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NON-ALGEBRAIC EXAMPLES OF MANIFOLDS WITH THE VOLUME DENSITY PROPERTY

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Abstract. Some Stein manifolds (with a volume form) have a large group of (volume-preserving) automorphisms: this is formalized by the (volume) density property, which has remarkable consequences. Until now all known manifolds with the volume density property are algebraic, and the tools used to establish this property are algebraic in nature. In this note we adapt a known criterion to the holomorphic case, and give the first examples of non-algebraic manifolds with the volume density property: they arise as suspensions or pseudo-affine modifications over Stein manifolds satisfying some technical properties. As an application we show that there are such manifolds that are potential counterexamples to the Zariski Cancellation Problem, a variant of the Tóth-Varolin conjecture, and the problem of linearization of $\mathbb{C}^*$-actions on $\mathbb{C}^3$.

1. Introduction

The group of automorphisms of complex affine space $\mathrm{Aut}(\mathbb{C}^n)$ has been intensively studied, both from the algebraic as well as from the analytic point of view. A foundational observation by E. Andersén and L. Lempert [AL92], who proved that every polynomial vector field on $\mathbb{C}^n$ is a finite sum of complete vector fields, served as a starting point for new studies on $\mathrm{Aut}(\mathbb{C}^n)$ (because the algebraic flow of a complete algebraic vector field generates a $\mathbb{C}_+$-action on $\mathbb{C}^n$). This led F. Forstnerič and J.-P. Rosay [FR93] to the formulation which is now commonly called the Andersén-Lempert theorem: any local holomorphic flow defined near a holomorphically compact set can be approximated by global holomorphic automorphisms. Hence $\mathrm{Aut}(\mathbb{C}^n)$ is exceptionally large, and this result opens the possibility of constructing automorphisms with prescribed local behavior, with remarkable consequences, such as the existence of non-straightenable embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ (see Section 4), counterexamples to the holomorphic linearization problem [DK98], among many others (see e.g. [KK11]). This aspect of the study of the automorphism group may be referred to as Andersén-Lempert theory and is the subject of ongoing research.

In order to generalize those techniques to a wider class of manifolds, D. Varolin introduced in [Var01] the concept of the *density property*, which accurately captures the idea of a manifold having a "large" group of automorphisms. Examples include certain homogeneous spaces (see [DDK10]), Danilov-Gizatullin surfaces, as well as Danielewski surfaces (see below for a definition). Andersén considered even earlier,

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in [And90], the situation where the vector fields preserve the standard volume form on $\mathbb{C}^n$, obtaining similar results. There is a corresponding *volume density property* for manifolds equipped with a volume form, which has been substantially less studied. Beyond $\mathbb{C}^n$, only a few isolated examples were known to Varolin (see [Var99]), including $(\mathbb{C}^*)^n$ and $\mathrm{SL}_2(\mathbb{C})$. It took around ten years until new instances of these manifolds were found in [KK10]: all linear algebraic groups equipped with the left invariant volume form, as well as some algebraic Danielewski surfaces (see [KK15b] for an exhaustive list).

In this note we exhibit new manifolds with the volume density property. We prove a general result (see Theorem 4.1), from which we can deduce the following:

Theorem 1.1. *Let $n \geq 1$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be a non-constant holomorphic function with smooth reduced zero fiber $X_0 = f^{-1}(0)$, such that $\tilde{H}^{n-2}(X_0) = 0$ if $n \geq 2$ (here $\tilde{H}$ denotes the reduced cohomology). Then the hypersurface*

$$\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$$

has the volume density property with respect to the form $\bar{\omega}$ satisfying

$$d(uv - f) \wedge \bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n.$$

For $n = 1$ this manifold is called a *Danielewski surface*. Theorem 1.1 was known in the special case where f is a polynomial: this is due to S. Kaliman and F. Kutzschebauch; see [KK10]. Their proof heavily depends on the use of Grothendieck's spectral sequence and seems difficult to generalize to the non-algebraic case. Our method of proof is completely different. It relies on modifying and using a suitable criterion involving so-called semi-compatible pairs of vector fields, developed in [KK15a] for the algebraic setting. This method will be explained in Section 2. In Section 3 we study the suspension $\overline{X}$ (or pseudo-affine modification) of rather general manifolds X along $f \in \mathcal{O}(X)$. After some results concerning the topology and homogeneity of $\overline{X}$, we show that the structure of $\overline{X}$ makes it possible to lift compatible pairs of vector fields from X to $\overline{X}$, in such a way that a technical but essential generating condition on $T\overline{X} \wedge T\overline{X}$ is guaranteed (Theorem 3.4).

The term "non-algebraic" above and in the abstract refers, in a weak sense, to the defining equations being analytic and not necessarily polynomial. In a stronger sense, we say a manifold is *non-algebraic* if it is not biholomorphic to any algebraic manifold. A variety given by analytic equations is therefore a priori non-algebraic, but can certainly fail to be non-algebraic in the stronger sense. However, a truly non-algebraic manifold with the volume density property can actually be constructed: let (in the notation of Theorem 1.1) $n = 2$ and $f(z, w) = w(e^z - 1) - 1$. Then X_0 is smooth reduced and is biholomorphic to $\mathbb{C} \setminus \mathbb{Z}$, which is connected, hence $\tilde{H}^0(X_0) = 0$. It may be shown, by a calculation analogous to equation (3.4) below, that

$$H_3(\overline{\mathbb{C}^2_f}, \mathbb{Z}) \cong H_1(X_0, \mathbb{Z});$$

see also the Thom isomorphism in [KZ99, Prop. 4.1] for a direct proof. However $H_1(X_0, \mathbb{Z}) \cong H_1(\mathbb{C} \setminus \mathbb{Z}, \mathbb{Z})$ is a free abelian group of infinite rank, so $\overline{\mathbb{C}^2_f}$ cannot be homeomorphic, let alone biholomorphic, to an algebraic affine variety.

It is still unknown whether a contractible Stein manifold with the volume density property has to be biholomorphic to $\mathbb{C}^n$. It is believed that the answer is negative; see [KK10]. For instance the affine algebraic submanifold of $\mathbb{C}^6$ given by the equation $uv = x + x^2y + s^2 + t^3$ is such an example. Another prominent one is the Koras-Russell cubic threefold; see [Leu]. According to a theorem of Forstnerič [For99], there exist non-straightenable proper holomorphic embeddings of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$. In Section 4 we will show how to use this fact and Theorem 1.1 to produce an a priori non-algebraic manifold with the volume density property which is diffeomorphic to $\mathbb{C}^n$, which to our knowledge is the first of this kind. In fact we prove the following:

Theorem 1.2. *Let $\phi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ be a proper holomorphic embedding, and consider the manifold defined by $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \dots, z_n)\} \subset \mathbb{C}^{n+2}$, where $f \in \mathcal{O}(\mathbb{C}^n)$ generates the ideal of functions vanishing on $\phi(\mathbb{C}^{n-1})$. Then $\overline{\mathbb{C}^n_f}$ is diffeomorphic to $\mathbb{C}^{n+1}$ and has the volume density property with respect to the volume form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n$. Moreover $\overline{\mathbb{C}^n_f} \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$.*

This implies that the resulting manifold is a potential counterexample to the Zariski Cancellation Problem. In fact, S. Borell and Kutzschebauch have shown in [BK06] that there are families of non-straightenable embeddings which are pairwise non-equivalent. One could therefore make a stronger conjecture: pairwise non-equivalent embeddings ϕ lead to pairwise non-biholomorphic suspensions $\overline{\mathbb{C}^n_f}$, where f is a function vanishing on $\phi(\mathbb{C}^{n-1})$.

We end Section 4 with two more examples which are related to the problem of linearization of holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^n$.

