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FINITE GROUPS AND THEIR COPRIME AUTOMORPHISMS

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Abstract. Let p be a prime and A a finite group of exponent p acting by automorphisms on a finite p'-group G. Assume that A has order at least p^3 and $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. It is shown that G is nilpotent with class bounded solely in terms of c and p.

1. Introduction

Let a finite group A act by automorphisms on a group G. We denote by $C_G(A)$ the set $C_G(A) = \{g \in G \mid g^a = g \text{ for all } a \in A\}$, the centralizer of A in G (the fixed-point subgroup). In what follows we denote by $A^{\#}$ the set of non-trivial elements of A. The action of A on G is coprime if the groups A and G have coprime orders. It has been known for some time that centralizers of coprime automorphisms have strong influence on the structure of G. To exemplify this we cite the following results.

The first result is a celebrated theorem of Thompson [12].

1. If A has prime order and $C_G(A) = 1$, then G is nilpotent.

The next result is an easy consequence of the classification of finite simple groups [13].

2. If A is a group of automorphisms of G whose order is coprime to that of G and $C_G(A)$ is nilpotent or has odd order, then G is soluble.

3. If A is elementary abelian of rank at least 2 and if $C_G(a)$ is nilpotent for any $a \in A^{\#}$, then the group G is soluble.

Of course, this is immediate from the previous result. A classification-free proof of solubility of G can be found, for example, in [1, Corollary 3.5].

4. Ward showed that if A is an elementary abelian p-group of rank at least 3 and if $C_G(a)$ is nilpotent for any $a \in A^{\#}$, then the group G is nilpotent [14]. Later the second author showed that if, under these hypotheses, $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, then the group G is nilpotent with (c, p)-bounded nilpotency class [11].

Throughout the paper we use the expression "$(a, b, \dots)$-bounded" to abbreviate "bounded from above in terms of $a, b, \dots$ only". Many other results illustrating the influence of centralizers of automorphisms on the structure of G can be found in [7].

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The goal of the present article is to show that the above result can be strengthened as follows.

Theorem 1.1. *Let p be a prime and A a finite group of exponent p acting on a finite p'-group G. Assume that A has order at least p^3 and $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G is nilpotent with class bounded solely in terms of c and p.*

Thus, the novelty of the present paper is that we allow the acting group to be non-abelian. There are many results in the literature dealing with the situation where A is elementary abelian. Now one may wonder if some of those results can be extended to the action of non-abelian groups. For instance, it was shown in [5] that if A is an elementary abelian p-group of rank at least 3 and the commutator subgroup of $C_G(a)$ is small in some sense for all $a \in A^{\#}$, then the commutator subgroup G' must also be "small". It would be interesting to see if the results obtained in [5] can be extended to the case where A is non-abelian.

Throughout the paper we use without special references the well-known properties of coprime actions:

If A acts by automorphisms on a finite group G and $(|A|,|G|)=1$, then $C_{G/N}(A) = C_G(A)N/N$ for any A-invariant normal subgroup N.

If A is a non-cyclic abelian group acting coprimely on a finite group G, then G is generated by the subgroups $C_G(B)$, where A/B is cyclic.

2. Bounding nilpotency class of Lie algebras

Let p be a prime and K an associative ring with unity in which p is invertible. Let L be a Lie algebra over K. Assume that A is a finite group of exponent p and order at least p^3 acting by automorphisms on L in such a way that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. The purpose of the present section is to show that with these hypotheses L is nilpotent of class bounded solely in terms of c and p. If A contains an abelian subgroup of order p^3, the result is immediate from [11]. Therefore without loss of generality we assume that all subgroups of order p^3 in A are non-abelian. Thus, $p \neq 2$. Clearly, A must contain subgroups of order p^3. Thus, replacing if necessary A by one of its subgroups of order p^3 we may assume that A is an extra-special group of order p^3. As usual, we denote by A' the commutator subgroup of A. Since A is extra-special, $A' = Z(A)$. Set $L_0 = C_L(A')$.

