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BOUNDED H^∞-CALCULUS FOR THE HYDROSTATIC STOKES OPERATOR ON L^p-SPACES AND APPLICATIONS

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ABSTRACT. It is shown that the hydrostatic Stokes operator on $L^p_{\overline{\sigma}}(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a cylindrical domain subject to mixed periodic, Dirichlet and Neumann boundary conditions, admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ for $p \in (1,\infty)$ of H^∞-angle 0. In particular, maximal $L^q - L^p$-regularity estimates for the linearized primitive equations are obtained.

1. INTRODUCTION

The primitive equations for the ocean and atmosphere are considered to be a fundamental model for geophysical flows; see e.g. the survey article [25]. The mathematical analysis of the primitive equations has been commenced by Lions, Teman and Wang in their articles [22–24]. In contrast to the Navier-Stokes equations, it is known that for space dimension three this set of equations is globally strongly well posed for arbitrarily large initial data in a subspace of H^1; see the celebrated result by Cao and Titi [4]. For an extension of these results to the L^p-context, see [15,16].

For the analysis of the non-linear primitive equations, the study of its linear part – referred to as *hydrostatic Stokes equations* given here in (2.2) below – is of great relevance, too. In particular, the construction of a unique, mild solution to the non-linear primitive equations in the L^p-setting relies crucially on the fact that the solution of the linearized equations is governed by an analytic semigroup T_p on $L^p_{\overline{\sigma}}(\Omega)$ for $1 < p < \infty$; for details see [15]. Here $L^p_{\overline{\sigma}}(\Omega)$ denotes the hydrostatic solenoidal subspace of $L^p(\Omega)$, the definition of which is recapped in (2.3) below. The generator A_p of this semigroup is called the *hydrostatic Stokes operator*, and also it has been shown in [15] that $-A_p$ is a sectorial operator in $L^p_{\overline{\sigma}}(\Omega)$ of angle 0.

For the classical linear Stokes equations, maximal $L^q - L^p$-regularity properties of its solution have many implications in the study of fluid dynamics, for instance when considering free boundary value problems (see e.g. [27]) or fluid-structure interaction problems (see e.g. [18],[9]). Hence, maximal $L^q - L^p$-regularity investigations

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for the classical Stokes operator have a long tradition starting with the pioneering articles of Solonnikov [29] and Giga [11]. In fact, maximal $L^p - L^p$-estimates were proved first by Solonnikov [29], while maximal mixed $L^q - L^p$-estimates were obtained first by Giga and Sohr [14] by proving boundedness of the imaginary powers of the Stokes operator; see [12], [13]. In addition, the existence of a bounded H^∞-calculus for the Stokes operator plays an important role in this context. For a recent survey of results in this direction we refer to [17].

Starting from the fact that the negative hydrostatic Stokes operator $-A_p$ in $L^p_{\overline{\sigma}}(\Omega)$ for $1 < p < \infty$ is a sectorial operator of spectral angle 0, it is an interesting question to ask whether $-A_p$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$. Here we consider the situation where the underlying domain is a cylindrical domain with laterally periodic boundary conditions and with Dirichlet and/or Neumann boundary conditions on the bottom and top parts of $\partial\Omega$.

In this note we give an affirmative answer to this question and show in particular that $-A_p$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with H^∞-angle equal to 0 by means of a perturbation argument. Our approach allows us further to prove sectoriality of $-A_p$ and the fact that A_p is the generator of an analytic semigroup on $L^p_{\overline{\sigma}}(\Omega)$ in a way which is much shorter than the one presented first in [15]. As a consequence, we obtain maximal $L^q - L^p$-regularity estimates for the linearized primitive equations.

Combining a recent result on the explicit description of the complex interpolation spaces $[L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\theta$ given in [16] with the existence of a bounded H^∞-calculus for $-A_p$ implies further that the domains of the fractional powers $(-A_p)^\theta$ can be characterized explicitly as Bessel potential spaces satisfying appropriate boundary conditions depending on the value of the interpolation parameter $\theta \in [0, 1]$.

We finally show that the hydrostatic Stokes semigroup satisfies global $L^p - L^q$-smoothing estimates similar to the ones known for the classical Stokes semigroup.

This note is organized as follows: After introducing the basic setting in the subsequent Section 2, Section 3 contains the main results of this note concerning the existence of a bounded H^∞-functional calculus for $-A_p$ on $L^p_{\overline{\sigma}}(\Omega)$, the maximal L^q-regularity estimates for the linearized primitive equations, the characterization of the domains of the fractional powers of $-A_p$ as well as the $L^p - L^q$-smoothing properties of the hydrostatic Stokes semigroup. We then present the proofs of the main results in Section 4.