2. A criterion for volume density property

Let X be a complex manifold of dimension n. We implicitly identify $T^{1,0}X$ with the real bundle TX; the global holomorphic sections of TX are called holomorphic vector fields, and for simplicity we denote by $\mathrm{VF}(X)$ the $\mathcal{O}(X)$-module of all such fields (we also treat a holomorphic vector field as a derivation Θ taking $f \in \mathcal{O}(X)$ to $\Theta[f] \in \mathcal{O}(X)$; therefore the kernel of a vector field is well defined). In the sequel we drop the adjective *holomorphic* since we only deal with such objects. Of particular interest to us are the *complete* vector fields on X: these are defined to be those whose flow, starting at any point in X, exists for all complex times, and hence generate one-parameter group automorphisms of X. We denote by $\mathrm{CVF}(X)$ the vector space of such fields, and note that given $\Theta \in \mathrm{CVF}(X)$, if either f or $\Theta[f]$ lies in $\mathrm{Ker}(\Theta)$, then $f\Theta \in \mathrm{CVF}(X)$ (see e.g. [Var99, 3.2]). Observe that sums or Lie combinations of elements in $\mathrm{CVF}(X)$ are in general not complete; denote by $\mathrm{Lie}(X)$ the Lie algebra generated by the elements in $\mathrm{CVF}(X)$.

Global holomorphic sections of the bundle $\wedge^j T^*X$ are called holomorphic j-forms, and we denote by $\Omega^j(X)$ the vector space of all such forms. We let $\mathcal{Z}^j(X)$ (resp. $\mathcal{B}^j(X)$) be the vector space of d-closed (resp. d-exact) holomorphic j-forms on X. Denote by $H^j(X)$ the associated cohomology groups (the holomorphic de Rham cohomology) and notice that if X is a Stein manifold, these can be computed using smooth differential forms, so in particular they are isomorphic to the standard de Rham cohomology groups: see for example [GR79, Thm 7 §V.4]. In the rest of the paper we drop as well the adjective *holomorphic* for j-forms.

Assume X is equipped with a *volume form* ω, that is, a non-degenerate n-form. Recall that the divergence of a vector field Θ on X with respect to ω is the unique complex-valued function $\operatorname{div}_\omega \Theta$ such that

$$(\operatorname{div}_\omega \Theta)\omega = \mathcal{L}_\Theta \omega$$

where $\mathcal{L}_\Theta$ is the Lie derivative in the direction of Θ. We can consider vector fields Θ of zero divergence with respect to ω: $\mathcal{L}_\Theta\omega = 0$, which is equivalent to $\phi_t^*\omega = \omega$, where ϕ_t is the time t map of the local flow of Θ. Denote by $\mathrm{VF}_\omega(X)$ the vector space of all such fields, which we also call *volume-preserving* (note that this is not an $\mathcal{O}(X)$-module anymore), We denote by $\mathrm{Lie}_\omega(X)$ the Lie algebra generated by elements in $\mathrm{CVF}_\omega(X) = \mathrm{VF}_\omega(X) \cap \mathrm{CVF}(X)$. The following is a definition of Varolin, making explicit the essential property of $\mathbb{C}^n$ necessary for the Andersén-Lempert behavior described in Section 1.

Definition 2.1. Let X be a Stein manifold. We say that X has the density property (in short DP) if $\mathrm{Lie}(X)$ is dense in $\mathrm{VF}(X)$ in the compact-open topology. If moreover X is equipped with a volume form ω, we say that X has the volume density property with respect to ω (in short ω-VDP) if $\mathrm{Lie}_\omega(X)$ is dense in $\mathrm{VF}_\omega(X)$.

A manifold may have the VDP with respect to one form but not with respect to another one. It does not imply in general the DP: take for instance $(\mathbb{C}, dz)$; less trivially, $(\mathbb{C}^*)^k$ has the VDP with respect to the Haar form but it is unknown if it has the DP for $k \geq 2$. These definitions can be modified to the algebraic setting: if we consider only algebraic vector fields on an affine algebraic variety, (and an algebraic volume form, respectively), and replace density by equality, the definitions above are that of the algebraic (ω-V)DP. It should be noted that the algebraic DP implies the DP as defined above; similarly, the algebraic VDP implies the holomorphic VDP, although this is not a trivial fact (see [KK10, Prop. 4.1]).

An effective criterion for the algebraic density property was found by Kaliman and Kutzschebauch in [KK08a]. The idea is to find a non-zero $\mathbb{C}[X]$-module in the Lie algebra generated by complete *algebraic* vector fields on X; this algebra can be "enlarged" in the presence of a certain homogeneity condition to the space of all algebraic vector fields on X. The module can be found as soon as there is a pair of complete fields which is "compatible" in a certain technical sense. The algebraic VDP was first thoroughly studied in [KK10], and a corresponding criterion in the holomorphic setting was subsequently developed in [KK15a], wherein the notion of "semi-compatible" vector fields is central. In what follows, we make explicit the corresponding criterion in the holomorphic setting. The proof and ideas are that of [KK15a], with the only difference that in contrast to the algebraic case, the VDP is about an *approximation* of fields: we must therefore use some simple methods in sheaf theory – Cartan's theorem B, approximation near Runge sets – not necessary in the algebraic case.

Given a vector field $\Theta \in \mathrm{VF}(X)$, there is a $\wedge$-antiderivation ι_Θ of degree -1 on the graded algebra of forms $\Omega(X)$ called *interior product*, defined by the relation

$$(\iota_\Theta\omega)(\nu) = \omega(\Theta \wedge \nu), \quad \omega \in \Omega^{k+1}(X), \nu \in \Gamma(\wedge^k TX, X).$$

Its relationship to the exterior derivative d is expressed through Cartan's formula

$$\mathcal{L}_\Theta\omega = d\iota_\Theta\omega + \iota_\Theta d\omega.$$

In the particular case where ω is a volume form on X, non-degeneracy implies that vector fields and $(n-1)$-forms are in one-to-one correspondence via $\Theta \mapsto \iota_\Theta\omega$, which by Cartan's formula restricts to an isomorphism

$$\Phi : \mathrm{VF}_\omega(X) \to \mathcal{Z}^{n-1}(X).$$

In the same spirit, there is an isomorphism of $\mathcal{O}(X)$-modules

$$\Psi : \mathrm{VF}(X) \wedge \mathrm{VF}(X) \to \Omega^{n-2}(X), \quad \nu \wedge \mu \mapsto \iota_\nu \iota_\mu \omega \tag{2.1}$$

and it is straightforward that $\iota_\mu \iota_\nu \omega = \iota_{\nu\wedge\mu}\omega$. We can deduce from the easily verified relation $[\mathcal{L}_\nu, \iota_\mu] = \iota_{[\nu,\mu]}$ that for $\nu, \mu \in \mathrm{VF}_\omega(X)$,

$$\iota_{[\nu,\mu]}\omega = d\iota_\nu \iota_\mu \omega. \tag{2.2}$$

Hence by restricting the isomorphism in equation (2.1) to $\wedge^2 \mathrm{CVF}_\omega(X)$ and composing with the exterior differential $d : \Omega^{n-2} \to \mathcal{B}^{n-1}$ we obtain a mapping

$$d \circ \Psi : \mathrm{CVF}_\omega(X) \wedge \mathrm{CVF}_\omega(X) \to \mathcal{B}^{n-1}, \quad \nu \wedge \mu \mapsto i_{[\mu,\nu]}\omega,$$

whose image is in fact contained in $\Phi(\mathrm{Lie}_\omega(X))$.

Suppose we want to approximate $\Theta \in \mathrm{VF}_\omega(X)$ on $K \subset X$ by a Lie combination of elements in $\mathrm{CVF}_\omega(X)$. Consider the closed form $\iota_\Theta\omega$ and assume for the time being that it is exact. Then by equation (2.1) there is $\gamma \in \mathrm{VF}(X) \wedge \mathrm{VF}(X)$ such that $\iota_\Theta\omega = d(\Psi(\gamma))$. It now suffices to approximate γ by a sum of the form $\sum \alpha_i \wedge \beta_i \in \mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. Indeed, by equation (2.2), $\iota_\Theta\omega = d \circ \Psi(\gamma)$ would then be approximated by elements

$$d \circ \Psi(\sum \alpha_i \wedge \beta_i) = \sum \iota_{[\alpha_i,\beta_i]}\omega \in \Phi(\mathrm{Lie}_\omega(X)),$$

which implies that Θ is approximated uniformly on K by elements of the form $\sum[\alpha_i, \beta_i] \in \mathrm{Lie}_\omega(X)$, as desired. We therefore concentrate on this approximation on $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$. We will assume that *(a)* there are $\nu_1, \dots, \nu_k, \mu_1, \dots, \mu_k \in \mathrm{CVF}_\omega(X)$ such that the submodule of $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$ generated by the elements $\nu_j \wedge \mu_j$ is contained in the closure of $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. We may assume K to be $\mathcal{O}(X)$-convex, and let us suppose *(b)* that for all p in a Runge Stein neighborhood U of K, $\nu_j(p) \wedge \mu_j(p)$ generate the vector space $T_pX \wedge T_pX$. We then proceed with standard methods in sheaf cohomology: let $\mathfrak{F}$ be the coherent sheaf corresponding to the vector bundle $TX \wedge TX$. Condition *(b)* translates to the fact that the images of $\nu_j \wedge \mu_j$ generate the fibers of the sheaf, so by Nakayama's Lemma they lift to a set of generators for the stalks $\mathfrak{F}_p$ for all $p \in U$. Therefore, by Cartan's theorem B, the sections of $\mathfrak{F}$ on U are of the form

$$\sum h_i(\nu_j \wedge \mu_j) \tag{2.3}$$

with $h_j \in \mathcal{O}(U)$. Since U is Runge, we conclude that every element $\gamma \in \mathrm{VF}(X) \wedge \mathrm{VF}(X)$ may be uniformly approximated on K by elements as in equation (2.3) with $h_j \in \mathcal{O}(X)$. By assumption *(a)* γ may be approximated uniformly on K by elements in $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$.

