with converse:

Theorem 16. *Statements* (1-5) *of Lemma* 5 *are equivalent to the next four.*

(6) $C_c(X)$ *admits an algebra quotient isomorphic to c.*
(7) $C_c(X)$ *admits a separable Banach algebra quotient.*
(8) $C_c(A)$ *is isomorphic to c for some* $A \subset X$.
(9) $C_c(A)$ *is a separable Banach space for some* $A \subset X$.

Proof. If A is closed in X and $C_c(X)/\Im_A$ is normable, then for some compact K the quotient map q takes $[K, 1]$ into a bounded set. If we suppose that $K \not\supset A$, then Lemma 4 provides $f \in \Im_K \setminus \Im_A$. But then the span $\pounds$ of f is in $[K, 1]$ and

$q(f) \neq 0$, so the unbounded line $q(\pounds)$ is in $q([K,1])$, a contradiction. Therefore K must contain the closed set A, and A must be compact. We combine this with Lemma 15 to see that (6) $\Leftrightarrow$ (8) and (7) $\Leftrightarrow$ (9).

If A consists of a non-trivial convergent sequence and its limit, then it is clear from Case 1 that $C_c(A) \approx c$; *i.e.*, (1) $\Rightarrow$ (8). Trivially, (8) $\Rightarrow$ (9).

Finally, the Krein-Krein criterion [17] merely says (9) $\Leftrightarrow$ (5). $\square$

Lemma 17. *If X has no closed denumerable sets, then $C_p(X)$ is not separable.*

Proof. Let $f_1, f_2, \dots \in C(X)$ be arbitrary. We desire $y_1 \neq y_2$ in X with

$$|f_n(y_1) - f_n(y_2)| \leq 1$$

for all $n \in \mathbb{N}$. By hypothesis, every denumerable set has more than one cluster point in X. Fix a cluster point y_1 in X. Continuity allows us to choose a strictly decreasing sequence of closed neighborhoods V_n of y_1 so that each $f_n(V_n)$ has diameter no larger than 1. We choose $x_n \in V_n \backslash V_{n+1}$ and let y_2 be a cluster point of $\{x_n\}_n$ distinct from y_1. Since all but finitely many of the x_k are in a given V_n, this closed set contains the cluster point y_2. Indeed, then, the displayed inequality holds for each n.

Lemma 4 provides $h \in C(X)$ with $h(y_1) = 5$ and $h(y_2) = 9$. If we assume some $f_n \in h + [\{y_1, y_2\}, 1]$, we have $|f_n(y_1) - f_n(y_2)| \geq (9-5) - 1 - 1 = 2$, a contradiction. Thus the arbitrary sequence is not dense in $C_p(X)$. $\square$

Theorem 18. *The following three statements are equivalent.*

(1) *X admits a closed denumerable set D.*
(2) *$C_c(X)$ admits a separable algebra quotient.*
(3) *$C_p(X)$ admits a separable algebra quotient.*

Proof. [(1) $\Rightarrow$ (2)]. If D admits a compact infinite subset, the previous theorem ensures c is a (separable) algebra quotient of $C_c(X)$. If D has no such subset, we may assume D has no cluster points (Lemma 5), so that $C_c(X)/\Im_D = C_p(X)/\Im_D$ (Lemma 4), which is isomorphic to a dense subspace of the metrizable separable $\mathbb{R}^D$ by Lemma 15. Hence the algebra quotient is separable.

[(2) $\Rightarrow$ (3)]. If $C_c(X)/\Im_A$ is separable, so is $C_p(X)/\Im_A$.

[(3) $\Rightarrow$ (1)]. If A is closed in X and $C_p(X)/\Im_A$ is separable, then so is $C_p(A)$ by Lemma 15. Since A is infinite, (the contrapositive of) Lemma 17 shows A has a closed denumerable subset D. Thus D is closed in X, and (1) holds. $\square$

Thus $C_c(X)$ and $C_p(X)$ have separable algebra quotients if X has an infinite closed subset that is metrizable, *e.g.*, if X is a tvs. Since $\beta\mathbb{N}$ lacks a closed denumerable set, $C_c(\beta\mathbb{N})$ and $C_p(\beta\mathbb{N})$ lack separable algebra quotients, although $C_c(\beta\mathbb{N})$ contains a copy of c, as do all Banach spaces of the form $C_c(X)$.

