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ON THE ELLIPTIC HARNACK INEQUALITY

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To the memory of Ennio De Giorgi on the 20th anniversary of his passing

Abstract. A brief exposition on some tools for proving the elliptic Harnack inequality is presented.

1. Introduction

Fix $0 < \lambda \leq \Lambda$ and let $A(x) = \{a_{ij}(x)\}_{i,j=1}^n$ be a measurable uniformly elliptic matrix, with constants λ and Λ, in a domain $\Omega \subset \mathbb{R}^n$. Let $\mathcal{B}_\Omega$ denote the collection of Euclidean balls $B = B_r(x)$ such that $10B := B_{10r}(x) \Subset \Omega$.

Jürgen Moser's celebrated Harnack inequality for non-negative (weak) solutions $u \in W^{1,2}(\Omega)$ of $\operatorname{div}(A\nabla u) = 0$ in Ω establishes the existence of a constant $C_H > 0$, depending only on λ, Λ, and n, such that

$$\sup_B u \leq C_H \inf_B u \quad \forall B \in \mathcal{B}_\Omega. \tag{1.1}$$

In a recent article [24], the authors presented a direct proof of the Harnack inequality (1.1) based on the following two results.

Theorem 1. *There exists a constant $C_2 > 0$, depending only on λ, Λ, and n, such that every non-negative sub-solution u satisfies*

$$\sup_{\frac{1}{2}B} u \leq C_2 \left(\frac{1}{|B|} \int_B u^2 \, dx \right)^{\frac{1}{2}} \quad \forall B \in \mathcal{B}_\Omega. \tag{1.2}$$

Theorem 2 (Compare to [26, Theorem 2])**.** *For every $\delta \in (0,1)$ there exists $M > 0$, depending only on δ, λ, Λ, and n, such that for every non-negative super-solution u and every ball $B \in \mathcal{B}_\Omega$ the following implication holds true:*

$$|\{x \in B : u(x) > M\}| \geq \delta |B| \quad \Rightarrow \quad \inf_{3B} u \geq 1. \tag{1.3}$$

Lest there might be a misunderstanding of the current state of affairs related to such a direct proof, a brief clarification appears to be appropriate. That is the sole purpose of this note.

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2. A two-step method towards the Harnack inequality

Direct proofs of the Harnack inequality (1.1) from the inequality (1.2) and implications like (1.3) have appeared multiple times in diverse PDE contexts to the extent of having become standard and even axiomatized in the context of metric spaces admitting a doubling measure.

The first step in this usual procedure, as followed in [24], is to show that the weak reverse-Hölder inequality (1.2) self-improves to admit every exponent $p \in (0, 2)$ on its right-hand side (and with a new constant C_p, depending only on C_2, p, and n).

The second step is to prove that an implication like (1.3) yields constants $N > 1$ and $\varrho \in (0, 1)$ (depending only on δ, M, and n), such that given a ball $B \in \mathcal{B}_\Omega$ with $\inf_{3B} u \leq 1$ the distribution function of u in B satisfies the following power-like decay estimate:

$$|\{x \in B : u(x) > N^k\}| \leq \varrho^k |B| \quad \forall k \in \mathbb{N}. \tag{2.4}$$

From the decay estimate (2.4) one immediately gets the existence of $p_0 > 0$ and $C_0 > 0$ (depending only on ϱ, N, and n) such that non-negative super-solutions satisfy the weak Harnack inequality

$$\left(\frac{1}{|B|} \int_B u^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq C_0 \inf_B u \quad \forall B \in \mathcal{B}_\Omega. \tag{2.5}$$

Thus, the self-improved version of (1.2) chosen with exponent p_0 together with the weak Harnack inequality (2.5) proves (1.1).

3. Critical-density estimates and the Harnack inequality

Implications of the type (1.3), as well as weaker versions where instead of holding for *every* $\delta \in (0, 1)$ the implication only holds for *some* $\delta \in (0, 1)$, are known in the literature as "critical-density properties" or "measure-to-point estimates" (or "point-to-measure estimates" when expressed as the contrapositive), and they lie at the core of the regularity theory for elliptic PDEs (with corresponding versions in the parabolic setting as well).