To find the pairs $\nu_j \wedge \mu_j$, observe that if $\nu, \mu \in \mathrm{CVF}_\omega(X)$, and $f \in \operatorname{Ker}\nu, g \in \operatorname{Ker}\mu$, then $f\nu, g\mu \in \mathrm{CVF}_\omega(X)$. By linearity, any element in the span of $(\operatorname{Ker}\nu \cdot \operatorname{Ker}\mu) \cdot (\nu \wedge \mu)$ lies in $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. By considering the closures, we see that if I is a non-zero ideal contained in the closure of $\mathrm{Span}_{\mathbb{C}}(\operatorname{Ker}\nu \cdot \operatorname{Ker}\mu)$, then $I \cdot (\nu \wedge \mu)$ generates a submodule of $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$ which is contained in $\overline{\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)}$. This motivates the following definition.

Definition 2.2. Let ν, μ be non-trivial complete vector fields on X. We say that the pair (ν, μ) is *semi-compatible* if the closure of the span of $\operatorname{Ker}\nu \cdot \operatorname{Ker}\mu$ contains a non-zero ideal of $\mathcal{O}(X)$. We call such an ideal $I \subset \overline{\operatorname{Span}_{\mathbb{C}}(\operatorname{Ker}\nu \cdot \operatorname{Ker}\mu)}$ an ideal of the pair (ν, μ).

To reduce to the special case just treated (where $\iota_\Theta \omega$ is exact), we must further assume that given $\Theta \in \mathrm{VF}_\omega(X)$, it is possible to obtain the zero class in $H^{n-1}(X)$ by subtracting an element of $\Phi(\mathrm{Lie}_\omega(X))$; however, equation (2.2) implies that Lie brackets represent the zero class in $H^{n-1}(X)$, so it is enough to subtract elements from $\Phi(\mathrm{CVF}_\omega(X))$. The preceding discussion then shows that the existence of "enough" semi-compatible pairs of volume-preserving vector fields, along with this condition, suffices to establish the VDP. We have thus proved the following criterion:

Proposition 2.3. *Let X be a Stein manifold of dimension n with a volume form ω, satisfying the following condition:*

every class of $H^{n-1}(X)$ contains an element in the closure of $\Phi(\mathrm{CVF}_\omega(X))$.

Suppose there are finitely many semi-compatible pairs of volume-preserving vector fields (ν_j, μ_j) with ideals I_j such that for all $x \in X$,

$$\{I_j(x)(\nu_j(x) \wedge \mu_j(x))\}_j \text{ generates } \wedge^2 T_x X.$$

Then X has the ω-VDP.

It is also possible to adapt the criterion for the algebraic DP of [KK08a], using so-called compatible vector fields, which satisfy a stronger condition. Namely, a semi-compatible pair of (complete) vector fields (ν, μ) is called *compatible* if there exists $h \in \mathcal{O}(X)$ such that $\nu[h] \in \operatorname{Ker}\nu$ and $h \in \operatorname{Ker}\mu$. Let $f \in \operatorname{Ker}\nu$ and $g \in \operatorname{Ker}\mu$. Then $f\nu, fh\nu, g\mu, gh\mu$ are complete vector fields and a simple calculation shows that

$$fg\nu[h]\mu = [f\nu, gh\mu] - [fh\nu, g\mu] \in \mathrm{Lie}(X).$$

In other words, if I is an ideal associated to the pair (ν, μ), then $I \cdot \nu[h] \cdot \mu$ generates a submodule of $\mathrm{VF}(X)$ which is contained in $\mathrm{Lie}(X)$. So by an obvious variant of the above discussion, we obtain the following generalized criterion for the DP.

Proposition 2.4. *Let X be a Stein manifold. Suppose there are finitely many compatible pairs of vector fields (ν_j, μ_j) such that $I_j(x) \cdot \nu_j[h_j](x)$ generate $T_x X$ for all $x \in X$. Then X has the DP.*

3. Suspensions

Let X be a connected Stein manifold of dimension n, and let $f \in \mathcal{O}(X)$ be a non-constant holomorphic function with a smooth reduced zero fiber X_0 (this means that df is not identically 0 at any point of X_0). To it we associate the space $\overline{X}$, called the *suspension* over X along f, which is defined as

$$\overline{X} = \{(u, v, x) \in \mathbb{C}^2 \times X; uv - f(x) = 0\}.$$

Since X_0 is reduced, $d(uv - f) \neq 0$ in a neighborhood of $\overline{X}$, so $\overline{X}$ is smooth. Hence $\overline{X}$ is a Stein manifold of dimension $n + 1$.

Suppose X has a volume form ω. Then $\Omega = du \wedge dv \wedge \omega$ is a volume form on $\mathbb{C}^2 \times X$. There exists a canonical volume form $\overline{\omega}$ on $\overline{X}$ such that

$$d(uv - f) \wedge \overline{\omega} = \Omega|_{\overline{X}}.$$

Moreover, any vector field $\overline{\Theta}$ on $\overline{X}$ has an extension Θ to $\mathbb{C}^2 \times X$ with $\Theta[uv - f] = 0$ (as a derivation), and we have $\operatorname{div}_{\overline{\omega}} \overline{\Theta} = \operatorname{div}_\Omega \Theta|_{\overline{X}}$ (see [KK08b, 2.2,2.4]). In view of our criterion we now investigate the existence of sufficient semi-compatible fields, as well as the topology of $\overline{X}$.

Let $\Theta \in \mathrm{VF}(X)$. There exists an extension $\tilde{\Theta} \in \mathrm{VF}(\mathbb{C}^2 \times X)$ such that $\tilde{\Theta}[u] = \tilde{\Theta}[v] = 0$ and $\tilde{\Theta}[\tilde{g}] = \Theta[g]$ for all $g \in \mathcal{O}(X)$ (here $\tilde{g}$ is an extension of g not depending on u, v). Clearly, $\operatorname{div}_\Omega \tilde{\Theta} = \pi^*(\operatorname{div}_\omega \Theta)$, where $\pi : \mathbb{C}^2 \times X \to X$ is the natural projection. We may "lift" Θ to a field in $\overline{X}$ in two different ways. Consider the fields on $\mathbb{C}^2 \times X$

$$\Theta_u = v \cdot \tilde{\Theta} + \tilde{\Theta}[\tilde{f}] \frac{\partial}{\partial u}, \qquad \Theta_v = u \cdot \tilde{\Theta} + \tilde{\Theta}[\tilde{f}] \frac{\partial}{\partial v},$$

which are clearly tangent to $\overline{X}$; we may therefore consider the corresponding fields (restrictions) on $\overline{X}$, which we denote simply Θ_u and Θ_v.

Lemma 3.1. *If Θ is ω-volume-preserving, then Θ_u and Θ_v are of $\overline{\omega}$-divergence zero. Moreover, if Θ is complete, then Θ_u and Θ_v are also complete.*

Proof. The completeness of the lifts is clear, but it will be useful for the sequel to compute explicitly their flows. Denote by $\phi^t(x)$ the flow of Θ on X, and let $g : X \times \mathbb{C} \to \mathbb{C}$ satisfy

$$f(\phi^t(x)) = f(x) + t g(x, t). \tag{3.1}$$

Since f is holomorphic, g is well defined and holomorphic on $X \times \mathbb{C}$. We claim that $\Phi : \overline{X} \times \mathbb{C}_t \to \overline{X}$ defined by

$$\Phi^t(u, v, x) = (u + t g(x, tv), v, \phi^{tv}(x)) \tag{3.2}$$

is the flow of Θ_u, which is therefore defined for all t.