If countable intersections of open sets are open, X is called a *P-space*; then denumerable sets are closed, not compact, so one may apply Theorem 18, not 16:

Corollary 19. *If X is a P-space, then $C_c(X)$ and $C_p(X)$ admit separable algebra quotients.*

Corollary 20. *Both $C_c(X)$ and $C_p(X)$ have properly separable quotients when $L_m(X)$ does not.*

Proof. By hypothesis, dense subspaces of $L_m(X)$ are primitive [26], including $L_m(X)$. Therefore X is a P-space [9, 10] and the previous corollary applies. $\square$

X is *of pointwise countable type* (Arkhangel'skii) if every point in X is in some compact set K for which there exists a sequence of open sets U_n in X with the properties that (i) each U_n contains K and (ii) some U_n is contained in U whenever U is an open set containing K. Obviously, X is of pointwise countable type if X is first countable, and conversely when every compact set K is finite. Only the most extreme P-spaces are of pointwise countable type. Indeed,

Theorem 21. *Assume X is of pointwise countable type. The following five statements are equivalent.*

(1) *X is discrete.*
(2) *X is a P-space.*
(3) *No compact set in X is infinite.*
(4) *No compact set in X is denumerable, and X is first countable.*
(5) $C_c(X) = C_p(X) = \mathbb{R}^X$.

Proof. Trivially, (1) $\Rightarrow$ (2). Suppose (2) holds. Then every denumerable set in X is closed and not compact. Therefore there are no denumerable subsets of compact sets, thus no infinite compact sets in X; i.e., (2) $\Rightarrow$ (3). Since X is of pointwise countable type, (3) $\Rightarrow$ (4).

[(4) $\Rightarrow$ (1)]. Suppose (4) holds and not (1). Then there is some $x_0 \in X$ such that $\{x_0\}$ is not open in X. First countability posits a countable base $\{V_n\}_n$ of open neighborhoods of x_0. We may assume each $V_n \supset V_{n+1}$ and inductively choose distinct points $x_1, x_2, \dots$ with each $x_n \in V_n$. Clearly, this sequence converges to x_0, and $\{x_0, x_1, x_2, \dots\}$ is a denumerable compact set in X, a contradiction of (4); the desired implication follows.

We now have (1)-(4) are equivalent. Since [(1) $\Rightarrow$ (5)] and [(5) $\Rightarrow$ (3)] are obvious, the proof is complete. □

Corollary 22. *If X is of pointwise countable type, then $C_c(X)$ has a quotient isomorphic to either $\mathbb{R}^{\mathbb{N}}$, c, or ℓ^2.*

Proof. Clearly, $\mathbb{R}^{\mathbb{N}}$ is a quotient of $\mathbb{R}^X$. If $C_c(X) \neq \mathbb{R}^X$, then X contains an infinite compact set Y, and Corollary 10 applies. □

The weak and strong duals of $C_c(X)$ have separable quotients [16], but not always *properly* separable quotients (*e.g.*, when X is discrete). We re-examine the dual of $C_p(X_\kappa)$, adding new analytic P-space characterizations for the special Tichonov spaces X_κ defined below (see [2, 9, 10]).