2. Preliminaries

For $a, b \in \mathbb{R}$ with $a < b$ consider the domain $\Omega = G \times (a, b) \subset \mathbb{R}^3$ with $G = (0, 1) \times (0, 1)$. The original primitive equations for the velocity in the isothermal setting given in [22–24], i.e. omitting temperature and salinity, can be equivalently reformulated as

$$
\left\{
\begin{array}{rll}
\partial_t v + v \cdot \nabla_H v + w(v) \cdot \partial_z v - \Delta v + \nabla_H \pi_s & = f, & \text{in } \Omega \times (0, T), \\
\mathrm{div}_H \overline{v} & = 0, & \text{in } \Omega \times (0, T), \\
v(0) & = v_0, & \text{in } \Omega.
\end{array}
\right. \tag{2.1}
$$

Here $v = (v_1, v_2)$ denotes the horizontal velocity of the fluid and π_s the surface pressure. Moreover, denoting the horizontal coordinates by $(x, y) \in G$ and the vertical one by $z \in (a,b)$, we use the notation

$$\nabla_H = (\partial_x, \partial_y)^T, \quad \operatorname{div}_H v = \partial_x v_1 + \partial_y v_2 \quad \text{and} \quad \overline{v} := \frac{1}{b-a}\int_a^b v(\cdot,\cdot,\xi)d\xi,$$

whereas Δ denotes the three dimensional Laplacian. The vertical component of the velocity $w = w(v)$ is determined by

$$w(x,y,z) = -\int_a^z \operatorname{div}_H v(x,y,\xi)d\xi.$$

Furthermore, the full pressure π is determined by the surface pressure $\pi_s \colon G \to \mathbb{R}$, since in the original model one has $\partial_z \pi = 0$. For details, we refer to [15].

The linearization of equation (2.1) comprises the *hydrostatic Stokes equations*, which are given by

$$\left\{\begin{aligned} \partial_t v - \Delta v + \nabla_H \pi_s &= f, && \text{in } \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} &= 0, && \text{in } \Omega \times (0,T), \\ v(0) &= v_0 && \text{in } \Omega. \end{aligned}\right. \tag{2.2}$$

The name 'hydrostatic Stokes equations' is motivated by the assumption of a hydrostatic balance when deriving the full primitive equations; see [22–24] for details. The equations (2.2) are supplemented by the mixed boundary conditions on

$$\Gamma_a = G \times \{a\}, \quad \Gamma_b = G \times \{b\} \quad \text{and} \quad \Gamma_l = \partial G \times (a,b),$$

i.e. the bottom, upper and lateral parts of the boundary $\partial\Omega$, respectively, given by

$$\begin{aligned} v, \pi_s \text{ are periodic on } \Gamma_l \times (0,\infty), \\ v = 0 \text{ on } \Gamma_D \times (0,\infty) \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N \times (0,\infty), \end{aligned}$$

where Dirichlet, Neumann and mixed boundary conditions are given by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_a \cup \Gamma_b) \setminus \Gamma_D.$$

Note that in the literature several sets of boundary conditions are considered. In [22, equation (1.37) and (1.37)'] Dirichlet and mixed Dirichlet-Neumann boundary conditions are considered, respectively, while in [4] Neumann boundary conditions are assumed. This system was first analyzed by Ziane in [31, 32] within the L^2 context.

In the following, horizontal periodicity is modeled by the function spaces $C^\infty_{per}(\Omega)$ and $C^\infty_{per}(G)$ defined as in [15, Section 2], where smooth functions are periodic only with respect to x, y coordinates and not in the z direction.

For $p \in (1,\infty)$ and $s \in [0,\infty)$ one defines the spaces

$$H^{s,p}_{per}(\Omega) := \overline{C^\infty_{per}(\Omega)}^{\|\cdot\|_{H^{s,p}(\Omega)}} \quad \text{and} \quad H^{s,p}_{per}(G) := \overline{C^\infty_{per}(G)}^{\|\cdot\|_{H^{s,p}(G)}},$$

where $H^{0,p}_{per} := L^p$. Here $H^{s,p}(\Omega)$ denotes the Bessel potential spaces, which are defined as restrictions of Bessel potential spaces on the whole space; compare e.g. [30, Definition 3.2.2]. It is well known that the space $H^{s,p}(\Omega)$ coincides with the classical Sobolev space $W^{m,p}(\Omega)$ provided $s = m \in \mathbb{N}$.