Fix $p = (u, v, x) \in \overline{X}$ and assume $\phi^0(x) = x$. Clearly $\Phi^0(p) = p$ and we must check that

$$\frac{\partial}{\partial t}\Big|_{t=t_0} \Phi^t(p) = \Theta_u(\Phi^{t_0}(p)) \quad \forall t_0 \in \mathbb{C}.$$

Recall that $\Theta(\cdot)$ refers to the evaluation of the field Θ at a point, and $\Theta[\cdot]$ to the action of Θ as a derivation. The right-hand side is equal to

$$v \cdot \Theta(\phi^{t_0 v}(x)) + \Theta[f]\left(\phi^{t_0 v}(x)\right) \frac{\partial}{\partial u}\Big|_{\Phi^{t_0}(p)}$$

as an element in $T_{\Phi^{t_0}(p)}\overline{X}$, while the left-hand side evaluates to

$$\frac{\partial}{\partial t}\Big|_{t=t_0} (t g(x, tv)) \frac{\partial}{\partial u}\Big|_{\Phi^{t_0}(p)} + \frac{\partial}{\partial t}\Big|_{t=t_0} \left(\phi^{tv}(x)\right).$$

The second summand is equal to $v \cdot \Theta(\phi^{t_0 v}(x))$. Consider equation (3.1) with t replaced by tv, then take the derivative with respect to t evaluated at t_0:

$$v \cdot \frac{\partial}{\partial t}\Big|_{t=t_0 v} (f \circ \phi^t(x)) = v \cdot \frac{\partial}{\partial t}\Big|_{t=t_0} (t g(x, tv)).$$

But by definition the left-hand side term is $v \cdot \Theta[f](\phi^{t_0 v}(x))$, so the claim is proved.

Since $\operatorname{div}_{\overline{\omega}} \Theta_u = \operatorname{div}_\Omega \Theta_u|_{\overline{X}}$, and because divergence (with respect to any volume form) is linear and satisfies $\operatorname{div}(h \cdot \Theta) = h \operatorname{div} \Theta + \Theta[h]$, we get

$$\operatorname{div}_{\overline{\omega}} \Theta_u = v \cdot \operatorname{div}_\Omega \tilde{\Theta}|_{\overline{X}} + \tilde{\Theta}[v] + \tilde{\Theta}[\tilde{f}] \cdot \operatorname{div}_\Omega \left(\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial u}\left[\tilde{\Theta}[\tilde{f}]\right] = v \cdot \operatorname{div}_\Omega \tilde{\Theta}|_{\overline{X}}$$

and as noted above $\operatorname{div}_\Omega \tilde{\Theta} = \pi^*(\operatorname{div}_\omega \Theta) = 0$. $\square$

Lemma 3.2. *Suppose* (ν, μ) *is a semi-compatible pair of vector fields on* X. *Then* (ν_u, μ_v) *and* (ν_v, μ_u) *are semi-compatible pairs on* $\overline{X}$.

Proof. By Lemma 3.1, the lifted and extended fields are complete. It then suffices to show that (ν_u, μ_v) is a semi-compatible pair in $\mathbb{C}^2 \times X$, because we may restrict the elements in an ideal to $\overline{X}$: by the Cartan extension theorem, this set forms an ideal in $\mathcal{O}(\overline{X})$.

Let I be an ideal of the pair (ν, μ). For any function $h \in \mathcal{O}(X)$, denote by $\tilde{h}$ the trivial extension as above and let

$$\tilde{I} = \left\{\tilde{h} \cdot F(u, v); h \in I, F \in \mathcal{O}(\mathbb{C}^2)\right\} \subset \mathcal{O}(\mathbb{C}^2 \times X).$$

This is clearly a non-zero ideal. An element in $\tilde{I}$ can be approximated uniformly on a given compact of $\mathbb{C}^2 \times X$ by a finite sum

$$(\sum_k \tilde{n}_k \tilde{m}_k) \sum_{i,j} a_{i,j} u^i v^j = \sum_{i,j,k} a_{i,j} (\tilde{n}_k v^j)(\tilde{m}_k u^i)$$

where $n_k \in \mathrm{Ker}(\nu), m_k \in \mathrm{Ker}(\mu)$ for all k. Since $\tilde{n}_k v^j \in \mathrm{Ker}(\nu_u)$ for all $j, k \geq 0$ and $\tilde{m}_k u^i \in \mathrm{Ker}(\mu_v)$ for all $i, k \geq 0$, it follows that $\tilde{I}$ is contained in the closure of $\mathrm{Span}_{\mathbb{C}}(\mathrm{Ker}(\nu_u) \cdot \mathrm{Ker}(\mu_v))$. $\square$

The topology of the suspension $\overline{X}$ is closely related to that of X. In the case where X is the affine space, this relationship is computed in detail in [KZ99, §4]. For more general X we have the following.

Proposition 3.3. *Assume* X *is a Stein manifold of dimension* $n \geq 2$. *If* $H^n(X) = H^{n-1}(X) = 0$ *and* $\tilde{H}^{n-2}(X_0) = 0$, *where* $\tilde{H}$ *means the reduced cohomology, then* $H^n(\overline{X}) = 0$.

Proof. Recall that by the remark at the beginning of Section 2, the holomorphic de Rham cohomology is isomorphic to standard cohomology, and so we may use without further comment the usual tools of algebraic topology. Consider the long exact sequence of the pair $(\overline{X}, \overline{X} \setminus U_0)$ in cohomology, where U_0 is the subspace of $\overline{X}$ where u vanishes:

$$\cdots \to H^n(\overline{X}, \overline{X} \setminus U_0) \to H^n(\overline{X}) \to H^n(\overline{X} \setminus U_0) \to \dots . \tag{3.3}$$

The term on the right vanishes, because $\overline{X} \setminus U_0$ is biholomorphic to $\mathbb{C}^* \times X$ via $(u, x) \mapsto (u, f(x)/u, x)$, so

$$H^n(\overline{X} \setminus U_0) = (H^1(\mathbb{C}^*) \otimes H^{n-1}(X)) \oplus (H^0(\mathbb{C}^*) \otimes H^n(X)) = 0.$$

To evaluate the term on the left-hand side of equation (3.3), we borrow an idea from M. Zaidenberg (see [Zai96]). Consider the normal bundle $\pi : N \to U_0$ of the closed submanifold U_0 in $\overline{X}$, with zero section $N_0 \cong U_0$. Fix a tubular neighborhood W of U_0 in $\overline{X}$ such that the pair (W, U_0) is diffeomorphic to (N, N_0). Then by excision of $\overline{X} \setminus W$ out of the pair $(\overline{X}, \overline{X} \setminus U_0)$, we have that

$$\tilde{H}^*(\overline{X}, \overline{X} \setminus U_0) \cong \tilde{H}^*(W, W \setminus U_0) \cong \tilde{H}^*(N, N \setminus N_0).$$

Let $t \in H^2(N, N \setminus N_0)$ be the Thom class of U_0 in $\overline{X}$, that is, the unique cohomology class taking value 1 on any oriented relative 2-cycle in $H_2(N, N \setminus N_0)$ defined by a fiber F of the normal bundle N (see e.g. [MS74, §9–10] for details). Then, by taking the cup-product of the pullback under π of a cohomology class with t, we obtain the Thom isomorphisms

$$H^i(U_0) \cong H^{i+2}(N, N \setminus N_0) \cong H^{i+2}(\overline{X}, \overline{X} \setminus U_0) \quad \forall i. \tag{3.4}$$

Since $U_0 \cong X_0 \times \mathbb{C}$, U_0 is homotopy equivalent to X_0, and we have $H^n(\overline{X}, \overline{X} \setminus U_0) \cong H^{n-2}(X_0)$. If $n \geq 3$, reduced cohomology coincides with standard cohomology, and therefore $H^n(\overline{X}) = 0$ by exactness of equation (3.3). If $n = 2$, that sequence becomes

$$\cdots \to H^1(\overline{X} \setminus U_0) \to H^2(\overline{X}, \overline{X} \setminus U_0) \to H^2(\overline{X}) \to 0.$$