If $\{f_n\}_n$ is a sequence in the dual E' of an lcs E, the *eventually zero subspace* $\mathrm{ez}\{f_n\}_n$ is that subspace of E defined by writing

$$ez\{f_n\}_n = \{x \in E : f_n(x) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

Two facts are useful, now and later:

(*) [29, Theorem 3.11(a)] *Let F be a dense primitive subspace of an lcs E. Every subspace between F and E is primitive if and only if $\mathrm{ez}\{f_n\}_n = E$ whenever $\{f_n\}_n \subset E'$ and $\mathrm{ez}\{f_n\}_n \supset F$.*

(**) [26, Theorem 1(iii)] *E has a properly separable quotient if and only if $ez\{f_n\}_n$ is dense and proper in E for some $\{f_n\}_n$ in E'.*

If κ is an infinite cardinal, let X_κ denote the closed interval $[0, \kappa]$ of ordinals with the topology whose open sets are precisely those which either omit κ or contain the closed interval $[\alpha, \kappa]$ for some ordinal $\alpha < \kappa$. Certainly, X_κ is an infinite completely regular Hausdorff space. The *cofinality* of κ, denoted $\mathrm{cof}(\kappa)$, is the least cardinality of the cofinal subsets of the well-ordered interval $[0,\kappa)$.

By Theorem 18, all $C_p(X_\kappa)$ and $C_c(X_\kappa)$ admit separable algebra quotients. Easily, [X_κ is a P-space] $\Leftrightarrow$ [$\mathrm{cof}(\kappa) \neq \aleph_0$] $\Leftrightarrow$ [X_κ has no infinite compact set] $\Leftrightarrow$ [X_κ has no denumerable compact set], and a real-valued funtion h on X_κ is continuous if it is continuous at κ. If $\mathrm{cof}(\kappa) \neq \aleph_0$, then h is continuous if and only if h is constant on the closed interval $[\alpha,\kappa]$ for some $\alpha < \kappa$.

Theorem 23. *The following six assertions are equivalent.*

(1) *X_κ is a P-space.*
(2) $\mathrm{cof}(\kappa) \neq \aleph_0$
(3) *$L_m(X_\kappa)$ is primitive.*
(4) *Every dense subspace of $L_m(X_\kappa)$ is primitive.*
(5) *Every dense subspace of $L_m(X_\kappa)$ is inductive.*
(6) *No quotient of $L_m(X_\kappa)$ is properly separable.*

Proof. Obviously, (1) $\Leftrightarrow$ (2), and (1) $\Leftrightarrow$ (3) by [9, Theorem 6]. Always, *inductive* $\Rightarrow$ *primitive*, and for Mackey spaces, *primitive* $\Leftrightarrow$ *inductive* [29, box 4 of Fig. 3], so Theorem 3.12 of [29] yields (4) $\Leftrightarrow$ (5). And [(4) $\Leftrightarrow$ (6)] is a part of [26, Theorem 1]. Trivially, (4) $\Rightarrow$ (3).

We are left to prove [(2) $\Leftrightarrow$ (6)]. In the usual manner, we identify X_κ with a Hamel basis for $L(X_\kappa)$ and $C(X_\kappa)$ with the dual of $L_m(X_\kappa)$. Let $\{f_n\}_n \subset C(X_\kappa)$ be given with eventually zero subspace $Z = \mathrm{ez}\{f_n\}_n$ in $L_m(X_\kappa)$. Now (2) implies that for each $n \in \mathbb{N}$ there is some $\alpha_n < \kappa$ such that f_n is constant on $[\alpha_n,\kappa]$, and $\sup\{\alpha_n\}_n = \alpha < \kappa$. Thus each f_n is constant on the infinite closed interval $[\alpha,\kappa]$. This ensures the vector $\beta - \alpha$ is in Z for every $\beta \in [\alpha,\kappa]$. If h is a linear form on $L_m(X_\kappa)$ that vanishes on Z, then h is constant on $[\alpha,\kappa]$, hence continuous on X_κ, thus also on $L_m(X_\kappa)$. Therefore the kernel $h^\perp$ is closed, which implies Z is closed, as well, and cannot be both dense and proper in $L_m(X_\kappa)$. By (**), then, (6) holds. □

Theorem 24. *X_κ is a P-space if and only if the weak dual $C_c(X_\kappa)'_\sigma$ of $C_c(X_\kappa)$ has no properly separable quotient.*

Proof. Always, $L_p(X)$ is a dense subspace of $C_c(X)'_\sigma$, so the latter has a properly separable quotient if the former does. Thus by Theorem 23 and duality invariance, $C_c(X_\kappa)'_\sigma$ has a properly separable quotient if X_κ is *not* a P-space.