Similarly to the case of the Stokes equation, for $1 < p < \infty$ we consider a solenoidal subspace of $L^p(\Omega)^2$, which, following the approach developed in [15, Sections 3 and 4], is defined by

$$L^p_{\overline{\sigma}}(\Omega) = \overline{\{v \in C^\infty_{per}(\Omega)^2 : \operatorname{div}_H \overline{v} = 0\}}^{L^p(\Omega)^2}. \tag{2.3}$$

Then there exists a continuous projection P_p, called the *hydrostatic Helmholtz projection*, from $L^p(\Omega)^2$ onto $L^p_{\overline{\sigma}}(\Omega)$. Using the decomposition

$$v = \overline{v} + \tilde{v}, \text{ where } \tilde{v} := v - \overline{v}, \quad \tilde{v} = \overline{v} \text{ if and only if } v = 0,$$

we have

$$P_p v = Q_p \overline{v} + \tilde{v}, \quad \text{where} \quad Q_p : L^p(G)^2 \to L^p_\sigma(G),$$

and Q_p denotes the classical Helmholtz projection onto the solenoidal space $L^p_\sigma(G)$ on the unit square with horizontal periodicity. The projector onto the horizontal gradient fields is given by

$$(\mathbb{1} - P_p)v = (\mathbb{1} - Q_p)\overline{v} = \nabla_H \Delta_H^{-1} \mathrm{div}_H \overline{v},$$

where Δ_H denotes the two dimensional Laplacian defined on $H^{2,p}_{per}(G)^2$. Its inverse is considered as an operator in $L^p_0(G) = \{\overline{v} \in L^p(G)^2 \mid \int_G \overline{v} = 0\}$. Here, $\nabla_H \Delta_H^{-1} \mathrm{div}_H$ denotes actually the closure of the composition which is a bounded operator in $L^p(G)^2$. This projector can also be represented in terms of Riesz transforms on the torus, which can be defined by means of Fourier series; compare [26, Chapter 9], where a slightly more general situation is considered. In particular, P_p annihilates the pressure term $\nabla_H \pi_s$.

Following [15], we define the *hydrostatic Stokes operator* A_p in $L^p_{\overline{\sigma}}(\Omega)$ by

$$A_p v := P_p \Delta v, \quad D(A_p) := \{v \in H^{2,p}_{per}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\, v\big|_{\Gamma_D} = 0\} \cap L^p_{\overline{\sigma}}(\Omega).$$

Recall that $\Gamma_D \neq \emptyset$ means that Dirichlet conditions are imposed on either Γ_a, Γ_b or $\Gamma_a \cup \Gamma_b$ with Neumann conditions on the remaining part of $\Gamma_a \cup \Gamma_b$. Considering the case $\Gamma_D = \Gamma_a$ and $\Gamma_N = \Gamma_b$ it was proved in [15, Sections 3 and 4] that $-A_p$ is invertible and sectorial with spectral angle zero. This carries over to the situations where $\Gamma_D \neq \emptyset$. For $\Gamma_D = \emptyset$, the Laplacian and the projector P_p actually commute, and therefore $A_p v = \Delta v$ for $v \in D(A_p)$. In particular, for $\Gamma_D = \emptyset$, zero is an eigenvalue of A_p. Since the definition of sectoriality requires injectivity (compare e.g. [5, Definition 1.1]), we consider the operator $-A_p + \nu$ for some $\nu > 0$ which is – as it turns out – sectorial. Note that from sectoriality of $-A_p + \nu$ and the compact embedding $D(A_p) \subset H^{2,p}(\Omega)^2 \hookrightarrow L^p(\Omega)^2$, it follows that the spectrum of A_p is purely discrete.

Let us recall from [5] or [21] that a sectorial operator A in a Banach space X with spectral angle ϕ_A is said to admit a *bounded H^∞-calculus* or $A \in \mathcal{H}^\infty(X)$ if there exist an angle $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that

$$\|f(A)\| \leq K_\phi \|f\|_\infty^\phi, \quad f \in H_0(\Sigma_\phi). \tag{2.4}$$

Here $\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$, $\|f\|_\infty^\phi =: \sup\{\|f(\lambda)\| : |\arg \lambda| < \phi\}$, and $H_0(\Sigma_\phi)$ and

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \quad f \in H_0(\Sigma_\phi)$$

are defined as in [5, Sections 1.4 and 2.4] with line integral along a negatively oriented path Γ surrounding Σ_ϕ. The *H^∞-angle of A* is given by $\phi_A^\infty := \inf\{\phi > \phi_A \mid (2.4) \text{ is valid}\}$.