Let γ be an oriented 2-cycle in $\overline{X}$ whose boundary $\partial\gamma$ lies in $\overline{X} \setminus U_0$ (a disk transversal to U_0). A one-dimensional subspace of $H^1(\overline{X} \setminus U_0)$ is generated by a 1-cocycle taking value 1 on $\partial\gamma$, and this cocycle is sent via the coboundary operator (which is the first map in the above sequence) to a 2-cocycle taking value 1 on γ, i.e., to the Thom class t described previously, which is also a generator of a one-dimensional subspace of $H^2(\overline{X}, \overline{X} \setminus U_0)$. However, $H^1(\overline{X} \setminus U_0) \cong H^1(\mathbb{C}^* \times X) \cong \mathbb{C}$ and $H^2(\overline{X}, \overline{X} \setminus U_0) \cong H^0(U_0) \cong \mathbb{C}$, so the coboundary map is an isomorphism, and by exactness it follows that $H^2(\overline{X}) = 0$. $\square$

Next, we show how to lift a collection of semi-compatible fields to the suspension and span $\wedge^2 T\overline{X}$ with semi-compatible fields.[1] We will denote by $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}_\omega(X)$) the group of holomorphic automorphisms of the manifold X (resp. the volume-preserving automorphisms).

Theorem 3.4. *Let X be a Stein manifold with a finite collection S of semi-compatible pairs (α, β) of vector fields such that for some $x_0 \in X$*

$$\{\alpha(x_0) \wedge \beta(x_0); (\alpha, \beta) \in S\} \text{ spans } \wedge^2 (T_{x_0} X). \tag{3.5}$$

Assume that $\mathrm{Aut}(\overline{X})$ acts transitively on $\overline{X}$. Then there exists a finite collection $\overline{S}$ of semi-compatible pairs (A_j, B_j) on $\overline{X}$ with corresponding ideals I_j such that

$$\mathrm{Span}_{\mathbb{C}}\{I_j(\bar{x}) A_j(\bar{x}) \wedge B_j(\bar{x})\}_j = \wedge^2(T_{\bar{x}}\overline{X}) \quad \forall \bar{x} \in \overline{X}. \tag{3.6}$$

Moreover, if X has a volume form ω and the fields in S preserve it, and $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ acts transitively, then the fields in $\overline{S}$ can be chosen to preserve the form $\bar{\omega}$

Proof. We claim that it is sufficient to show that the conclusion holds for a single $\bar{x}_0 \in \overline{X}$. Indeed, let C be the analytic set of points $\bar{x} \in \overline{X}$ where equation (3.6) does not hold, and decompose C into its (at most countably many) irreducible components C_i. For each i, let D_i be the set of automorphisms ϕ of $\overline{X}$ such that the image of $\overline{X} \setminus C_i$ under ϕ has a non-empty intersection with C_i. Clearly each D_i is open, and it is also dense: given $h \in \mathrm{Aut}(\overline{X})$ not in D_i, let $c \in C_i$, $d = h(c) \in C_i$ and $\gamma \in \mathrm{Aut}(\overline{X})$ mapping $\bar{x}_0$ to d. Now, since the assumption in equation (3.6) implies that the tangent space at $\bar{x}_0$ is spanned by complete fields, there exists a complete field α from the collection $\overline{S}$ such that $\gamma_*(\alpha)$ is not tangent to C_i. If φ is the flow of $\gamma_*(\alpha)$, then $\varphi^t \circ h$ is an automorphism arbitrarily close to h mapping c out of C_i for t small enough. By the Baire Category Theorem, since $\mathrm{Aut}(\overline{X})$ admits a complete metric inducing the compact-open topology, there exists a $\psi \in \bigcap D_i$. By expanding $\overline{S}$ to $\overline{S} \cup \{(\psi_*\alpha, \psi_*\beta); (\alpha, \beta) \in \overline{S}\}$, we obtain a finite collection of semi-compatible fields which fail to satisfy equation (3.6) in an exceptional variety

[1] A simpler algebraic case has been treated by J. Josi (Master thesis, 2013, unpublished).

of dimension strictly lower than that of C. The conclusion follows from the finite iteration of this procedure.

By the previous lemmas, if $(\alpha,\beta)\in S$, then (α_u,β_v) and (α_v,β_u) are semi-compatible pairs in $\overline{X}$. We let $\overline{S}$ consist of all those pairs. We will also add two pairs to $\overline{S}$, of the form $(\phi^*\alpha_u,\phi^*\beta_v)$, where ϕ is an automorphism of $\overline{X}$ (preserving a volume form, if necessary) to be specified later. We now select an appropriate $\bar{x}_0=(u_0,v_0,x_0)\in\overline{X}$ by picking any element from the complement of finitely many analytic subsets which we now describe. The first analytic subset of $\overline{X}$ to avoid is the locus where any of the (finitely many) associated ideals I_j vanish. Note that equation (3.5) is in fact satisfied everywhere on X except an analytic variety C: the second closed set in $\overline{X}$ to avoid is the preimage of C under the projection. Finally, we avoid $u=0, v=0$ and $\mathrm{d}_{x_0}f=0$. In short, we pick a $\bar{x}_0=(u_0,v_0,x_0)\in\overline{X}$ with $u_0\neq 0, v_0\neq 0, \mathrm{d}_{x_0}f\neq 0$, such that equation (3.5) is satisfied at x_0, and such that none of the ideals $I_j(\bar{x}_0)$ vanish. Because of this last condition, it will suffice to show that $\{A(\bar{x}_0)\wedge B(\bar{x}_0);(A,B)\in\overline{S}\}$ spans $\wedge^2(T_{x_0}X)$.

Consider $\pi:\overline{X}\to X\times\mathbb{C}_u$, which at $\bar{x}_0$ induces an isomorphism $\mathrm{d}_{\bar{x}_0}\pi:T_{\bar{x}_0}\overline{X}\to T_{x_0}X\times T_{u_0}\mathbb{C}$. Denote by $\partial_u=\frac{\partial}{\partial u}$ the basis of $T_{u_0}\mathbb{C}$, and consider the linear map P induced by $\mathrm{d}_{\bar{x}_0}\pi$

$$P:\wedge^2(T_{\bar{x}_0}\overline{X})\to\wedge^2(T_{x_0}X\oplus\langle\partial_u\rangle)=\wedge^2(T_{x_0}X)\oplus(T_{x_0}X\otimes\langle\partial_u\rangle).$$

Since P is an isomorphism, it now suffices to show that the direct sum on the right-hand side equals $P(\Lambda)$, where

$$\Lambda=\mathrm{Span}_{\mathbb{C}}\{A(\bar{x}_0)\wedge B(\bar{x}_0);(A,B)\in\overline{S}\}.$$

We will prove (i) that $\wedge^2(T_{x_0}X)\subseteq P(\Lambda)$, and (ii) that $T_{x_0}X\otimes\langle\partial_u\rangle\subseteq P(\Lambda)$.