Conversely, if X_κ *is* a P-space, then all compact sets in X_κ are finite, so that $C_c(X_\kappa)'_\sigma = L_p(X)_\kappa$ has no properly separable quotient, again by Theorem 23 and duality invariance. □

Precisely the X_κ that are P-spaces provide a wealth of simple lcs's $L_m(X_\kappa)$ to which the proof of Example 3 applies:

Example 25. Suppose X_κ is a P-space; equivalently, every dense subspace of $L_m(X_\kappa)$ is primitive. The S_σ space $L_m(X_\kappa)$ [9] dominates a dense hyperplane H of some non-primitive lcs (E,τ) with $(H,\tau)' = L_m(X_\kappa)'$ [29, Theorem 3.2]. The non-primitive E must admit a properly separable quotient, but, via duality invariance, its hyperplane H cannot (see first paragraph of Section 2).

Cofinality is similarly crucial in [25]. If E is a linear space with infinite Hamel basis B of size $|B|$, define the subspace $E^{B,|B|}$ of the algebraic dual E^* by writing

$$E^{B,|B|} = \{f \in E^* : |\{x \in B : f(x) \neq 0\}| < |B|\}.$$

The lcs that E becomes under the Mackey topology $\mu\left(E, E^{B,|B|}\right)$, denoted E_B, is never $\aleph_0$-barrelled and has dense subspaces of codimension $|B|$, the largest possible [25, Theorems 2, 3]. Moreover,

Theorem 26. *The following seven statements are equivalent.*

(1) $\operatorname{cof}(|B|) \neq \aleph_0$.
(2) E_B *is primitive.*
(3) [*Every*] ⟨*Some*⟩ *dense hyperplane in* E_B *is non-Mackey.*
(4) *Every dense proper subspace in* E_B *is non-Mackey.*
(5) *Every dense subspace in* E_B *is primitive.*
(6) *Every dense subspace in* E_B *is inductive.*
(7) E_B *does not admit a properly separable quotient.*

Proof. By [25, Theorems 2, 3], (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). In particular, the [*Every*] and ⟨*Some*⟩ versions of (3) are equivalent.

[(3) $\Leftrightarrow$ (4)]. If F is a dense proper subspace in E_B, there is a hyperplane H in E_B with $F \subset H$; if F is also Mackey, so is H, routinely, which contrapositively proves [(3) $\Rightarrow$ (4)]. The converse is trivial.

[(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7)]. Suppose (5) holds. Then the primitive Mackey space E_B is inductive [29], which, combined with (5), implies (6) [29, Theorem 3.12]. The converse is obvious, so (5) $\Leftrightarrow$ (6). Theorem 1(ii) of [26] equates (5) and (7).

[(5) $\Rightarrow$ (2)]. Trivially. To complete the proof, we show:

[(1) $\Rightarrow$ (7)]. Given any $\{f_n\}_n \subset E'_B$,

$$B_0 = \{x \in B : f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}$$

is a countable union of sets of size $< |B|$, so if (1) holds, then $|B_0| < |B|$ and all superspaces of $\operatorname{sp}(B \backslash B_0)$ are closed by definition of E'_B. In particular, $\operatorname{ez}\{f_n\}_n$ is closed and cannot be dense and proper in E_B; thus (7) holds via (**). ☐

Since each E_B is S_σ, the process of Examples 3 and 25 applies to precisely those $E_B = H$ with $\operatorname{cof}(|B|) \neq \aleph_0$. Conversely, no other process avails, since

Theorem 27. *If an lcs E admits a properly separable quotient and a dense hyperplane H does not, then H is S_σ and E is not primitive.*

Proof. Theorem 1 implies H has a separable quotient Q. Thus H must be S_σ, since Q would otherwise be properly separable, contrary to hypothesis.