If $X, Y, X_1, \dots X_s$ are subsets of L we use $[X, Y]$ to denote the subspace of L spanned by the set $\{[x,y] | x \in X, y \in Y\}$. If $t \geq 2$ we write $[X,_t Y]$ for $[[X,_{t-1} Y], Y]$ and $[X_1, \dots, X_t]$ for $[[X_1, \dots, X_{t-1}], X_t]$.

Lemma 2.1. *There exists a (c,p)-bounded number u such that*

$$[L, \underbrace{L_0, \dots, L_0}_{u}] = 0.$$

Proof. Of course, for any non-cyclic group B of order p^2 acting on L and any B-invariant subspace V of L we have

$$V = \sum_{b \in B^{\#}} C_V(b).$$

The group A contains precisely $p+1$ subgroups of order p^2 and we denote them by $B_1, B_2, \dots, B_{p+1}$. Set $V_i = C_L(B_i)$ for $i = 1, 2, \dots, p+1$. Denote by $\bar{A}$ the quotient-group A/A'. Since $\bar{A}$ is non-cyclic and L_0 is A-invariant, it follows that $L_0 = \sum_{\bar{a}} C_{L_0}(\bar{a})$ where $\bar{a} \in \bar{A}^{\#}$. Of course, an alternative way of expressing this is to write that $L_0 = \sum_i V_i$.

Further, for each i we have $L = \sum_{a \in B_i^{\#}} C_L(a)$. Thus, taking into account that all centralizers $C_L(a)$ are nilpotent of class c and $V_i \leq C_L(a)$ whenever $a \in B_i^{\#}$, we write

$$[L, \underbrace{V_i, \ldots, V_i}_{c}] = \sum_{a \in B_i^{\#}} [C_L(a), \underbrace{V_i, \ldots, V_i}_{c}] = 0.$$

Let M be an A-invariant subalgebra of L_0 and let $M_i = M \cap V_i$ for $i = 1, 2, \ldots, p+1$. It follows that $M = \sum_i M_i$. By induction on the nilpotency class of M we will show that there exists a (c, p)-bounded number u such that $[L,_u M] = 0$. Since $[M, M]$ is nilpotent of smaller class, there exists a (c, p)-bounded number u_1 such that $[L,_{u_1} [M, M]] = 0$.

Set $r = (c-1)(p+1) + 1$ and $W = [L, M_{i_1}, \ldots, M_{i_r}]$ for some choice of $M_{i_1}, \ldots, M_{i_r} \in \{M_1, \ldots, M_{p+1}\}$. It is clear that for any permutation π of the symbols $i_1, \ldots, i_r$ we have $W \leq [L, M_{\pi(i_1)}, \ldots, M_{\pi(i_r)}] + [L, [M, M]]$. Also, note that the number r is big enough to ensure that some M_i occurs in the list $M_{i_1}, \ldots, M_{i_r}$ at least c times. Thus, we deduce that $W \leq [L,_c M_i, *, \ldots, *] + [L, [M, M]]$, where the asterisks denote some of the subalgebras M_j, which, in view of the fact that $[L,_c M_i] = 0$, are of no consequence. Hence, $W \leq [L, [M, M]]$.

Further, for any choice of $M_{i_1}, \ldots, M_{i_r} \in \{M_1, \ldots, M_{p+1}\}$ the same argument shows that

$$[W, M_{i_1}, \ldots, M_{i_r}] \leq [W, [M, M]] \leq [L, [M, M], [M, M]].$$

More generally, for any k and any $M_{i_1}, \ldots, M_{i_{kr}} \in \{M_1, \ldots, M_{p+1}\}$ we have

$$[L, M_{i_1}, \ldots, M_{i_{kr}}] \leq [L, \underbrace{[M, M], \ldots, [M, M]}_{k}].$$

Put $u = u_1 r$. The above shows that

$$[L, M_{i_1}, \ldots, M_{i_u}] \leq [L, \underbrace{[M, M], \ldots, [M, M]}_{u_1}] = 0.$$

Of course, this implies that $[L,_u M] = 0$. The lemma is now straightforward from the case where $M = L_0$. □

A well-known theorem of Kreknin [9] says that if a Lie ring H admits a fixed-point-free automorphism of finite order n, then H is soluble and the derived length of H is bounded by a function of n. We will require the following extension of this result [8].