Let us first show (i). Let $\alpha(x_0)\wedge\beta(x_0)\in\wedge^2(T_{x_0}X)$. Since $\{\alpha(x_0)\wedge\beta(x_0)\}_{(\alpha,\beta)\in S}$ spans $\wedge^2(T_{x_0}X)$, we can assume that (α,β) is a pair of vector fields lying in S (for simplicity we often omit to indicate the point x_0 at which these fields are evaluated). Then $(\alpha_u,\beta_v)\in\overline{S}$, so $P(\Lambda)$ contains

$$P(\alpha_u\wedge\beta_v)=P((v\tilde{\alpha}+\alpha[f]\partial_u)\wedge(u\tilde{\beta}+\beta[f]\partial_v))=uv(\alpha\wedge\beta)-u\alpha[f](\beta\wedge\partial_u). \tag{3.7}$$

At the point $\bar{x}_0$, we have assumed that u and v are both non-zero. If $\alpha[f]$ happens to vanish at x_0, then $\alpha(x_0)\wedge\beta(x_0)$ is in $P(\Lambda)$, as desired. Otherwise, consider the vector field $(u-u_0)\alpha_v$ on $\overline{X}$. Since α_v is complete and $(u-u_0)$ lies in the kernel of α_v, $(u-u_0)\alpha_v$ is a complete (and $\bar{\omega}$-divergence free) vector field on $\overline{X}$. Quite generally one can compute, in local coordinates for example, that the flow at time 1 of the field $g\Theta$, where $\Theta\in\mathrm{CVF}(M)$ and $g\in\mathrm{Ker}(\Theta)$ with $g(p)=0$, is a map ϕ whose derivative at $p\in M$ is given by:

$$w\mapsto w+\mathrm{d}_pg(w)\Theta(p)\quad w\in T_pM.$$

Therefore, for a vector field $\mu\in\mathrm{VF}(M)$, we have

$$(\phi^{-1})^*(\mu)(p)=(\mathrm{d}_p\phi)(\mu(p))=\mu(p)+(\mu[g])(p)\Theta(p). \tag{3.8}$$

Apply this in the case of $M=\overline{X}$, $p=\bar{x}_0$, $\Theta=\alpha_v$ and $g=u-u_0$. For the vector fields $\mu=\beta_v$, this equals β_v; for $\mu=\alpha_u$, it equals $\alpha_u+\alpha[f]\alpha_v$. Hence, if we add $((\phi^{-1})^*\alpha_u,(\phi^{-1})^*\beta_v)$ to $\overline{S}$, we obtain that $P(\Lambda)$ contains

$$P((\phi^{-1})^*\alpha_u\wedge(\phi^{-1})^*\beta_v-\alpha_u\wedge\beta_v)=P(\alpha[f]\alpha_v\wedge\beta_v)=\alpha[f]u^2(\alpha\wedge\beta).$$

We now show (ii). It will be useful to distinguish elements in $T_{x_0}X$ according to whether they belong to $K=\mathrm{Ker}(\mathrm{d}_{x_0}f)$ or not. Since we have assumed $\mathrm{d}_{x_0}f\neq 0$,

$T_{x_0}X$ splits as $K \oplus V$, where V is a vector space of dimension 1, which may be spanned by some ξ satisfying $\mathrm{d}_{x_0} f(\xi) = \xi[f] = 1$. The isomorphism is given by the unique decomposition $v = (v - v[f]\xi) + v[f]\xi$. This induces another splitting

$$\begin{aligned} &\wedge^2(T_{x_0}X) \to \wedge^2(K) \oplus (K \otimes V) \\ &\quad \alpha \wedge \beta \mapsto (\alpha - \alpha[f]\xi) \wedge (\beta - \beta[f]\xi) + (\alpha[f]\beta - \beta[f]\alpha) \wedge \xi. \end{aligned}$$

Since the left-hand side is generated by $\{\alpha \wedge \beta; (\alpha, \beta) \in S\}$, $K \otimes V$ is generated by $\{(\alpha[f]\beta - \beta[f]\alpha) \wedge \xi; (\alpha, \beta) \in S\}$, and therefore K by $\{\alpha[f]\beta - \beta[f]\alpha; (\alpha, \beta) \in S\}$. Consider equation (3.7) and subtract $P(\alpha_v \wedge \beta_u) = uv(\alpha \wedge \beta) + u\beta[f](\alpha \wedge \partial_u)$: recalling that $u_0 \neq 0$, we see that

$$\{u(\beta[f]\alpha - \alpha[f]\beta) \wedge \partial_u; (\alpha, \beta) \in S\} = K \otimes \langle \partial_u \rangle \subset P(\Lambda). \tag{3.9}$$

It remains to show that $V \otimes \langle \partial_u \rangle \subset P(\Lambda)$. By linearity, since V is of dimension 1, it suffices to find a single element in $P(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$. In fact since we have already proven (i), it suffices to find an element in $P_2(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$, where P_2 is the second component of the map P. If it were the case that for some pair $(\alpha, \beta) \in S$ both $\alpha[f]$ and $\beta[f]$ are non-zero at x_0, then by equation (3.7) $-u\alpha[f]\beta \wedge \partial_u$ is such an intersection element. In the other case, there is at least a pair $(\alpha, \beta) \in S$ for which $\alpha[f](x_0) = 0$ and both $\beta[f](x_0) \neq 0$ and $\alpha(x_0) \neq 0$, for otherwise the spanning condition implied by equation (3.9) would fail to be satisfied. As in the proof of (i), we will add to $\overline{S}$ the pair $(\phi^*(\alpha_u), \phi^*(\beta_v))$, where ϕ is the time 1 map of the flow of the complete (volume-preserving) field $\Theta = g(x)(u\partial_u - v\partial_v)$, and $g \in \mathcal{O}(X)$ vanishes at x_0. By equation (3.8), we have that

$$\phi^*(\alpha_u) = \alpha_u + \alpha_u[g]\Theta = v\alpha + \alpha[f]\partial_u + v\alpha[g](u\partial_u - v\partial_v)$$

which by assumption simplifies to

$$\phi^*(\alpha_u) = v\alpha + uv\alpha[g]\partial_u - v^2\alpha[g]\partial_v.$$

Similarly we have

$$\phi^*(\beta_v) = u\beta + u^2\beta[g]\partial_u + (\beta[f] - uv\beta[g])\partial_v.$$

Hence

$$P_2(\phi^*(\alpha_u) \wedge \phi^*(\beta_v)) = u^2v\beta[g]\alpha \wedge \partial_u - u^2v\alpha[g]\beta \wedge \partial_u.$$

By assumption, the first summand lies in $K \otimes \langle \partial_u \rangle$, which we have already shown to be contained in $P(\Lambda)$. Since $\beta[f] \neq 0$, the second summand, if non-zero, lies in $P_2(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$. But it is clear that we may find a $g \in \mathcal{O}(X)$ such that $\alpha[g](x_0) \neq 0$. □

Finally, we show how the transitivity requirement for the previous proposition can be inherited from the base space X. We say that a point x in a Stein manifold X (resp. the Stein manifold X) is *holomorphically (volume) flexible* if the complete (volume-preserving) vector fields span the tangent space T_xX (resp. for all $x \in X$) - see [AFK13, §6]. Clearly, a manifold X is holomorphically (volume) flexible if one point $x \in X$ is, and $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}_\omega(X)$) acts transitively. Moreover, holomorphic (volume) flexibility implies the the transitive action of $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}_\omega(X)$) on X.

Lemma 3.5. *If X is holomorphically flexible, then* $\mathrm{Aut}(\overline{X})$ *acts transitively. Moreover, if X is holomorphically volume flexible at a point $x \in X$ and* $\mathrm{Aut}_\omega(X)$ *acts transitively, then* $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ *acts transitively.*

Proof. For simplicity we prove the first statement: the second is proven in an exactly analogous manner. Let $\bar{x}_0 = (u_0, v_0, x_0) \in \overline{X}$ with $u_0 v_0 \neq 0$, and let us determine the orbit of $\bar{x}_0$ under $\mathrm{Aut}(\overline{X})$. Given $\Theta \in \mathrm{VF}(X)$, by equation (3.2) we have, for each t, an automorphism of $\overline{X}$ of the form

$$(u, v, x) \mapsto (u + tg(x, tv), v, \phi^{tv}(x)). \tag{3.10}$$

The orbit of $\bar{x}_0$ must hence contain the hypersurface $\{v = v_0\} \subset \overline{X}$ (because $\mathrm{Aut}(X)$ acts transitively on X), and analogously, since $u_0 \neq 0$, the orbit contains $\{u = u_0\} \subset \overline{X}$. Let $(u_1, v_1, x_1) \in \overline{X}$ be another point with $u_1 v_1 \neq 0$. Note that the non-constant function $f : X \to \mathbb{C}$ can omit at most one value ξ. Indeed, by flexibility there is a complete vector field which at x_0 points in a direction where f is not constant; precomposition with its flow map at x_0 gives an entire function which must omit at most one value. By definition ξ is not $u_0 v_0$ nor $u_1 v_1$; assume it is not 0: for if it were, then $\overline{X}$ would be biholomorphic to $\mathbb{C}^* \times X$ (and the biholomorphism sends $\bar{\omega}$ to the standard form $z^{-1} dz \wedge \omega$), and in this case the lemma is trivial. Follow the orbit of $\bar{x}_0$ along the hypersurface $\{u = u_0\} \cap \overline{X}$ until (u_0, v_1, x'), for some x' in X, then along $\{v = v_1\} \cap \overline{X}$ until (u_1, v_1, x_1) (if $\xi = u_0 v_1$ replace v_1 by $2v_1$). So the orbit contains all points $(u, v, x) \in \overline{X}$ with $uv \neq 0$ and by equation (3.10) also those with either u or v non-zero. Consider now a point of the form $(0, 0, x_0) \in \overline{X}$. Since $x_0 \in X_0$ and X_0 is reduced, $\mathrm{d}_{x_0} f \neq 0$, so there is a tangent vector evaluating to a non-zero number, which since X is flexible can be taken to be of the form $\Theta(x_0)$ for a complete field Θ. By lifting Θ we obtain in particular an automorphism of $\overline{X}$ of the form $(u, v, x) \mapsto (u + g(x, v), v, \phi^v(x))$. Since

$$g(x_0, 0) = \lim_{t \to 0} \frac{f(\phi^t(x_0)) - f(\phi^0(x_0))}{t} = (f \circ \phi)'(0) = \mathrm{d}_{x_0} f(\Theta(x_0)) \neq 0,$$

this automorphism maps $(0, 0, x_0)$ to a point of non-zero u coordinate, and we are done. □

