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TRACIAL APPROXIMATION IS STABLE

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ABSTRACT. Let Ω be a class of unital C*-algebras such that Ω is closed under tensoring with matrix algebras and taking unital hereditary C*-subalgebras and such that $tsr(B) = 1$ and the Cuntz semigroup $\mathrm{Cu}(B)$ is almost unperforated for any $B \in \Omega$. Then $A \in \mathrm{TA}\Omega$ for any unital C*-algebra $A \in \mathrm{TA}(\mathrm{TA}\Omega)$. As an application, this result can be used to study tracially quasidiagonal C*-algebra extensions of tracial topological rank no more than one.

1. INTRODUCTION

The Elliott conjecture asserts that all nuclear, separable C*-algebras are classified up to isomorphism by an invariant, called the Elliott invariant. A first version of the Elliott conjecture might be said to have begun with the K-theoretical classification of AF-algebras in [2]. Since then, many classes of C*-algebras have been found to be classified by the Elliott invariant. Among them, one important class is the class of simple unital AH-algebras without dimension growth dealt with in [13] and [6]. A very important axiomatic version of the classification of AH-algebras without dimension growth was given by H. Lin in [15]. Instead of assuming inductive limit structure, he started with a certain abstract approximation property and showed that C*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras (presaged, in a more concrete way, in the decomposition result of [13]).

Following the notion of Lin of tracial approximation by interval algebras, G. A. Elliott and Z. Niu in [7] considered simple unital C*-algebras admitting tracial approximation by C*-algebras in just some given class. Let Ω be a class of unital C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$, is defined as follows. A simple unital C*-algebra A belongs to the class $\mathrm{TA}\Omega$ if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any element $a \geq 0$, there exist a projection $p \in A$ and a sub-C*-algebra B of A with $1_B = p$ and $B \in \Omega$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$,

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

The second author of the present note and Fang considered in [9] the nonsimple unital C*-algebras tracially approximated by a special class of C*-algebras. Let Ω be a class of unital C*-algebras. Then the class of C*-algebras which can be

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tracially approximated by C*-algebras in Ω, denoted again by $\mathrm{TA}\Omega$, was defined as follows. A unital C*-algebra A belongs to the class $\mathrm{TA}\Omega$ if, for any positive numbers $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any positive element a, and any integer $n > 0$, there exist a projection $p \in A$ and a sub-C*-algebra B of A with $B \in \Omega$ and $1_B = p$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$ (the definition of $f_{\sigma_2}^{\sigma_1}$ will appear in section 2).

Let $\mathfrak{I}^k$ ($\mathfrak{I}^0$ denotes the class of unital finite dimensional algebras). Denote the class of unital C*-algebras which are unital hereditary sub-C*-algebras of C*-algebras of the form $C(X) \otimes F$ where X is a k-dimensional finite CW complex and F is a finite dimensional C*-algebra. A is said to have tracial topological rank no more than k if $A \in \mathrm{TA}\mathfrak{I}^k$, thus $\mathrm{TR}(A) \leq k$ (this property was introduced by Lin in [15]).

Let

$$(\star) \qquad 0 \longrightarrow I \longrightarrow A \xrightarrow{\ \pi\ } A/I \longrightarrow 0$$

be a short exact sequence extension of C*-algebras. The extension $(\star)$ will be denoted by the pair (A, I).

A notion of tracially quasidiagonal extension was introduced by Hu, Lin, and Xue in [14], generalizing the notion of quasidiagonal extension. Also in [14] they proved that the class of C*-algebras of tracial topological rank zero is closed under passing to tracially quasidiagonal extensions. Let us state this result more formally.

Theorem 1.1 ([14]). *Let I and A be as in $(\star)$. Suppose that the extension (A, I) is tracially quasidiagonal. If $TR(I) = 0$ and $TR(A/I) = 0$, then $TR(A) = 0$.*

The authors introduce a class of C*-algebras with what they call property P_k. They show that property P_0 is equivalent to tracial topological rank zero. Using this result they show the theorem above.

In [27], Fang and Zhao proved the following theorem.

Theorem 1.2 ([27]). *Let I and A be as in $(\star)$. Suppose that the extension (A, I) is tracially quasidiagonal. If $TR(I) \leq 1$ and $TR(A/I) \leq 1$, then A has the property P_1.*

In this paper, we shall show that if Ω is a class of unital C*-algebras which is closed under tensoring with matrix algebras and taking unital hereditary C*-subalgebras, and $tsr(B) = 1$ and the Cuntz semigroup $\mathrm{Cu}(B)$ is almost unperforated for any $B \in \Omega$, then $A \in \mathrm{TA}\Omega$ for any unital C*-algebra $A \in \mathrm{TA}(\mathrm{TA}\Omega)$. In particular, we show that if A has property P_1, then $\mathrm{TR}(A) \leq 1$.