Instances of the presence of measure-to-point estimates in the study of a number of linear or nonlinear, degenerate or singular, homogeneous or non-homogeneous, elliptic or elliptic-type PDE problems, all having as a common feature the property that positive multiples of (sub-, super-) solutions are also (sub-, super-) solutions, include the following:

(i) Lemmas 2.1 and 2.2 from [28], Theorems 2.1.1 and 2.1.2 from [13] in the context of uniformly elliptic PDEs in non-divergence form;
(ii) Lemma 3.4 from [10] in the context of the De Giorgi classes of order p (which includes the p-Laplacian for $1 < p < \infty$);
(iii) Lemma 10.1 from [6] in the context of fully nonlinear integro-differential PDEs, respectively;
(iv) Theorems 1 and 2 from [5], Lemma 4.1 and Proposition 4.1 from [23] in the context of the linearized Monge-Ampère equation, and Theorem 12.2 from [25] for its fractional non-local counterpart;
(v) Lemma 7 from [16] in the context of degenerate/singular fully nonlinear PDEs;
(vi) Lemma 5.1 from [2], Lemma 3.1 from [18], and Proposition 4.1 from [32] in the context of non-divergence elliptic equations on Riemannian manifolds;

(vii) Theorem 2 from [31] and Lemma 3.3 from [12] in the contexts of quasi-linear elliptic PDEs and adjoints of uniformly elliptic operators in non-divergence form, respectively;
(viii) Lemmas 6.1 and 6.2 from [19] in the context of regularity of quasi-minimizers on metric spaces;
(ix) Theorems 3.3 and 4.1 from [14] and Theorem 2.4 from [30] in the context of degenerate operators in Heisenberg and H-type groups.

More geometric versions of the critical-density property can be found for instance in [29, Lemma 2.2] and [9, Lemma 3.1]. The previous list is by no means comprehensive and it is intended only to represent a sample of older and more recent results. Parabolic counterparts are plentiful as well.

It is by now well understood that, when combined with a covering lemma, measure-to-point estimates of the type (1.3) imply a weak Harnack inequality. For example, in the contexts listed above the types of covering lemmas include: Calderón-Zygmund ([2, 6, 18]), Vitali ([29]), Besicovitch ([5]), and the "ink spot crawling lemma" ([19, 23, 25, 28, 31]).

The bottom line is that, whether a measure-to-point estimate like (1.3) is proved in the context of fully nonlinear elliptic PDEs (e.g. Lemma 5 from [3], Lemma 4.5 from [4]) or in the context of divergence-form elliptic PDEs (from which [24] cites (1.3)) or in any of the contexts above, *it is the measure-to-point estimates themselves that imply, via a covering lemma, the weak Harnack inequality* (2.5).

This approach to a weak Harnack inequality through measure-to-point estimates stems from the work of the 1970's-80's Russian school; see [20–22, 27, 28] ([28] is the English translation of [27]), where implications of the type (1.3) are usually referred to as *growth lemmas*, which, in turn, stem from [22] in the context of elliptic and parabolic PDEs in both divergence and non-divergence forms.

4. An axiomatic approach in metric spaces

The proof that a critical-density or measure-to-point estimate like (1.3) implies, via a covering lemma, the weak Harnack property (2.5) has become standard and it has been axiomatized within the context of metric spaces as follows. Let d be a distance and μ be a Borel measure on a set X satisfying the following *doubling property*: there exists a positive constant $C_d > 1$ such that

$$0 < \mu(B_{2r}(x)) \le C_d\, \mu(B_r(x)) < \infty, \quad \forall x \in X, r > 0, \tag{4.6}$$

where $B_r(x) := \{y \in X : d(x, y) < r\}$.

Definition 1. Fix an open set $\Omega \subset X$. Following [1, 3, 4, 11, 17], let $\mathbb{K}_\Omega$ denote a family of μ-measurable, locally bounded, non-negative functions defined in Ω such that $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, that is, $\tau u \in \mathbb{K}_\Omega$ whenever $u \in K_\Omega$ and $\tau > 0$. (Think of $\mathbb{K}_\Omega$ as a class of non-negative (sub-, super-) solutions.)

Given $M \ge 1$ and $\varepsilon \in (0, 1)$, $\mathbb{K}_\Omega$ is said to possess the *critical-density property* with constants M and ε if for every ball $B_{2R}(x_0) \Subset \Omega$ and for every $u \in \mathbb{K}_\Omega$ the following implication holds true:

$$\mu(\{x \in B_R(x_0) \mid u(x) \ge M\}) \ge \varepsilon\mu(B_R(x_0)) \quad \Rightarrow \quad \inf_{B_{R/2}(x_0)} u > 1. \tag{4.7}$$

Given $\gamma \in (0,1)$, $\mathbb{K}_\Omega$ is said to possess the *double-ball property* with constant γ if for every ball $B_{2R}(x_0) \Subset \Omega$ and for every $u \in \mathbb{K}_\Omega$ the following implication holds true:

$$\inf_{B_{R/2}(x_0)} u \geq 1 \quad \Rightarrow \quad \inf_{B_R(x_0)} u \geq \gamma. \tag{4.8}$$