The more particular class $\mathcal{RH}^\infty(X)$ of operators admitting an $\mathcal{R}$-bounded H^∞-calculus with angle $\phi_A^{\mathcal{R}\infty}$ is defined as in [5, Definition 4.9] or [21]. For the notion of

$\mathcal{R}$-boundedness, we refer e.g. to [5, Definition 3.1] or [21]. In general, $\mathcal{RH}^\infty(X) \subset \mathcal{H}^\infty(X)$; however, if the Banach space X satisfies the property (α), both notions agree and angles coincide (compare e.g. [20, Theorem 5.3.1]).

For $1 < q < \infty$ and $0 < T \leq \infty$ consider in a Banach space X the abstract Cauchy problem

$$u'(t) + Au(t) = f(t), \quad t \in (0,T), \quad u(0) = u_0, \tag{2.5}$$

for closed operator A in X and $u_0 \in X_\gamma = (X, D(A))_{1/q',q}$, where $1/q' + 1/q = 1$ and $(\cdot,\cdot)_{1/q',q}$ denotes the real interpolation functor. This problem admits *maximal L^q-regularity* or $A \in \mathcal{M}_q(0,T;X)$ if for each $f \in L^q(0,T;X)$ and $u_0 \in X_\gamma$, the equation (2.5) admits a unique solution u satisfying $u \in W^{1,q}((0,T);X)$ and $Au \in L^q((0,T);X)$. In this case there exists a constant $C > 0$ such that

$$\|u\|_{W^{1,q}((0,T);X)} + \|Au\|_{L^q((0,T);X)} \leq C \left(\|f\|_{L^q((0,T);X)} + \|u_0\|_{X_\gamma} \right).$$

For details see [2] or [7].

3. Main results

We are now in a position to formulate the main results of this note.

Theorem 3.1. *Let $p \in (1,\infty)$ and $\nu \geq 0$. Then the operator $-A_p + \nu$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with $\phi^\infty_A = 0$ provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

The Banach space $L^p_{\overline{\sigma}}(\Omega)$ satisfies property (α), since $L^p(\Omega)^2$ as an L^p space on a finite measure space with values in a Hilbert space already has property (α). For details about this property we refer e.g. to [21] or [8]. Hence, we obtain the following corollary.

Corollary 3.2. *Let $p \in (1,\infty)$ and $\nu \geq 0$. Then the operator $-A_p + \nu$ admits a bounded $\mathcal{R}H^\infty$-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with $\phi^{\mathcal{R}\infty}_A = 0$ provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

The existence of the bounded $\mathcal{H}^\infty$-calculus for $-A_p$ implies that $D((-A_p)^\theta) = [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\theta$ for $\theta \in [0,1]$, where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor; compare e.g. [5, Theorem 2.5]. Since $D(A_p) \subset H^{2,p}(\Omega)^2$, we may conclude analogously to [15, Lemma 4.6 (a)] that $D(-A_p^\theta) \subset H^{2\theta,p}(\Omega)^2$. A suitable retract to compute the interpolation spaces in terms of boundary conditions has been given recently in [16, Section 4] by using an argument due to Amann [1] and by using a localization procedure. There, mixed boundary conditions have been considered. Adapting this to the present situation allows us to characterize the domains of $(-A_p)^\theta$ for $\theta \in [0,1]$ as follows.

Corollary 3.3. *Let $1 < p < \infty$ and $\theta \in [0,1]$ with $\theta \notin \{1/2p, 1/2 + 1/2p\}$. Then*

$$D((-A_p)^\theta) = \begin{cases} \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 1/2 + 1/2p < \theta \leq 1, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\big|_{\Gamma_D} = 0\}, & 1/2p < \theta < 1/2 + 1/2p, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)\}, & \theta < 1/2p. \end{cases}$$

As a further consequence, we obtain maximal $L^q - L^p$-regularity estimates for the linearized primitive equations.

Corollary 3.4. *Let $p, q \in (1,\infty)$ and $T \in (0,\infty)$. Then $-A_p \in \mathcal{M}_q((0,T); L^p_{\overline{\sigma}}(\Omega))$. In particular, A_p is the generator of an analytic semigroup on $L^p_{\overline{\sigma}}(\Omega)$. If $\Gamma_D \neq \emptyset$, then the above assertion also holds true for $T = \infty$.*

Maximal L^q-regularity implies that the solution operator $(f(\cdot), v_0) \mapsto v$ for the abstract Cauchy problem (2.5) defines a bounded isomorphism provided $f \in L^q((0,T); L^p_{\overline{\sigma}}(\Omega))$ and $v_0 \in (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1/q',q}$ with $1/q + 1/q' = 1$; compare e.g. [2, Section 1].