Assume E is primitive. Let $\{f_n\}_n \subset E'$ with $Z = \operatorname{ez}\{f_n\}_n$ dense in E. Let $\overline{H \cap Z}^H$ and $\overline{H \cap Z}^E$ denote the closure of $H \cap Z$ in H and E, respectively. Clearly,

$$\operatorname{codim}_H\left(\overline{H \cap Z}^H\right) \leq \operatorname{codim}_E\left(\overline{H \cap Z}^E\right).$$

If $Z \subset H$, then by density both codimensions are null. If there exists $x \in Z \backslash H$, then $Z = H \cap Z + \operatorname{sp} x$, and $E = \overline{Z} = \overline{H \cap Z}^E + \operatorname{sp} x$: Both codimensions are ≤ 1. In every case, then,

$$\operatorname{codim}_H\left(\overline{H \cap Z}^H\right) \leq \operatorname{codim}_E\left(\overline{H \cap Z}^E\right) \leq 1.$$

By hypothesis, every dense subspace of H is primitive. If G is a subspace between $H \cap Z$ and $\overline{H \cap Z}^H$, it has codimension ≤ 1 in a dense, hence primitive, subspace of H, and therefore G itself is primitive [29, Theorem 2.9]. Application of (*) to the primitive subspace $H \cap Z$, dense in $\overline{H \cap Z}^H$, yields the fact that $Z = \operatorname{ez}\{f_n\} \supset \overline{H \cap Z}^H$. Consequently, $\overline{H \cap Z}^H = H \cap Z$ and

$$\operatorname{codim}_E(Z) \leq \operatorname{codim}_H(H \cap Z) + \operatorname{codim}_E(H) = \operatorname{codim}_H\left(\overline{H \cap Z}^H\right) + 1 \leq 2.$$

Therefore every subspace between the dense Z and the primitive E has codimension ≤ 2 and is, itself, primitive. Now (*) implies $Z = \operatorname{ez}\{f_n\}_n = E$. Hence there is no $\{f_n\}_n \subset E'$ with $\operatorname{ez}\{f_n\}_n$ dense and proper in E; i.e., E has no properly separable quotient, a contradiction of the hypothesis. We must conclude the assumption is false; E is not primitive. □

4. Remaining questions

Despite the strong dual solution [1], the Banach problem remains. Analogs implicate P-spaces and weak barrelledness. We have clear answers as to when

- the strong and weak duals of $C_c(X)$ have separable quotients (always [16])
- the weak dual of $C_c(X_\kappa)$ has a properly separable quotient (when X_κ is not a P-space, Theorem 24)
- the weak dual of $C_p(X_\kappa)$ has a separable quotient (always [9]) or a properly separable quotient (when X_κ is not a P-space, Theorem 23)
- proper (LF)-spaces have separable quotients (always [28]) or properly separable quotients (almost always [26])
- non-normable Fréchet spaces have separable quotients (always [6, Satz 2])
- GM-spaces have properly separable quotients (never) or separable quotients (when they are S_σ [16])
- $C_c(X)$, $C_p(X)$ have separable algebra quotients (Theorem 18)
- barrelled $C_c(X)$, $C_p(X)$ have separable quotients (always [16])

Q1. Must arbitrary $C_c(X)$ have separable quotients? (Rosenthal and Theorem 18 leave only the case where X is countably compact and not compact.)

Q2. If $C_c(X)$ has separable quotients, must $C_p(X)$? (See Corollary 11.)

Q3. If X is compact, must $C_p(X)$ have separable quotients?

So far, only certain GM-spaces have been shown to lack separable quotients [16]. Are there Schwartz spaces or nuclear spaces, for example, that lack separable quotients? Our answer is positive, albeit pedestrian: Any non-trivial variety $\mathcal{V}$ of lcs's, the nuclear and Schwartz varieties included, must contain the smallest non-trivial variety $\mathcal{W}$ of all lcs's having their weak topology [3]. Let E be a non-S_σ GM-space. Then $(E, \sigma(E, E'))$ is in $\mathcal{W} \subset \mathcal{V}$ and does not admit separable quotients.

Q4. Does some Schwartz or nuclear space not in $\mathcal{W}$ lack separable quotients? (Both varieties have separable universal generators not in $\mathcal{W}$ [12].)

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