Given $\varrho \in (0,1)$ and $N > 1$, $\mathbb{K}_\Omega$ is said to possess the *power-like decay property* with constants N and ϱ if for every ball $B_{2R}(x_0) \Subset \Omega$ and every $u \in \mathbb{K}_\Omega$ with

$$\inf_{B_R(x_0)} u \leq 1,$$

it follows that

$$\mu(\{x \in B_{R/2}(x_0) : u(x) > N^k\}) \leq \varrho^k \mu(B_{R/2}(x_0)), \quad \forall k \in \mathbb{N}. \tag{4.9}$$

Given $C'_H > 1$ and $\beta > 0$, $\mathbb{K}_\Omega$ is said to possess the *weak Harnack property* with constants C'_H and β if for every ball $B_{2R}(x_0) \Subset \Omega$ and for every $u \in \mathbb{K}_\Omega$,

$$\left(\frac{1}{\mu(B_R(x_0))} \int_{B_R(x_0)} u^\beta d\mu \right)^{\frac{1}{\beta}} \leq C'_H \inf_{B_R(x_0)} u. \tag{4.10}$$

Given $C_H \geq 1$, $\mathbb{K}_\Omega$ is said to possess the *Harnack property* with constant C_H if for every $u \in \mathbb{K}_\Omega$ it follows that

$$\sup_B u \leq C_H \inf_B u \quad \forall B \in \mathcal{B}_\Omega.$$

Under different sets of assumptions on the doubling metric space (X, d, μ) (e.g. ring condition, non-empty annuli, unboundedness, segment properties, etc.) the axiomatic approaches developed in [1, 11, 17] yield the following result.

Theorem 3 (See [1, Theorem 3.1(d)], [11, Theorem 4.7], [17, Theorem 7])**.** *Suppose that $\mathbb{K}_\Omega$ possesses the critical-density property* (4.7) *with some constants $M > 1$ and $\varepsilon \in (0,1)$ and the doubling-ball property* (4.8) *with some constant $\gamma \in (0,1)$.*

Then, $\mathbb{K}_\Omega$ also possesses the power-like decay property (4.9) *with some constants $N > 1$ and $\varrho \in (0,1)$ depending only on ε, γ, M, and C_d.*

Equivalently, $\mathbb{K}_\Omega$ possesses the weak Harnack property (4.10) *with constants C'_H and β depending only on ε, γ, M, and C_d.*

Notice that by combining the critical-density estimate (4.7) with iterations of the double-ball property (4.8) one gets, for instance, that every $u \in \mathbb{K}_\Omega$ satisfies

$$\mu(\{x \in B_R(x_0) \mid u(x) \geq M\}) \geq \varepsilon \mu(B_R(x_0)) \quad \Rightarrow \quad \inf_{B_{3R}(x_0)} u > \gamma^3. \tag{4.11}$$

In the context of elliptic and parabolic PDEs, the implication (4.11) is also known as an *expansion of positivity* (see for instance [9, Chapter 4]).

The key to the implication (4.11) is that the ball on the left-hand side of the implication is smaller than the one on the right-hand side (think of a Calderón-Zygmund cube and its predecessor). Then, a covering lemma will allow for the application of (4.11) at every scale in order to produce the weak Harnack inequality (4.10).

As it turns out, if the critical-density estimate (4.7) is sensitive enough, then it will imply the double-ball property (4.8). More precisely, we have

Theorem 4 (See [11, Proposition 4.3]). *If $\mathbb{K}_\Omega$ possesses the critical-density property with some constants $M > 1$ and $\varepsilon \in (0,1)$ with $\varepsilon < C_d^{-2}$, where $C_d > 0$ is the doubling constant from* (4.6)*, then $\mathbb{K}_\Omega$ possesses the doubling-ball property with some constant $\gamma \in (0,1)$ depending only on ε, M and C_d.*

This is why Theorem 2 from [26], which represents a critical-density estimate like (4.7) with an arbitrary constant $\varepsilon \in (0,1)$ (that constant is denoted as c_0^{-1} in [26, p. 463]), yields the implication (1.3). Notice that (1.3) is just a case of (4.11).

Thus, putting things together we deduce that a sensitive-enough critical-density estimate (4.7) (e.g. $0 < \varepsilon < C_d^{-2}$) implies (4.11), which, via a covering lemma, implies the weak Harnack inequality (4.10).