In [2, Section 4] Amann considered real interpolation spaces of second order elliptic operators on smooth domains subject to Dirichlet and/or Neumann boundary conditions on disjoint sets of the boundary and was able to characterize them in terms of boundary values. Using the same retract and co-retract as defined in [16, Section 4] this characterization carries over to the present situation.

Corollary 3.5. *Let $p, q \in (1,\infty)$ with $1/p + 2/q \notin \{1, 2\}$ and $1/q + 1/q' = 1$. Then*

$$(L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1/q',q}$$
$$= \begin{cases} \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\,|_{\Gamma_N} = 0, v\,|_{\Gamma_D} = 0\}, & 1 + 1/p < 2 - 2/q \leq 2, \\ \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\,|_{\Gamma_D} = 0\}, & 1/p < 2 - 2/q < 1 + 1/p, \\ \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)\}, & 0 < 2 - 2/q < 1/p. \end{cases}$$

Here $B^s_{p,q}(\Omega)$ denote Besov spaces defined by restrictions of functions in $B^s_{p,q}(\mathbb{R}^3)$ to Ω; compare e.g. [30, Definitions 3.2.2]. Since Besov spaces are compatible with localization by smooth cut-off functions (compare [30, Section 3.1.3]), the same retract and co-retract given in [16, Section 4] are well defined here and yield the result given above, where $B^s_{p,q,per}(\Omega)$ denotes the subspace of $B^s_{p,q}(\Omega)$ of horizontally periodic distributions. The space $B^s_{p,q,per}(\Omega)$ for $p, q \in (0, \infty]$ and $s \in \mathbb{R}$ can be defined as the closure of smooth horizontally periodic functions, i.e.

$$B^s_{p,q,per}(\Omega) := \overline{C^\infty_{per}(\Omega)}^{\|\cdot\|_{B^s_{p,q}(\Omega)}}$$

since $C^\infty_{per}(\overline{\Omega}) \subset B^s_{p,q}(\Omega)$ is dense; compare e.g. [30, Proposition 3.2.4].

Considering $(-A_p)^{1/2}$ we obtain from Corollary 3.3 the L^p-boundedness of the *hydrostatic Riesz transformations* associated with A_p.

Corollary 3.6. *Let $1 < p < \infty$. Then the hydrostatic Riesz transform*

$$R_p\colon L^p_{\overline{\sigma}}(\Omega) \to L^p(\Omega)^{2\times 2} \quad \textit{given by} \quad R_p v := \nabla(-A_p)^{-1/2} v$$

is bounded provided $\Gamma_D \neq \emptyset$.

We next state that the hydrostatic semigroup admits global $L^p - L^q$-smoothing properties.

Theorem 3.7. *Let $\Gamma_D \neq \emptyset$ and $p, q \in (1,\infty)$ such that $p \leq q$. Then there exists a constant $C > 0$ such that*

$$\|e^{tA_p} P_p f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\Omega)^2}, \qquad \textit{for } f \in L^p(\Omega)^2, \qquad t > 0,$$

$$\|\nabla e^{tA_p} P_p f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L^p(\Omega)^2}, \qquad \textit{for } f \in L^p(\Omega)^2, \qquad t > 0,$$

$$\|e^{tA_p} P_p \operatorname{div} f\|_{L^q(\Omega)^2} \leq C t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|f\|_{L^p(\Omega)^{2\times 2}}, \quad \textit{for } f \in L^p(\Omega)^{2\times 2}, \quad t > 0.$$

4. Proofs of the main theorems

The proof of Theorem 3.1 is based on perturbation techniques. We start by solving the equation (2.2) for the pressure and obtain the key observation that

$$A_p v = \Delta v + \nabla_H \Delta_H^{-1} \mathrm{div}_H D_z v\,|_{\Gamma_D}, \quad v \in D(A_p).$$

Here, for brevity we set

$$D_z v\,|_{\Gamma_D} = \tfrac{1}{b-a}\left(\gamma(b)\partial_z v\,|_{\Gamma_b} - \gamma(a)\partial_z v\,|_{\Gamma_a}\right),$$

and for $c \in \{a,b\}$ we set $\gamma(c) = 1$ if $\Gamma_c \subset \Gamma_D$ and $\gamma(c) = 0$ otherwise.