Using this result, in combination with a result of Fang and Zhao, we have the following result.

Theorem 1.3. *Let I and A be as in $(\star)$. Suppose that the extension (A, I) is tracially quasidiagonal. If $TR(I) \leq 1$ and $TR(A/I) \leq 1$, then $TR(A) \leq 1$.*

2. Preliminaries and definitions

Let a and b of two positive elements of a C*-algebra A. We shall write $[a] \leq [b]$ (cf. Definition 3.5.2 of [16]) if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in Her(a)$, v^*c, $cv \in A$, $vv^* = P_a$, where P_a is the range projection of a

in A^{**}, and $v^*cv \in Her(b)$. We shall write $[a] = [b]$ if $v^*Her(a)v = Her(b)$. Let n be a positive integer. Let us write $n[a] \leq [b]$ if there are n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in Her(b)$ such that $[a] \leq [b_i]$, $i = 1, 2, \ldots, n$.

Let $0 < \sigma_2 < \sigma_1 \leq 1$ be two positive numbers. Define

$$f_{\sigma_2}^{\sigma_1}(t) = \begin{cases} 1 & \text{if } t \geq \sigma_1, \\ \frac{t-\sigma_2}{\sigma_1-\sigma_2} & \text{if } \sigma_2 \leq t \leq \sigma_1, \\ 0 & \text{if } 0 < t \leq \sigma_2. \end{cases}$$

Let Ω be a class of unital C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by TAΩ, is defined as follows.

Definition 2.1 ([9])**.** A unital C*-algebra A belongs to the class TAΩ if for any positive numbers $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a of A, and any integer $n > 0$, there exist a nonzero projection $p \in A$ and a C*-subalgebra B of A with $B \in \Omega$ and $1_B = p$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$.

Let us recall the property P_k in [14].

Definition 2.2 ([14])**.** A unital C*-algebra A has the property P_k if the following holds: for any $\epsilon > 0$, any integer $n > 0$, any finite subset $F \subset A$, any nonzero positive element a of A, and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exist a projection $p \in A$ and a sub-C*-algebra C of A with $1_C = p$ and $\mathrm{TR}(C) \leq k$ such that:

(1) $\|px - xp\| < \epsilon$, for all $x \in F$;
(2) $pxp \in_\epsilon C$, for all $x \in F$;
(3) $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq 2[f_{\sigma_4}^{\sigma_3}(pap)]$.

We introduce the following definition.

Definition 2.3. Let Ω be a class of unital C*-algebras such that Ω is closed under tensoring with matrix algebras and under taking unital hereditary sub-C*-algebras. Let A be a unital C*-algebra. A has the property P_Ω if the following statement holds: for any $\epsilon > 0$, any integer $n > 0$, any finite subset $F \subset A$ containing a nonzero positive element a, and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exist a projection $p \in A$ and a sub-C*-algebra C of A with $1_C = p$ and $C \in$ TAΩ such that

(1) $\|px - xp\| < \epsilon$, for all $x \in F$;
(2) $pxp \in_\epsilon C$, for all $x \in F$; and
(3) $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq 2[f_{\sigma_4}^{\sigma_3}(pap)]$.

Let A be a C*-algebra, and let $M_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A), \phi_n)$ where $\phi_n : M_n(A) \to M_{n+1}(A)$ is given by $a \mapsto diag(a, 0)$. Let $M_\infty(A)_+$ (resp. $M_n(A)_+$) denote the positive elements in $M_\infty(A)$ (resp. $M_n(A)$). For positive elements a and b in $M_\infty(A)$, write $a \oplus b$ to denote the element $diag(a, b)$, which is also positive in $M_\infty(A)$. Given $a, b \in M_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $M_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalent class of a.

We define $\mathrm{W}(A) := M_\infty(A)_+/\sim$, and $\mathrm{Cu}(A) := W(A \otimes \mathcal{K})$ will be called the Cuntz semigroup of A. Observe that $\mathrm{Cu}(A)$ becomes a positively ordered abelian monoid when equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \lesssim b.$$

We remind the reader that an ordered abelian semigroup M is almost unperforated if whenever $(k+1)x \leq ky$ for $k \in \mathbb{N}$, it follows that $x \leq y$.