5. On the self-improving of weak reverse-Hölder inequalities

On the other hand, regarding the self-improving properties of the inequality (1.2), in a metric space with a doubling measure (X, d, μ) and a fixed open subset $\Omega \subset X$ we have the following: if there exist constants $C > 0$ and $p > 0$ such that a weight u satisfies the weak reverse-Hölder inequality

$$\sup_{\frac{1}{2}B} u \le C \left(\frac{1}{\mu(B)} \int_B u^p \, d\mu \right)^{\frac{1}{p}} \quad \forall B \in \mathcal{B}_\Omega, \tag{5.12}$$

then for every $q \in (0,p)$ there is a constant $C_q > 0$, depending only on C, p, and the doubling constant C_d in (4.6), such that

$$\sup_{\frac{1}{2}B} u \le C_q \left(\frac{1}{\mu(B)} \int_B u^q \, d\mu \right)^{\frac{1}{q}} \quad \forall B \in \mathcal{B}_\Omega. \tag{5.13}$$

This is a consequence of Young's inequality and a well-known real-analysis lemma (see for instance [15, Lemma 4.3], [19, Lemma 3.2]). Details can be found, for instance, in [19, Remarks 4.4].

6. Conclusions

Direct proofs of the implication (1.3) + (1.2) ⇒ (1.1) follow from the implications (5.12) ⇒ (5.13) and (1.3) ⇒ (2.5) (i.e., steps 1 and 2 from Section 2, respectively). In turn, these implications are by now well understood and even axiomatized in the context of metric spaces admitting doubling measures. Indeed, in such context the self-improving property (5.12) ⇒ (5.13) holds true by simple real analysis (see for instance Remarks 4.4 from [19]), and extensions of the implication (1.3) ⇒ (2.5) have been illustrated in Theorems 3 and 4.

Except for the additional convenience of considering the homogeneous PDE (that is, $f \equiv 0$) and the freedom to choose $\delta = 1/2$, the proof of (1.3) ⇒ (2.5) in [24] reads almost verbatim as the proof of Lemma 4.6 in [4, p. 33]. This is only natural since such a proof is by now standard; see for instance the proofs of Lemma 6 from [3] and Lemma 5.14 from [15]. In addition, notice that the critical-density estimate in Lemma 4.5 from [4] holds for *some* $\delta \in (0,1)$, which suffices for the proof of (1.3) ⇒ (2.5) in [4, Lemma 4.5].

Finally, as L. Caffarelli points out in [7, p. 44], the fact that De Giorgi's arguments from [8] can be modified to prove the Harnack inequality (1.1) has been known for a long time. For instance, true to the spirit of De Giorgi's approach in [8], such line of thought was successfully developed in great generality in [10] (by means of the critical-density property in [10, Lemma 3.4] and the "crawling ink spots lemma" [10, Lemma 3.5]), where the functions at hand are assumed to belong to De Giorgi's classes (which contain the solutions to a number of elliptic PDEs). These ideas have also been carried out in the context of doubling metric spaces admitting a first-order calculus (see for instance [19]).

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References

[1] H. Aimar, L. Forzani, and R. Toledano, *Hölder regularity of solutions of PDE's: a geometrical view*, Comm. Partial Differential Equations **26** (2001), no. 7-8, 1145–1173, DOI 10.1081/PDE-100106130. MR1855275

[2] Xavier Cabré, *Nondivergent elliptic equations on manifolds with nonnegative curvature*, Comm. Pure Appl. Math. **50** (1997), no. 7, 623–665, DOI 10.1002/(SICI)1097-0312(199707)50:7⟨623::AID-CPA2⟩3.3.CO;2-B. MR1447056

[3] Luis A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) **130** (1989), no. 1, 189–213, DOI 10.2307/1971480. MR1005611

[4] Luis A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR1351007

[5] Luis A. Caffarelli and Cristian E. Gutiérrez, *Properties of the solutions of the linearized Monge-Ampère equation*, Amer. J. Math. **119** (1997), no. 2, 423–465. MR1439555

[6] Luis Caffarelli and Luis Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), no. 5, 597–638, DOI 10.1002/cpa.20274. MR2494809

[7] Ennio De Giorgi, *Selected papers*, edited by Luigi Ambrosio, Gianni Dal Maso, Marco Forti, Mario Miranda and Sergio Spagnolo, Springer-Verlag, Berlin, 2006. MR2229237

[8] Ennio De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari* (Italian), Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR0093649

[9] E. Di Benedetto, U. Gianazza, and V. Vespri, Harnack's inequality for degenerate and singular parabolic equations, Springer Monographs in Mathematics, Springer, 2011.