Define the Laplace operator Δ_p on cylindrical domains with mixed boundary conditions by

$$\Delta_p v := \Delta v, \quad D(\Delta_p) := \{v \in H^{2,p}_{per}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\, v\big|_{\Gamma_D} = 0\}.$$

Such cylindrical mixed boundary value problems have been studied in detail in [26, Sections 6 - 8]. In particular, the following result was proved there.

Lemma 4.1 ([26])**.** *Let $p \in (1,\infty)$ and $\nu \geq 0$. Then the operator $-\Delta_p + \nu$ admits a bounded $\mathcal{R}H^\infty$-calculus on $L^p(\Omega)$ with $\phi^{\mathcal{R}\infty}_{\Delta_p} = 0$ provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

Proof. For studying spectral properties of Δ_p one may consider the Laplacian on the three dimensional torus, and then by taking odd and even parts iteratively, one arrives at the setting discussed here.

Consider $\Delta_{p,\mathbb{T}^3}$ defined by

$$\Delta_{p,\mathbb{T}^3} u = \Delta u, \quad D(\Delta_{p,\mathbb{T}^3}) = H^{2,p}(\mathbb{T}^3),$$

where $H^{2,p}(\mathbb{T}^3)$ denotes periodic functions in $H^{2,p}((0,2\pi)^3)$ with periodic derivatives in all three directions.

For $-\Delta_{p,\mathbb{T}^3}$, its formal symbol is $|\xi|^2$, $\xi \in \mathbb{R}^3$, and this is a parameter elliptic symbol with parameter ellipticity angle zero in the classical sense; compare [26, Definition 6.4]. The notion of parameter-ellipticity is adapted in [26, Definition 7.4] to tori by considering restrictions of classical symbols in $\mathbb{R}^n$ to $\mathbb{Z}^n$, and in this sense $-\Delta_{p,\mathbb{T}^3}$ is parameter elliptic with parameter ellipticity angle $\phi_P = 0$ with discrete symbol $P(k) = |k|^2$, $k \in \mathbb{Z}^3$. Using this it follows from [26, Definition 7.13] and [26, Theorem 7.15] that $-\Delta_{p,\mathbb{T}^3} + \nu$, $\nu > 0$, with the domain specified above is a closed boundedly invertible operator in $L^p(\mathbb{T}^3)$. It follows from [26, Proposition 7.25] together with the reasoning on property (α) that $-\Delta_{p,\mathbb{T}^3} + \nu$, $\nu > 0$, admits a bounded $\mathcal{R}H^\infty$ calculus with angle $\phi^{\mathcal{R}\infty}_{\Delta_{p,\mathbb{T}^3}} \leq \phi_P = 0$. The actual formulation [26, Proposition 7.25] includes perturbations of the Laplacian, but these can be set to become zero.

In [26, Proposition 7.26] the corresponding result is given on the cube with mixed Dirichlet and Neumann boundary conditions. Using the reflexion arguments only in the z-direction one arrives at the situation considered here up to scaling of the length, which can be achieved by a linear coordinate transformation.

For the case $\Gamma_D \neq \emptyset$ one can shift the operator back as long as it is invertible to obtain the claim. $\square$

It follows from Lemma 4.1 that the domains of the fractional powers of the Laplacian can be computed by complex interpolation using arguments similar to the ones yielding Corollary 3.3.

Lemma 4.2. *Let $1 < p < \infty$ and $\theta \in [0,1]$ with $\theta \notin \{1/2p, 1/2+1/2p\}$. Then*

$$D((-\Delta_p)^\theta) = \begin{cases} \{v \in H^{2\theta,p}_{per}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 1/2+1/2p < \theta \le 1, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 : v\big|_{\Gamma_D} = 0\}, & 1/2p < \theta < 1/2+1/2p, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2\}, & \theta < 1/2p. \end{cases}$$

Proof of Theorem 3.1. For $\lambda \in \Sigma_\pi$ and $f \in L^p(\Omega)$ consider the resolvent problem for the hydrostatic Stokes equation which is given by

$$\begin{aligned} \lambda v - \Delta v + \nabla_H \pi &= f, \quad \text{in } \Omega, \\ \mathrm{div}_H \overline{v} &= 0, \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

subject to boundary conditions given by

$$v, \pi \text{ are periodic on } \Gamma_l,$$
$$v = 0 \text{ on } \Gamma_D \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N,$$

where Dirichlet, Neumann and mixed boundary conditions are given by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_a \cup \Gamma_b) \setminus \Gamma_D.$$