Theorem 2.4 ([7], [9])**.** *If the class Ω is closed under tensoring with matrix algebras or closed under taking unital hereditary* C^**-subalgebras, then $TA\Omega$ is closed under passing to matrix algebras or unital hereditary* C^**-subalgebras.*

Lemma 2.5 ([14])**.** *For any $0 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$, there exists $\eta = \eta(\delta_2, \delta_1) > 0$ such that if a and b are positive elements of A with $\|a - b\| < \eta$ and $\|a\|,\ \|b\| \leq 1$, then $[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\delta_2}^{\delta_1}(b)] \leq [b]$.*

The following lemmas are all well known.

Lemma 2.6 ([24], [23])**.** *$C(X)$ has stable rank one and the Cuntz semigroup is almost unperforated if X is a compact space with covering dimension one.*

Lemma 2.7 ([19])**.** *Let A be a* C^**-algebra with stable rank one and let a and b be positive elements of A. Then the following are equivalent:*

(1) *$[a] \leq [b]$,*
(2) *$a \lesssim b$, and*
(3) *there exist $x \in A$ such that $x^*x = a$ and $xx^* \in Her(b)$.*

3. The main results

Theorem 3.1. *Let Ω be a class of unital* C^**-algebras such that Ω is closed under tensoring with matrix algebras and passing to unital hereditary sub-*C^**-algebras, and such that $tsr(B) = 1$ and the Cuntz semigroup $Cu(B)$ is almost unperforated for any $B \in \Omega$. Then we have $A \in TA\Omega$ for any unital* C^**-algebra $A \in TA(TA\Omega)$.*

Proof. We need to show that for any positive numbers $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $F \subseteq A$ containing a nonzero positive element a, any integer $n > 0$, and any sufficiently small $\varepsilon > 0$, there exist a nonzero projection $p \in A$ and a sub-C^*-algebra B of A with $B \in \Omega$ and $1_B = p$, such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$.

Without loss of generality, we may assume that $F = \{x_1,\ x_2,\ \cdots,\ x_k\} \cup \{a\}$. Let $0 < \sigma_4 < \sigma_3 < \delta_{26} < \delta_{25} < \delta_{24} < \delta_{23} < \cdots < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$. Let $\varepsilon' > 0$ be such that $\max(n(n+2)\varepsilon', 10\varepsilon') < \eta$, $\max(n(n+2)\varepsilon', 10\varepsilon') < \varepsilon$, $\varepsilon'/5n(n+2) < \max(\delta, \delta')$ where $\eta = \min\{\eta(\sigma_4, \sigma_3), \eta(\delta_{26}, \delta_{25}), \cdots, \eta(\delta_2, \delta_1)\}$ as required in Lemma 2.5,

$$\delta = \min\{\delta(f_{\delta_6}^{\delta_5}, \varepsilon'/10), \delta(f_{\delta_20}^{\delta_19}, \varepsilon'/10), \delta(f_{\delta_{20}}^{\delta_{19}})^{1/2}, \varepsilon'/10)\}$$

and

$$\delta' = \min\{\delta'(f_{\delta_6}^{\delta_5}), \delta'(f_{\delta_{20}}^{\delta_{19}}), \delta'((f_{\delta_{20}}^{\delta_{19}})^{1/2})\}$$

as required in Lemma 2.5.11 in [16]. Since $A \in \mathrm{TA}(\mathrm{TA}\Omega)$ there exist a sub-C*-algebra C of A and a nonzero projection $p' \in A$ with $C \in \mathrm{TA}\Omega$ and $1_C = p'$, such that

(1′) $\|x_i p' - p' x_i\| < \varepsilon'$ for all $1 \le i \le k$,

(2′) $p' x_i p' \in_{\varepsilon'} C$, $p' a p' \in_{\varepsilon'} C$ for all $1 \le i \le k$,

(3′) $n(n+2)[f_{\delta_8}^{\delta_7}((1-p')a(1-p'))] \le [f_{\delta_{10}}^{\delta_9}(p'ap')]$.