[10] E. DiBenedetto and Neil S. Trudinger, *Harnack inequalities for quasiminima of variational integrals* (English, with French summary), Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 295–308. MR778976

[11] Giuseppe Di Fazio, Cristian E. Gutiérrez, and Ermanno Lanconelli, *Covering theorems, inequalities on metric spaces and applications to PDE's*, Math. Ann. **341** (2008), no. 2, 255–291, DOI 10.1007/s00208-007-0188-x. MR2385658

[12] E. B. Fabes and D. W. Stroock, *The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** (1984), no. 4, 997–1016, DOI 10.1215/S0012-7094-84-05145-7. MR771392

[13] Cristian E. Gutiérrez, *The Monge-Ampère equation*, Progress in Nonlinear Differential Equations and their Applications, 44, Birkhäuser Boston, Inc., Boston, MA, 2001. MR1829162

[14] Cristian E. Gutiérrez and Federico Tournier, *Harnack inequality for a degenerate elliptic equation*, Comm. Partial Differential Equations **36** (2011), no. 12, 2103–2116, DOI 10.1080/03605302.2011.618210. MR2852071

[15] Qing Han and Fanghua Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, vol. 1, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. MR1669352

[16] Cyril Imbert, *Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations*, J. Differential Equations **250** (2011), no. 3, 1553–1574, DOI 10.1016/j.jde.2010.07.005. MR2737217

[17] Sapto Indratno, Diego Maldonado, and Sharad Silwal, *On the axiomatic approach to Harnack's inequality in doubling quasi-metric spaces*, J. Differential Equations **254** (2013), no. 8, 3369–3394, DOI 10.1016/j.jde.2013.01.025. MR3020880

[18] Seick Kim, *Harnack inequality for nondivergent elliptic operators on Riemannian manifolds*, Pacific J. Math. **213** (2004), no. 2, 281–293, DOI 10.2140/pjm.2004.213.281. MR2036921

[19] Juha Kinnunen and Nageswari Shanmugalingam, *Regularity of quasi-minimizers on metric spaces*, Manuscripta Math. **105** (2001), no. 3, 401–423, DOI 10.1007/s002290100193. MR1856619

[20] N. V. Krylov and M. V. Safonov, *An estimate for the probability of a diffusion process hitting a set of positive measure* (Russian), Dokl. Akad. Nauk SSSR **245** (1979), no. 1, 18–20. MR525227

[21] N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 161–175, 239. MR563790

[22] E. M. Landis, *Second order equations of elliptic and parabolic type*, translated from the 1971 Russian original by Tamara Rozhkovskaya, with a preface by Nina Ural′tseva, Translations of Mathematical Monographs, vol. 171, American Mathematical Society, Providence, RI, 1998. MR1487894

[23] N. Q. Le, *On the Harnack inequality for degenerate and singular elliptic equations with unbounded lower order terms via sliding paraboloids*, Comm. Contemp. Math., to appear. Available at arXiv:1603.04763v2 [math.AP].

[24] Dongsheng Li and Kai Zhang, *A note on the Harnack inequality for elliptic equations in divergence form*, Proc. Amer. Math. Soc. **145** (2017), no. 1, 135–137, DOI 10.1090/proc/13174. MR3565366

[25] D. Maldonado and P. Stinga, *Harnack inequality for the fractional nonlocal linearized Monge-Ampère equation*, available at arXiv:1605.06165 [math.AP].

[26] Jürgen Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468. MR0170091

[27] M. V. Safonov, *Harnack's inequality for elliptic equations and Hölder property of their solutions* (Russian), Boundary value problems of mathematical physics and related questions in the theory of functions, 12, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 272–287, 312. MR579490

[28] M. Safonov, *Harnack's inequality for elliptic equations and the Hölder property of their solutions*, J. Math. Sci. **21** (1983), 851–863. DOI 10.1007/BF01094448.

[29] Ovidiu Savin, *Small perturbation solutions for elliptic equations*, Comm. Partial Differential Equations **32** (2007), no. 4-6, 557–578, DOI 10.1080/03605300500394405. MR2334822

[30] Giulio Tralli, *A certain critical density property for invariant Harnack inequalities in H-type groups*, J. Differential Equations **256** (2014), no. 2, 461–474, DOI 10.1016/j.jde.2013.09.008. MR3121702

[31] Neil S. Trudinger, *Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations*, Invent. Math. **61** (1980), no. 1, 67–79, DOI 10.1007/BF01389895. MR587334

[32] Yu Wang and Xiangwen Zhang, *An Alexandroff-Bakelman-Pucci estimate on Riemannian manifolds*, Adv. Math. **232** (2013), 499–512, DOI 10.1016/j.aim.2012.09.009. MR2989991

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