In particular, by the Andersén-Lempert theorem (see e.g. [KK11, §2.B]), the assumptions hold if X has the ω-VDP and is of dimension $n \geq 2$.

4. Examples or applications

The following theorem summarizes the previous discussion and gives conditions under which the suspension over a manifold has a VDP.

Theorem 4.1. *Let X be a Stein manifold of dimension $n \geq 2$ such that $H^n(X) = H^{n-1}(X) = 0$. Let ω be a volume form on X and suppose that $\mathrm{Aut}_\omega(X)$ acts transitively. Assume that there is a finite collection S of semi-compatible pairs (α, β) of volume-preserving vector fields such that for some $x_0 \in X$, $\{\alpha(x_0) \wedge \beta(x_0); (\alpha, \beta) \in S\}$ spans $\wedge^2 T_{x_0} X$. Let $f : X \to \mathbb{C}$ be a non-constant holomorphic function with smooth reduced zero fiber X_0 and $\tilde{H}^{n-2}(X_0) = 0$. Then the suspension $\overline{X} \subset \mathbb{C}^2_{u,v} \times X$ of X along f has the VDP with respect to a natural volume form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = (du \wedge dv \wedge \omega)|_{\overline{X}}$.*

Proof. The spanning condition on $\wedge^2 TX$ implies holomorphic volume flexibility at x_0. So by Lemma 3.5, $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ acts transitively, and therefore Theorem 3.4 may be applied. By assumption and Proposition 3.3, the topological condition of Proposition 2.3 is also trivially satisfied. □

Corollary 4.2. *Let $n \geq 1$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be a non-constant holomorphic function with smooth reduced zero fiber X_0, such that $\tilde{H}^{n-2}(X_0) = 0$ if $n \geq 2$. Then the hypersurface $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$ has the volume density property with respect to the form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n$.*

Proof. If $n \geq 2$ this follows immediately from the previous theorem, since in $\mathbb{C}^n$ the standards derivations ∂_{z_j} generate $\wedge^2 TX$. If $n = 1$, there are no semi-compatible pairs on $\mathbb{C}$, but it is possible to show the VDP directly. Given $\Theta \in \mathrm{VF}_\omega(\overline{\mathbb{C}_f})$ and a compact K of $\overline{\mathbb{C}_f}$, we must approximate Θ on K by a finite Lie combination of some complete volume-preserving fields. It will suffice to suppose that the coefficients of Θ are in $\mathcal{O}(\mathbb{C}) \otimes \mathbb{C}[u, v]$ and to express Θ as a finite Lie combination of complete volume-preserving fields (see [KK08b, §3]). Those will be the known Θ_u, Θ_v, and $h(u\frac{\partial}{\partial_u} - v\frac{\partial}{\partial_v})$, where $h \in \mathcal{O}(\mathbb{C}_z)$. The details of the calculation can be found in Proposition 3.3 of [KK10], but the setup is slightly different. We express the vector field Θ in coordinates of $\mathbb{C}^3$

$$\left(\mu(z) + \sum_{i=1}^{N} s_i(z)u^i + \sum_{i=1}^{M} t_i(z)v^i\right)\frac{\partial}{\partial_z} + A(z,u,v)\frac{\partial}{\partial_u} + B(z,u,v)\frac{\partial}{\partial_v},$$

where the coefficients μ, s_i, t_i are in $\mathcal{O}(\mathbb{C})$ and $A, B \in \mathcal{O}(\mathbb{C}) \otimes \mathbb{C}[u, v]$. By a general argument $\mu(z)\frac{\partial}{\partial_z}$ must be a divergence-free vector field on $\mathbb{C}$ (that is, a constant field) vanishing in the zero set of f, which we assume to be non-empty (otherwise $\overline{\mathbb{C}_f}$ is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$ and by [Var01, §4] we are done), so $\mu = 0$. The rest of the proof is now identical to that of Proposition 3.3 of [KK10]. □

Let $\phi : \mathbb{C}^{n-1} \to \mathbb{C}^n$ be a proper holomorphic embedding, and consider the closed subset $Z = \phi(\mathbb{C}^{n-1}) \subset \mathbb{C}^n$. It is a standard result that every multiplicative Cousin distribution in $\mathbb{C}^n$ is solvable: this is implied by $H^2(\mathbb{C}^n, \mathbb{Z}) = 0$, since the manifold is Stein. Therefore the divisor associated to Z is principal: in other words, there exists a holomorphic function f on $\mathbb{C}^n$ vanishing precisely on Z and such that $df \neq 0$ on Z. We may hence consider the suspension $\overline{\mathbb{C}^n_f}$ of $\mathbb{C}^n$ along f, which according to the above corollary must have the volume density property. The significance of this lies in the existence of non-straightenable embeddings. Recall that a proper holomorphic embedding $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is said to be *holomorphically straightenable* if there exists an automorphism α of $\mathbb{C}^n$ such that $\alpha(\phi(\mathbb{C}^k)) = \mathbb{C}^k \times \{0\}^{n-k}$. The existence of non-tame sets in $\mathbb{C}^n$, combined with an interpolation theorem, implies that there exists for each $k < n$ non-straightenable proper holomorphic embeddings $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$; see [For99]. Note that proper holomorphic embeddings are the holomorphic analogue of polynomial embeddings, and that the "classical" algebraic situation is in sharp contrast to the holomorphic one: for every $n > 2k+1$ polynomial embeddings $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ are algebraically straightenable (see [Kal92]), the case of real codimension 2 remaining notoriously open.

If the embedding ϕ is straightenable, it is clear that $\overline{\mathbb{C}^n_f}$ is trivially biholomorphic to $\mathbb{C}^{n+1}$, and a calculation shows that the form $\bar{\omega}$ is the standard one. So the result says something new only if ϕ is non-straightenable. Indeed, it is unknown whether $\overline{\mathbb{C}^n_f}$ is biholomorphic to $\mathbb{C}^{n+1}$. However, $\overline{\mathbb{C}^n_f} \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$ (see [DK98]), and is therefore a potential counterexample to the holomorphic version of the important Zariski Cancellation Problem: if X is a complex manifold of dimension n and $X \times \mathbb{C}$ biholomorphic to $\mathbb{C}^{n+1}$, does it follow that X is biholomorphic to $\mathbb{C}^n$?