Consider first the case where $\Gamma_D \neq \emptyset$. Taking the vertical average of (4.1) yields

$$\begin{aligned} \lambda \overline{v} - \Delta_H \overline{v} + \nabla_H \pi &= \overline{f} + D_z v \,|_{\Gamma_D}, \\ \mathrm{div}_H \overline{v} &= 0, \end{aligned} \tag{4.2}$$

and applying div_H implies that

$$\nabla_H \pi = \nabla_H \Delta_H^{-1} \,\mathrm{div}_H \overline{f} + \nabla_H \Delta_H^{-1} \,\mathrm{div}_H D_z v \,|_{\Gamma_D}. \tag{4.3}$$

Inserting this expression for $\nabla_H \pi$ into (4.1) yields

$$\lambda v - \Delta v + \nabla_H \Delta_H^{-1} \mathrm{div}_H D_z v \,|_{\Gamma_D} = f - \nabla_H \Delta_H^{-1} \,\mathrm{div}_H \overline{f}.$$

For $f \in L^p(\Omega)$ we interpret this equation now as an operator equation in $L^p(\Omega)$ as

$$\lambda v - \Delta_p v - B_p v = P_p f,$$

where P_p denotes the hydrostatic Helmholtz projection as described above and

$$B_p v := -\nabla_H \Delta_H^{-1} \mathrm{div}_H D_z v \,|_{\Gamma_D} \qquad \text{with } D(B_p) = H^{1+1/p+\delta,p}(\Omega)^2,$$

for some $\delta \in (0, 1-1/p)$. Obviously, $D(\Delta_p) \subset D(B_p)$. Moreover, we have

$$D(B_p) \xrightarrow{D_z \cdot|_{\Gamma_D}} B^\delta_{pp}(G)^2 \cong W^{\delta,p}(G)^2 \hookrightarrow L^p(G)^2 \xrightarrow{-\nabla_H \Delta_H^{-1} \mathrm{div}_H} L^p(G)^2 \hookrightarrow L^p(\Omega)^2,$$

where $W^{\delta,p}(G)$ denotes the Sobolev-Slobodeckii space on G of order δ. Boundedness of the trace operator is discussed e.g. in [30, Theorem 2.7.7] for half spaces, and the assertion cited carries over to the situation considered here since horizontally periodic functions can be extended periodically onto a layer and then cut off. Thus, by interpolation, Young's inequality and bounded invertibility of Δ_p, we have

$$\|B_p v\|_{L^p(\Omega)^2} \le \varepsilon \|\Delta_p v\|_{L^p(\Omega)^2} + C_\varepsilon \|v\|_{L^p(\Omega)^2}, \quad v \in D(\Delta_p),$$

for $\varepsilon > 0$ arbitrarily small and some $C_\varepsilon > 0$. Therefore, B_p is a relatively bounded perturbation of Δ_p. It follows that $\Delta_p + B_p$ generates an analytic semigroup $e^{t(\Delta_p + B_p)}$ on $L^p(\Omega)$. Inserting the expression (4.3) for $\nabla\pi$ into (4.2) yields that

$\operatorname{div}_H \overline{v} = 0$ for $v = (\lambda - \Delta_p - B_p)^{-1} P_p f$. Thus $(\lambda - \Delta_p - B_p)^{-1}$ maps $L^p_{\overline{\sigma}}(\Omega)$ into itself and

$$(\lambda - \Delta_p - B_p)^{-1}\big|_{L^p_{\overline{\sigma}}(\Omega)} = (\lambda - A_p)^{-1}, \tag{4.4}$$

where $\lambda \in \Sigma_\pi$ with sufficiently large $|\lambda|$ so that both resolvents exist.

Perturbation results for the H^∞-calculus were studied by many authors in [3, 5, 6, 19, 21, 28]. In order to deal with the present situation we apply [21, Proposition 13.1] and [5, Proposition 2.7]. To this end, consider the additive perturbation B_p defined as above. Considering the case $\Gamma_D \neq \emptyset$, we first show that

(i) $0 \in \rho(\Delta_p)$ and $D(\Delta_p) \subset D(B_p)$,
(ii) $\|B_p v\|_{L^p(\Omega)^2} \leq C\|(-\Delta_p)^{1-\delta} v\|_{L^p(\Omega)^2}$ for all $v \in D(\Delta_p)$ and for some $\delta \in (0,1)$ and some $C > 0$.

