

SELF-SIMILAR FUNCTIONS, FRACTALS AND ALGEBRAIC GENERICITY

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ABSTRACT. We introduce the class of everywhere like functions, which helps us to recover some known classes (such as that of everywhere surjective ones). We also study the algebraic genericity of this new class together with the class of fractal functions.

1. PRELIMINARIES

At the very beginning of the 2000's many authors became interested in the study of linearity within nonlinear settings or, in other words, the search for linear structures of mathematical objects enjoying certain special or, *a priori*, *unexpected* properties.

This trend has caught the eye of many experts in several fields of mathematics, from Linear Chaos to Real and Complex Analysis [4, 6, 7, 14, 24], passing through Set Theory [12, 17, 18] and Linear and Multilinear Algebra, or even Operator Theory [9], Topology, Measure Theory [7], and Abstract Algebra.

Recall that, as it nowadays is common terminology, a subset M of a topological vector space X is called *lineable* (respectively, *spaceable*) in X if there exists an infinite dimensional linear space (respectively, infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$. Moreover, given an algebra \mathfrak{A} , a subset $\mathcal{B} \subset \mathfrak{A}$ is said to be *algebrable* if there is a subalgebra \mathcal{C} of \mathfrak{A} such that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any generator of \mathcal{C} is infinite (see, e.g., [2, 4, 8, 25]). In case this set of generators were finite (for instance $N \in \mathbb{N}$), then the set would be said to be *N -algebrable*.

As we mentioned above, there have recently been many results regarding the linear structure of certain *special* subsets. One of the earliest results in this direction was provided by Gurariy, who showed that the set of Weierstrass' monsters is lineable [20]. Also, and more recently, Enflo et al. [14] proved that, for every infinite dimensional closed subspace X of $\mathcal{C}[0, 1]$, the set of functions in X having infinitely many zeros in $[0, 1]$ is spaceable in X (see, also, [10, 15]). A vast literature on this topic has been built during the last decade, and we refer the interested

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reader to the survey paper [8] or, for a much more detailed and thorough study, to the monograph [2].

This paper is focused on the study of some special types of surjections and self-similar functions. The set of surjections in $\mathbb{R}^{\mathbb{R}}$ has been intensely studied in the last years by many authors, and different new classes of surjections have been introduced. Here we are interested in the class of the so-called *everywhere surjective* functions.

Definition 1.1 (Everywhere surjective function [4, 25]). Let I_1, I_2 be non-void intervals in \mathbb{R} . We say that $f : I_1 \rightarrow I_2$ is everywhere surjective if, for any non-void interval $J \subseteq I_1$, we have $f(J) = I_2$.

The existence of such functions was, perhaps, first noticed by Lebesgue in [23] (see also [19]). In general, these functions are studied for the case $I_1 = I_2 = \mathbb{R}$ and their class is denoted *ES*. Moreover, a function f in *ES* can be chosen to enjoy, also, any of the following additional *pathologies*:

- (i) being 0 almost everywhere (see [8]),
- (ii) attaining every real value \mathfrak{c} times on every interval (see [16]),
- (iii) so that $f(P) = \mathbb{R}$ for any perfect subset P of \mathbb{R} (see [16]),
- (iv) or, for instance, f could be constructed in such a way that its graph intersects every subset of \mathbb{R}^2 with uncountable projection on the x -axis. These functions are also called *Jones functions* ([21]).

The class *ES* can also be studied in the complex plane by defining it within open subsets in \mathbb{C} in the natural fashion; that is, a function $f \in \mathbb{C}^{\mathbb{C}}$ would be called *ES* whenever $f(U) = \mathbb{C}$ for every open subset of \mathbb{C} .

This note is arranged as follows. Section 2 deals with the new concept of “everywhere like” function, in which several results (of independent interest) shall give us the main tools for Section 3, in which we study the lineability of the classes considered in Section 2. Fractal functions and their lineability will also be considered in Section 3. Finally, Section 4 focuses on algebraability of these fractal functions from Section 3, among other results.

2. CHARACTERISTIC AND “EVERYWHERE LIKE” FUNCTIONS

Let us begin this section giving some definitions and preliminary results that shall be needed later.

Definition 2.1. Define κ^+ as the smallest cardinal greater than κ .

Let us recall that [22, p. 79, Problem 4] states the following.

Proposition 2.2. *For every cardinal $\kappa \geq \aleph_0$ there are κ^+ distinct subsets of κ such that, if we choose any two of them, one always includes the other.*

Theorem 2.3. *Let $\{X_i, i \in I\}$ be a family of disjoint subsets of X , indexed by I with cardinality κ . For every ω subset of I define the characteristic function $\chi_\omega : X \rightarrow \{0, 1\}$ as*

$$\chi_\omega(x) = \begin{cases} 1, & \text{if } x \in X_i \text{ with } i \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Let V be a vector space over the field F and let $g: X \rightarrow V$ be a non-null function when restricted to each X_j . Then

$$\text{span}\{g(x)\chi_\omega(x) : \omega \subset I\}$$

has dimension greater than κ^+ .