Moreover, $\overline{\mathbb{C}^n_f}$ is diffeomorphic to complex affine space. This is best shown in the algebraic language of modifications, as follows. Given a triple (X, D, C) consisting of a Stein manifold X, a smooth reduced analytic divisor D, and a proper closed complex submanifold C of D, it is possible to construct the pseudo-affine modification of X along D with center C, denoted $\overline{X}$. It is the result of blowing up X along C and deleting the proper transform of D. We refer the interested reader to [KZ99] for a general discussion. In our situation we take $X = \mathbb{C}^n \times \mathbb{C}_u$, $D = \mathbb{C}^n \times \{0\}$, and $C = Z \times \{0\} = \phi(\mathbb{C}^{n-1}) \times \{0\}$: in this case, $\overline{X}$ is biholomorphic to $\overline{\mathbb{C}^n_f}$ (see Example 1.4 or Remark 4.1 in [KZ99]). We now invoke a general result giving sufficient conditions for a pseudo-affine modification to be diffeomorphic to affine space: since Z is contractible, Proposition 5.10 from [KK08b] is directly applicable, and therefore the following holds:

Corollary 4.3. *If $\phi : \mathbb{C}^{n-1} \to \mathbb{C}^n$ is a (possibly non-straightenable) holomorphic embedding, then the suspension $\overline{\mathbb{C}^n_f}$ along the function f defining the subvariety $\phi(\mathbb{C}^{n-1})$, is diffeomorphic to $\mathbb{C}^{n+1}$ and has the volume density property with respect to a natural volume form $\bar{\omega}$.*

Recall a conjecture of A. Tóth and Varolin [TV00] asking whether a complex manifold which is diffeomorphic to $\mathbb{C}^n$ and has the density property must be biholomorphic to $\mathbb{C}^n$. It is also unknown whether there are contractible Stein manifolds with the volume density property which are not biholomorphic to $\mathbb{C}^n$, and our construction provides a new potential counterexample.

To conclude, we give another example of an application. Consider a proper holomorphic embedding $\mathbb{D} \hookrightarrow \mathbb{C}^2_{x,y}$ (that this exists is a classical theorem of K. Kasahara and T. Nishino; see e.g. [Ste72]), and let f generate the ideal of functions vanishing on the embedded disk (which can be shown to be a non-algebraic manifold). Then $M = \overline{\mathbb{C}^2_f} \subset \mathbb{C}^2_{u,v} \times \mathbb{C}^2_{x,y}$ admits a $\mathbb{C}^*$-action, namely

$$\lambda \mapsto (\lambda u, \lambda^{-1} v, x, y),$$

whose fixed point set is biholomorphic to $\mathbb{D}$. Therefore, the action cannot be linearizable, i.e., there is no holomorphic change of coordinates after which the action is linear. Recall the problem of linearization of holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^k$ (see e.g. [DK98]): for $k = 2$, every action is linearizable; there are counterexamples for $k \geq 4$; and the problem remains open for $k = 3$. If M is biholomorphic to $\mathbb{C}^3$, there would be a negative answer. Otherwise, it resolves in the negative the Tóth-Varolin conjecture mentioned above. By a result of J. Globevnik [Glo97], it is also possible to embed arbitrarily small perturbations of a polydisk in $\mathbb{C}^n$ for any $n \geq 1$ into $\mathbb{C}^{n+1}$; by the same argument, we obtain for any $n \geq 3$, manifolds that are diffeomorphic to $\mathbb{C}^n$ with the volume density property.

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References

[AFK13] Ivan Arzhantsev, Hubert Flenner, Shulim Kaliman, Frank Kutzschebauch, and Mikhail Zaidenberg, *Infinite transitivity on affine varieties*, Birational geometry, rational curves, and arithmetic, Springer, New York, 2013, pp. 1–13, DOI 10.1007/978-1-4614-6482-2_1. MR3114921

[AL92] Erik Andersén and László Lempert, *On the group of holomorphic automorphisms of* $\mathbf{C}^n$, Invent. Math. **110** (1992), no. 2, 371–388, DOI 10.1007/BF01231337. MR1185588

[And90] Erik Andersén, *Volume-preserving automorphisms of* $\mathbf{C}^n$, Complex Variables Theory Appl. **14** (1990), no. 1-4, 223–235. MR1048723

[BK06] Stefan Borell and Frank Kutzschebauch, *Non-equivalent embeddings into complex Euclidean spaces*, Internat. J. Math. **17** (2006), no. 9, 1033–1046, DOI 10.1142/S0129167X06003795. MR2274009

[DDK10] F. Donzelli, A. Dvorsky, and S. Kaliman, *Algebraic density property of homogeneous spaces*, Transform. Groups **15** (2010), no. 3, 551–576, DOI 10.1007/s00031-010-9091-8. MR2718937

[DK98] Harm Derksen and Frank Kutzschebauch, *Nonlinearizable holomorphic group actions*, Math. Ann. **311** (1998), no. 1, 41–53, DOI 10.1007/s002080050175. MR1624259

[For99] Franc Forstneric, *Interpolation by holomorphic automorphisms and embeddings in* $\mathbf{C}^n$, J. Geom. Anal. **9** (1999), no. 1, 93–117, DOI 10.1007/BF02923090. MR1760722

[FR93] Franc Forstnerič and Jean-Pierre Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math. **112** (1993), no. 2, 323–349, DOI 10.1007/BF01232438. MR1213106

[Glo97] Josip Globevnik, *A bounded domain in* $\mathbf{C}^N$ *which embeds holomorphically into* $\mathbf{C}^{N+1}$, Ark. Mat. **35** (1997), no. 2, 313–325, DOI 10.1007/BF02559972. MR1478783

[GR79] Hans Grauert and Reinhold Remmert, *Theory of Stein spaces*, Classics in Mathematics, Springer-Verlag, Berlin, 1979, Translated from the German by Alan Huckleberry, Reprint of the 1979 translation. MR2029201

[Kal92] Shulim Kaliman, *Isotopic embeddings of affine algebraic varieties into* $\mathbf{C}^n$, The Madison Symposium on Complex Analysis (Madison, WI, 1991), Contemp. Math., vol. 137, Amer. Math. Soc., Providence, RI, 1992, pp. 291–295, DOI 10.1090/conm/137/1190990. MR1190990

[KK08a] Shulim Kaliman and Frank Kutzschebauch, *Criteria for the density property of complex manifolds*, Invent. Math. **172** (2008), no. 1, 71–87, DOI 10.1007/s00222-007-0094-6. MR2385667

[KK08b] Shulim Kaliman and Frank Kutzschebauch, *Density property for hypersurfaces* $UV = P(\overline{X})$, Math. Z. **258** (2008), no. 1, 115–131, DOI 10.1007/s00209-007-0162-z. MR2350038

[KK10] Shulim Kaliman and Frank Kutzschebauch, *Algebraic volume density property of affine algebraic manifolds*, Invent. Math. **181** (2010), no. 3, 605–647, DOI 10.1007/s00222-010-0255-x. MR2660454

[KK11] Shulim Kaliman and Frank Kutzschebauch, *On the present state of the Andersén-Lempert theory*, Affine algebraic geometry, CRM Proc. Lecture Notes, vol. 54, Amer. Math. Soc., Providence, RI, 2011, pp. 85–122. MR2768636

[KK15a] Sh. Kaliman and F. Kutzschebauch, *On algebraic volume density property*, Transform. Groups **21** (2016), no. 2, 451–478, DOI 10.1007/s00031-015-9360-7. MR3492044

[KK15b] Shulim Kaliman and Frank Kutzschebauch, *On the density and the volume density property*, Complex analysis and geometry, Springer Proc. Math. Stat., vol. 144, Springer, Tokyo, 2015, pp. 175–186, DOI 10.1007/978-4-431-55744-9_12. MR3446755

[KZ99] Shulim Kaliman and Mikhail Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), no. 1, 53–95. MR1669174 (2000f:14099)

[Leu] Matthias Leuenberger, *(Volume) density property of a family of complex manifolds including the Koras-Russell cubic threefold*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3887–3902, DOI 10.1090/proc/13030. MR3513546

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76. MR0440554

[Ste72] Jean-Luc Stehlé, *Plongements du disque dans C^2* (French), Séminaire Pierre Lelong (Analyse), Année 1970–1971, Springer, Berlin, 1972, pp. 119–130. Lecture Notes in Math., Vol. 275. MR0397012

[TV00] Arpad Toth and Dror Varolin, *Holomorphic diffeomorphisms of complex semisimple Lie groups*, Invent. Math. **139** (2000), no. 2, 351–369, DOI 10.1007/s002229900029. MR1738449

[Var99] Dror Varolin, *A general notion of shears, and applications*, Michigan Math. J. **46** (1999), no. 3, 533–553, DOI 10.1307/mmj/1030132478. MR1721579

[Var01] Dror Varolin, *The density property for complex manifolds and geometric structures*, J. Geom. Anal. **11** (2001), no. 1, 135–160, DOI 10.1007/BF02921959. MR1829353

[Zai96] Mikhail G. Zaidenberg, *On exotic algebraic structures on affine spaces*, Geometric complex analysis (Hayama, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 691–714. MR1453650

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