We already observed that $0 \in \varrho(\Delta_p)$ provided $\Gamma_D \neq \emptyset$ and that $D(\Delta_p) \subset D(B_p)$. Moreover, by the above considerations

$$\|B_p v\|_{L^p(\Omega)^2} \leq C\|v\|_{H^{1+1/p+\delta,p}(\Omega)^2} \leq C\|(-\Delta_p)^{1-\delta} v\|, \quad v \in D(\Delta_p), \tag{4.5}$$

for some $C > 0$, since by Lemma 4.2, $D((-\Delta_p)^{1-\delta}) \subset H^{1+1/p+\delta,p}(\Omega)^2$ and Δ_p is boundedly invertible. Now, combining Lemma 4.1 with [21, Proposition 13.1] it follows that for every $\phi > 0$ there exists $\mu = \mu(\phi) \geq 0$ sufficiently large such that

$$-\Delta_p - B_p + \mu \in \mathcal{H}^\infty(L^p(\Omega)^2) \quad \text{with} \quad \phi^\infty_{-\Delta_p + B_p + \mu} \leq \phi. \tag{4.6}$$

Finally, combining assertion (4.4) with [5, Proposition 2.7] yields that

$$-A_p \in \mathcal{H}^\infty(L^p_{\overline{\sigma}}(\Omega)) \quad \text{with} \quad \phi^\infty_{-A_p} = 0,$$

since we already know that $-A_p$ is invertible and sectorial with spectral angle 0; compare [15, Sections 3 and 4].

The case $\Gamma_N = \Gamma_a \cup \Gamma_b$, that is, $\Gamma_D = \emptyset$, can be proven directly by verifying that $(-A_p + \nu)v = (-\Delta_p + \nu)v$ for $v \in D(A_p)$ and $\nu > 0$, and hence as part of an operator it inherits directly without perturbation arguments the properties of $-\Delta_p + \nu$. □

For the proof of Theorem 3.7 we first need the following result.

Lemma 4.3. *There is a continuous extension operator $S\colon L^p(\Omega) \to L^p(\mathbb{R}^3)$, $p \in (1,\infty)$, which is also continuous with respect to the $H^{s,p}$-norm for all $s \in [0,\infty)$. In particular, $[L^p(\Omega), H^{2,p}(\Omega)]_\theta = H^{2\theta,p}(\Omega)$ for $\theta \in [0,1]$.*

Proof. Extension operators of the above type can be constructed for instance by odd and even extension in the z-direction for Dirichlet and Neumann conditions, respectively, where for mixed conditions even extension with respect to the Neumann boundary can be performed first, arriving at even functions with Dirichlet boundary conditions. Then these even and odd extensions can again be extended to odd and even functions on the three dimensional torus, respectively, and therefrom a suitable extension operator can be constructed using periodic extension and a suitable smooth cut-off function. The assertion concerning the interpolation follows from the fact that S and

$$R\colon L^p(\mathbb{R}^3) \to L^p(\Omega), \quad R(v) = v\big|_\Omega$$

define co-retraction and retraction. □

Proof of Theorem 3.7. Having Lemma 4.3 in hand, the proof of Theorem 3.7 is now analogous to the proof of [10, Proposition 3.1] for $n = 3$. Setting $\alpha := 3(\frac{1}{p} - \frac{1}{q})$ and assuming $|\frac{1}{p} - \frac{1}{q}| < \frac{2}{3}$, the first inequality follows from

$$\begin{aligned}\|e^{tA_p} P_p f\|_{L^q(\Omega)^2} &\leq C\|e^{tA_p} P_p f\|_{H^{\alpha,p}(\Omega)^2} \leq C\|e^{tA_p} P_p f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \|e^{tA_p} P_p f\|^{\frac{\alpha}{2}}_{H^{2,p}(\Omega)^2} \\ &\leq C\|f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \|A_p e^{tA_p} P_p f\|^{\frac{\alpha}{2}}_{L^p(\Omega)^2} \leq C\|f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \left(t^{-1}\|f\|_{L^p(\Omega)^2}\right)^{\frac{\alpha}{2}} \\ &= Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)^2}, \quad t > 0,\end{aligned}$$

where we made use of the Sobolev embedding $H^{\alpha,p}(\Omega) \hookrightarrow L^q(\Omega)$, Lemma 4.3 and the fact that the semigroup e^{tA_p} is bounded analytic. Iterating, we obtain the first inequality for all $1 < p \leq q < \infty$. The other inequalities follow similarly by using Corollary 3.6 and by writing

$$\nabla = \nabla(-A_p)^{-1/2}(-A_p)^{1/2} \quad \text{and analogously} \quad P_p\mathrm{div} = (-A_p)^{1/2}(-A_p)^{-1/2}P_p\mathrm{div}\,.$$

□

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