By (1′) and (2′), there exist $\tilde{a} \in C$ and $x_i' \in C$ for all $1 \le i \le k$ such that

$$\|p' x_i p' - x_i'\| < \varepsilon',\ i = 1,\ 2,\ \cdots,\ k,\ \|p'ap' - \tilde{a}\| < \varepsilon'.$$

Let $G = \{x_1',\ x_2',\ \cdots x_k'\} \cup \{\tilde{a}\}$. Since $C \in \mathrm{TA}\Omega$, there exist a nonzero projection $p \in A$ and a sub-C*-algebra B of A with $B \in \Omega$ and $1_B = p$, such that

(1″) $\|x_i' p - p x_i{}'\| < \varepsilon'$, $\|\tilde{a}p - p\tilde{a}\| < \varepsilon'$ for all $1 \le i \le k$,

(2″) $p x_i' p \in_{\varepsilon'} B$, $p\tilde{a}p \in_{\varepsilon'} B$ for all $1 \le i \le k$,

(3″) $(n+1)(n+1)[f_{\delta_{14}}^{\delta_{13}}((p'-p)\tilde{a}(p'-p))] \le [f_{\delta_{18}}^{\delta_{17}}(p\tilde{a}p)]$.

Therefore we have

$$\begin{aligned}
&\|x_i p - p x_i\| \\
&\le \|x_i p' p - p' x_i p\| + \|p' x_i p' p - x_i' p\| + \|p x_i p' - p p' x_i\| + \|x_i' p - p x_i' p\| \\
&< 5\varepsilon' \le \varepsilon, 1 \le i \le k.
\end{aligned} \tag{1}$$

Since $\|p x_i p - p x_i' p\| < \varepsilon'$, $p x_i' p \in_{\varepsilon'} B$. Therefore as $\varepsilon' < \varepsilon/2$, we have

$$p x_i p \in_{\varepsilon} B. \tag{2}$$

We also have

$$\begin{aligned}
&n(n+2)[f_{\delta_6}^{\delta_5}((1-p)a(1-p))] \\
&\le n(n+2)[f_{\delta_8}^{\delta_7}((1-p')a(1-p'))] + n(n+2)[f_{\delta_8}^{\delta_7}((p'-p)a(p'-p))] \\
&\le [f_{\delta_{10}}^{\delta_9}(p'ap')] + n(n+2)[f_{\delta_8}^{\delta_7}((p'-p)a(p'-p))] \\
&\le [f_{\delta_{12}}^{\delta_{11}}(\tilde{a})] + n(n+2)[f_{\delta_{10}}^{\delta_9}((p'-p)\tilde{a}(p'-p))] \\
&\le n(n+2)[f_{\delta_{10}}^{\delta_9}((p'-p)\tilde{a}(p'-p))] + [f_{\delta_{14}}^{\delta_{13}}((p'-p)\tilde{a}(p'-p))] + [f_{\delta_{14}}^{\delta_{13}}(p\tilde{a}p)] \\
&\le (n(n+2)+1)[f_{\delta_{14}}^{\delta_{13}}((p'-p)\tilde{a}(p'-p))] + [f_{\delta_{14}}^{\delta_{13}}(p\tilde{a}p)] \\
&\le [f_{\delta_{16}}^{\delta_{15}}(p\tilde{a}p)] + [f_{\delta_{16}}^{\delta_{15}}(p\tilde{a}p)] \\
&\le 2[f_{\delta_{18}}^{\delta_{17}}(pap)](*).
\end{aligned}$$

Therefore we have

$$n(n+2)[f_{\delta_6}^{\delta_5}((1-p)a(1-p))] \le 2[f_{\delta_{18}}^{\delta_{17}}(pap)].$$

There exists a partial isometry $v \in M_{n(n+2)}(A)$ such that

$$v^*v = diag(f_{\delta_6}^{\delta_5}((1-p)a(1-p)) \otimes 1_{n(n+2)})$$

and

$$vv^* \in Her(diag(f_{\delta_{18}}^{\delta_{17}}(pap), f_{\delta_{18}}^{\delta_{17}}(pap), 0, \cdots, 0)).$$

Let

$$e = diag(f_{\delta_{20}}^{\delta_{19}}(pap), f_{\delta_{20}}^{\delta_{19}}(pap), 0, \cdots, 0).$$

We have

$$vv^* = e^{1/2}vv^*e^{1/2}.$$

Let $H = \{(1-p)a(1-p),\ pap,\ v_{i,j},\ 1 \le i \le n(n+2),\ 1 \le j \le n(n+2)\}$. Using the same argument as above there exists a sub-C*-algebra E of A with $E \in \Omega$ and $1_E = q$ such that

(1‴) $\|yq - qy\| < \delta/n(n+2)$ for all $y \in H$,

(2‴) $qyq \in_{\delta/n(n+2)} E$ for all $y \in H$,

(3‴) $n[f_{\delta_2}^{\delta_1}((1-q)(1-p)a(1-p)(1-q))] \le 2[f_{\delta_4}^{\delta_3}(q(1-p)a(1-p)q)]$.