Theorem 2.2. *Let a Lie ring H admit an automorphism a of finite order n such that $[H, _t C_L(a)] = 0$ for some $t \geq 1$. Assume that $nH = H$. Then H is soluble with derived length at most $(t+1)^{n-1} + log_2 t$.*

Corollary 2.3. *The Lie algebra L is soluble with (c, p)-bounded derived length.*

Proof. We use Lemma 2.1 and invoke Theorem 2.2. It follows that the derived length of L is (c, p)-bounded. □

Let ω be a primitive pth root of unity. The tensor product $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ can be regarded as a Lie algebra over $K \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. The group A naturally acts on $\tilde{L}$ and the action satisfies the above hypotheses. Moreover, it is clear that L and $\tilde{L}$ have the same nilpotency class. Therefore it is sufficient to bound the class of $\tilde{L}$. Hence, from the outset we can assume that $\omega \in K$.

Recall that A' is cyclic of order p. Choose a generator φ of A'. For each $i = 0, \dots, p-1$ we denote by L_i the φ-eigenspace for the eigenvalue ω^i, that is, $L_i = \{l \in L | l^\varphi = \omega^i l\}$. We have

$$L = \bigoplus_{i=0}^{p-1} L_i \quad \text{and} \quad [L_i, L_j] \subseteq L_{i+j (\text{mod } p)}.$$

Note that $L_0 = C_L(\varphi)$.

Theorem 2.4. *The Lie algebra L is nilpotent of (c,p)-bounded class.*

Proof. By Corollary 2.3 L is soluble of (c,p)-bounded derived length. Now we argue by induction on the derived length of L. If L is abelian, there is nothing to prove. Assume that L is metabelian. In this case $[x,y,z] = [x,z,y]$ for every $x \in [L,L]$ and $y, z \in L$. For each $i = 0, \dots, p-1$ we denote $[L,L] \cap L_i$ by L'_i. Note that in general $L'_0 \neq [L_0, L_0]$. Put

$$t = max\{(c-1)(p+1) + p, u\},$$

where u is as in Lemma 2.1. Choose two indices $m, n \in \{0, \dots, p-1\}$ and consider the commutator $[L'_m, \underbrace{L_n, \dots, L_n}_{t}]$. It is sufficient to show that the commutator is zero.

First suppose that $m = 0$. If $n = 0$, then clearly the commutator is zero since L_0 is nilpotent of class at most c. Assume $n \neq 0$. Note that $L'_0 = \sum_i C_{L'_0}(B_i)$ where B_i ranges over the subgroups of order p^2 in A. Thus, it is sufficient to show that $[C_{L'_0}(B_i), \underbrace{L_n, \dots, L_n}_{t}] = 0$ for any i. Actually, we will show (and later use) that $[C_{L'_0}(B_i), \underbrace{L_n, \dots, L_n}_{t-(p-1)}] = 0$.

For each subgroup B_i we can write $L_n = \sum_{b \in B_i^\#} C_{L_n}(b)$. Therefore,

$$[C_{L'_0}(B_i), \underbrace{L_n, \dots, L_n}_{t-(p-1)}] = \sum_{b \in B_i^\#} [C_{L'_0}(B_i), \underbrace{C_{L_n}(b), \dots, C_{L_n}(b)}_{c}, *, \dots, *] = 0.$$

Here the asterisks denote some subspaces of L_n which are not important since the commutator is 0 anyway.