By (1‴) and (2‴) there exist $a',\ b' \in E$ such that

$$\|q(1-p)a(1-p)q - a'\| < \delta,\ \|qpapq - b'\| < \delta,$$

and

$$\|qv_{i,j}q - {v_{i,j}}'\| < \delta/n(n+2),\ 1 \le i \le n(n+2),\ 1 \le j \le n(n+2).$$

Set $v' = ({v_{i,j}}')_{i,j} \in M_{n(n+2)}(E)$. By functional calculus we have

$$\begin{aligned}
&\|diag(f_{\delta_8}^{\delta_7}(a') \otimes 1_{n(n+2)}) - v'^*v'\| \\
&\le \|diag(f_{\delta_8}^{\delta_7}(a') \otimes 1_{n(n+2)}) - diag(f_{\delta_8}^{\delta_7}(q(1-p)a(1-p)q) \otimes 1_{n(n+2)})\| \\
&+\|diag(f_{\delta_8}^{\delta_7}(q(1-p)a(1-p)q) \otimes 1_{n(n+2)}) \\
&\qquad\qquad -diag(qf_{\delta_8}^{\delta_7}((1-p)a(1-p))q \otimes 1_{n(n+2)})\| \\
&+\|diag(qf_{\delta_8}^{\delta_7}((1-p)a(1-p))q \otimes 1_{n(n+2)}) \\
&\qquad\qquad -diag(q \otimes 1_{n(n+2)})vv^*diag(q \otimes 1_{n(n+2)})\| \\
&+\|diag(q \otimes 1_{n(n+2)})vv^*diag(q \otimes 1_{n(n+2)}) - (qv_{i,j}q)_{i,j}(q{v_{i,k}}^*q)_{i,k}\| \\
&+\|(qv_{i,j}q)_{i,j}(q{v_{i,k}}^*q)_{i,k} - v'^*v'\| \\
&< \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 = \varepsilon'.
\end{aligned}$$

Therefore we have

$$\|diag(f_{\delta_8}^{\delta_7}(a') \otimes 1_{n(n+2)}) - v'^*v'\| < \varepsilon'.$$

By functional calculus we also have

$$\begin{aligned}
&\|(diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2} - v'v'^*\| \\
&\le \|(diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2} \\
&- (diag(f_{\delta_{20}}^{\delta_{19}}(qpapq) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(f_{\delta_{20}}^{\delta_{19}}(qpapq) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}\| \\
&+ \|(diag(f_{\delta_{20}}^{\delta_{19}}(qpapq) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(f_{\delta_{20}}^{\delta_{19}}(qpapq) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2} \\
&- (diag(qf_{\delta_{20}}^{\delta_{19}}(pap)q \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(qf_{\delta_{20}}^{\delta_{19}}(pap)q \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}\| \\
&+ \|(diag(qf_{\delta_{20}}^{\delta_{19}}(pap)q \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}v'v'^*(diag(qf_{\delta_{20}}^{\delta_{19}}(pap)q \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2} \\
&- q(diag(f_{\delta_{20}}^{\delta_{19}}(pap) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}qvv^*q(diag(f_{\delta_{20}}^{\delta_{19}}(pap) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}q\| \\
&+ \|q(diag(f_{\delta_{20}}^{\delta_{19}}(pap) \otimes 1_2,\ 0 \otimes 1_{n(n+2)-2}))^{1/2}qvv^*q(diag(f_{\delta_{20}}^{\delta_{19}}(pap) \otimes 1_2, 0 \otimes 1_{n(n+2)-2}))^{1/2}q \\
&- qvv^*q\| + \|qvv^*q - v'v'^*\| < \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 + \varepsilon'/5 = \varepsilon'.
\end{aligned}$$

Therefore we have

$$\|diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2})v'v'^*diag(f_{\delta_{20}}^{\delta_{19}}(b') \otimes 1_2, 0 \otimes 1_{n(n+2)-2}) - v'v'^*\| < \varepsilon'.$$

We have

$$
\begin{aligned}
&[diag(f_{\delta_8}^{\delta_7}(a')\otimes 1_{n(n+2)})]\\
&\le [f_{\delta_{10}}^{\delta_9}\circ f_{\delta_8}^{\delta_7}(v'^*v')]\\
&= [f_{\delta_{10}}^{\delta_9}\circ f_{\delta_8}^{\delta_7}(v'v'^*)]\\
&\le [f_{\delta_{14}}^{\delta_{13}}\circ (diag(f_{\delta_{20}}^{\delta_{19}}(b'), f_{\delta_{20}}^{\delta_{19}}(b'),0,\cdots,0)v'v'^*diag(f_{\delta_{20}}^{\delta_{19}}(b'), f_{\delta_{20}}^{\delta_{19}}(b'),0,\cdots,0))]\\
&\le [diag(f_{\delta_{20}}^{\delta_{19}}(b'), f_{\delta_{20}}^{\delta_{19}}(b'),0,\cdots,0)].
\end{aligned}
$$

So we have

$$n(n+2)[f_{\delta_8}^{\delta_7}(a')]\le 2[f_{\delta_{20}}^{\delta_{19}}(b')].$$

Since $\mathrm{diag}(f_{\delta_8}^{\delta_7}(a')\otimes 1_{n(n+2)})$, $\mathrm{diag}(f_{\delta_{20}}^{\delta_{19}}b'\otimes 1_2, 0\otimes 1_{n(n+2)-2})$ and v' are all in $M_{n(n+2)}(E)$, we consider the Cuntz comparison in $M_{n(n+2)}(E)$. Since $E\in\Omega$ and $\mathrm{Cu}(E)$ is almost unperforated, we have

$$(n+2)[f_{\delta_8}^{\delta_7}(a')]\le [f_{\delta_{20}}^{\delta_{19}}(b')].$$

Therefore

$$
\begin{aligned}
&n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))]\\
&\le n[f_{\delta_2}^{\delta_1}((1-q)(1-p)a(1-p)(1-q))]+n[f_{\delta_2}^{\delta_1}(q(1-p)a(1-p)q)]\\
&\le 2[f_{\delta_4}^{\delta_3}(q(1-p)a(1-p)q)]+n[f_{\delta_2}^{\delta_1}(q(1-p)a(1-p)q)]\\
&\le (n+2)[f_{\delta_4}^{\delta_3}(q(1-p)a(1-p)q)]\\
&\le (n+2)[f_{\delta_8}^{\delta_7}(a')]\\
&\le [f_{\delta_{20}}^{\delta_{19}}(b')]\\
&\le [f_{\delta_{22}}^{\delta_{21}}(q(pap)q)]\\
&\le [f_{\delta_{24}}^{\delta_{23}}(q(pap)q)]+[f_{\delta_{24}}^{\delta_{23}}((1-q)(pap)(1-q))]\\
&\le [f_{\delta_{26}}^{\delta_{25}}(pap)]\\
&\le [f_{\sigma_4}^{\sigma_3}(pap)].
\end{aligned}
$$

We have

$$n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))]\le [f_{\sigma_4}^{\sigma_3}(pap)]. \tag{3}$$

□

Corollary 3.2. *Let Ω be a class of unital C^*-algebra of tracial topological rank at most one. Then any unital C^*-algebra $A\in TA\Omega$ has tracial topological rank at most one.*

Proof. This follows from Theorem 3.1 and Theorem 2.4. □

Theorem 3.3. *Let Ω be a class of unital C^*-algebras such that Ω is closed under tensoring with matrix algebras and passing unital hereditary C^*-subalgebras, and $tsr(B)=1$ and the Cuntz semigroup $Cu(B)$ is almost unperforated for any $B\in\Omega$. Then $A\in TA\Omega$ for any unital C^*-algebra A with property P_Ω.*

Proof. The proof of this theorem is the same as the proof of Theorem 3.1. In fact, in the proof of Theorem 3.1, we also have (*) hold: if (3′) is replaced by $n(n+2)[f_{\delta_8}^{\delta_7}((1-p')a(1-p'))]\le 2[f_{\delta_{10}}^{\delta_9}(p'ap')]$, then the proof is the same as the proof of Theorem 3.1. □

Corollary 3.4. *Let A be a unital* C^**-algebra such that A has the property P_1. Then A has tracial topological rank at most one.*

Using the preceding result, combined with a result of Fang and Zhao studied as Theorem 1.2 above, we have the following result.

Theorem 3.5. *Let I and A be as in $(\star)$. Suppose the extension (A, I) is tracially quasidiagonal. If $TR(I) \leq 1$ and $TR(A/I) \leq 1$, then $TR(A) \leq 1$.*

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