Now suppose that $m \neq 0$. If $n = 0$, then we use Lemma 2.1. If $n \neq 0$, we find a positive integer $k < p$ such that $m + kn = 0 \pmod p$. We have $[L_m, \underbrace{L_n, \dots, L_n}_{t}] \subseteq [L'_0, \underbrace{L_n, \dots, L_n}_{t-k}]$. In the previous paragraph it was shown that the latter commutator is 0.

Now suppose that the derived length of L is at least 3. By induction hypothesis $[L,L]$ is nilpotent of (c,p)-bounded class. We already know that the quotient $L/[[L,L],[L,L]]$ is nilpotent of (c,p)-bounded class. The Lie algebra analogue [2] of P. Hall's theorem [6] now shows that L has bounded nilpotency class. □

3. Proof of the main result

Let A and G be as in Theorem 1.1. If A contains an abelian subgroup of order p^3, the result is immediate from [11]. Therefore without loss of generality we assume that all subgroups of order p^3 in A are non-abelian. Clearly, A must contain subgroups of order p^3. Thus, replacing if necessary A by one of its subgroups of order p^3 we may assume that A is an extra-special group of order p^3.

The result in [1] quoted in the introduction guarantees that G is soluble. We will require the following lemmas.

Lemma 3.1. *Let B be an extra-special p-group of exponent p and order p^3, and let D be a subgroup of order p^2 in B. If B acts on a finite group H in such a way that $C_H(D) = 1$ and $C_H(x)$ is nilpotent for all $x \in B \setminus D$, then G is nilpotent.*

Proof. This is immediate from [3, Corollary 1.2]. □

A proof of the next lemma can be found in [1, Lemma 3.4].

Lemma 3.2. *Let p be a prime and B an elementary group of order p^2 acting on a finite p'-group H in such a way that $C_H(b)$ is nilpotent for every non-trivial $b \in B$. Let r be a prime dividing $|C_H(B)|$ and x an r-element in $C_H(B)$. Then the index $[H : C_H(x)]$ is an r-power.*

We will also use the following lemma. It is given without a proof since it is straightforward from [10, Lemma 5].

Lemma 3.3. *Let p be a prime and B an elementary group of order p^2 acting on a finite metabelian p'-group H in such a way that $C_H(b)$ is nilpotent for every non-trivial $b \in B$. Then there exists a normal B-invariant subgroup N in H such that H/N is nilpotent and $C_N(B) = 1$.*

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Firstly, we prove that G is nilpotent. We argue by contradiction. Assume that G is not nilpotent and let G be a counter-example of minimal order. We know that G is soluble and so $G = VQ$, where V is a unique minimal normal A-invariant elementary abelian r-subgroup for a prime r and Q is an abelian A-invariant Sylow q-subgroup of G for some primes $q \neq r$.

Let B be a subgroup of order p^2 in A. Lemma 3.3 shows that $C_V(B) = 1$. Suppose that $C_Q(B) = 1$. Then $C_G(B) = 1$, and so by Lemma 3.1 we conclude that G is nilpotent, a contradiction. Thus, $C_Q(B) \neq 1$. Choose $1 \neq y \in C_Q(B)$. By Lemma 3.2 the index $[G : C_G(y)]$ is a power of q. It follows that y centralizes V. Thus, $C_Q(V) \neq 1$, and this is again a contradiction since V is a unique minimal A-invariant normal subgroup of G. Therefore, G is nilpotent.

Now, we wish to show that the nilpotency class of G is (c, p)-bounded. Consider the associated Lie ring of the group G

$$L(G) = \bigoplus_{i=1}^{n} \gamma_i/\gamma_{i+1},$$

where n is the nilpotency class of G and γ_i are the terms of the lower central series of G. The nilpotency class of G coincides with the nilpotency class of $L(G)$. The action of the group A on G induces naturally an action of A on $L(G)$. We observe that the subring $C_L(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Theorem 2.4 now tells us that L is nilpotent of (c, p)-bounded class. The proof is complete. □

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