Proof. By Proposition 2.2 there is a family \mathfrak{F} with κ^+ distinct subsets of I such that choosing any two of them, one includes the other. Let us prove that $\{g(x)\chi_\omega, \omega \in \mathfrak{F}\}$ is a linear independent set.

Let $\omega_1, \dots, \omega_n$ be different subsets of \mathfrak{F} . Without loss of generality, we may write

$$\omega_1 \subset \dots \subset \omega_n.$$

Consider the following null linear combination:

$$a_1g(x)\chi_{\omega_1} + \dots + a_ng(x)\chi_{\omega_n} = 0.$$

Since $\omega_n \neq \omega_{n-1}$ let $j \in \omega_n \setminus \omega_{n-1}$. Then for every $x \in X_j$,

$$0 = a_1g(x)\chi_{\omega_1}(x) + \dots + a_ng(x)\chi_{\omega_n}(x) = a_ng(x).$$

Since $g(x)$ is non-null over X_j , we obtain $a_n = 0$. By induction we obtain $0 = a_1 = a_2 = \dots = a_n$. □

Next, we introduce the concept of “everywhere like” function.

Definition 2.4. Let V be a vector space and $f, g: (0, 1) \rightarrow V$ be functions. We say that f is *everywhere like* g if for every open interval $(a, b) \subset (0, 1)$, there is an injection $h: (0, 1) \rightarrow (a, b)$ such that $f \circ h = g$.

Remark 2.5. Notice that f reproduces g in every interval (a, b) ; therefore it repeats every property that g possesses in its image, in every interval. For example, if $g: (0, 1) \rightarrow \mathbb{R}$ is surjective, then $f: (0, 1) \rightarrow \mathbb{R}$ is an *ES* function.

Next, we study a particular case of Definition 2.4, namely, functions that are *everywhere like an identity*.

Let \mathcal{A} be the set of sequences (a_n) of 0 and 1 with an infinite amount of coordinates equal to 1. It is clear that the cardinality of \mathcal{A} is \mathfrak{c} .

Definition 2.6. Let $(a_n) \in \mathcal{A}$. We define $S_{(a_n)}$ as the subset of the interval $(0, 1)$ formed by the numbers with one of the following decimal expansions:

- (1) $0, a_1c_1a_2c_2a_3 \dots,$
- (2) $0, b_1b_2 \dots b_n99a_1c_1a_2c_2a_3 \dots$

where b_i and c_i belong to $\{0, 1, \dots, 9\}$.

Let us now, in the following remark, state some basic properties of this set $S_{(a_n)}$ that we just defined.

Remark 2.7.

- (a) The numbers in $S_{(a_n)}$ have a unique decimal expansion, since they have an infinite number of 1’s in their expansion.
- (b) Since $a_i \in \{0, 1\}$, if a number in $S_{(a_n)}$ has a decimal expansion of the first type, then there are no couples of nines in this decimal expansion.
- (c) Since $a_i \in \{0, 1\}$, if a number in $S_{(a_n)}$ has a decimal expansion of the second type, then there are only finitely many couples of nines in its decimal expansion.

- (d) How to recognize a number of $S_{(a_n)}$? If there is a couple of nines in the decimal expansion of this number, then after the last couple of nines in this expansion, notice if the sequence a_n is shuffled with a sequence c_n .
- (e) If $(a_n) \neq (b_n)$ and $(a_n), (b_n) \in \mathcal{A}$, then $S_{(a_n)} \cap S_{(b_n)} = \emptyset$.

Definition 2.8.

(a) Let $(a_n) \in \mathcal{A}$. We define $f_{(a_n)}: (0, 1) \rightarrow (0, 1)$ by

$$f_{(a_n)}(x) = \begin{cases} 0, c_1c_2\dots, & \text{if } x = 0, a_1c_1a_2c_2a_3\dots \in S_{(a_n)}, \\ 0, c_1c_2\dots, & \text{if } x = 0, b_1\dots b_n99a_1c_1a_2c_2a_3\dots \in S_{(a_n)}, \\ 0, 5, & \text{if } x \notin S_{(a_n)}. \end{cases}$$

(b) We define $f_{\mathcal{A}}: (0, 1) \rightarrow (0, 1)$ by

$$f_{\mathcal{A}}(x) = \begin{cases} f_{(a_n)}(x), & \text{if } x \in S_{(a_n)} \text{ for some } (a_n) \in \mathcal{A}, \\ 0, 5, & \text{if } x \notin \bigcup_{(a_n) \in \mathcal{A}} S_{(a_n)}. \end{cases}$$

Lemma 2.9. *For every $(a_n) \in \mathcal{A}$ and every $(c, d) \subset (0, 1)$ there is an injection $h: (0, 1) \rightarrow (c, d) \cap S_{(a_n)}$ such that $f_{\mathcal{A}} \circ h = id: (0, 1) \rightarrow (0, 1)$.*

Proof. Considering the reduced decimal expansions of c and d , let k be the first position where they differ, that is,

$$c = 0, b_1b_2\dots b_{k-1}b_k\dots b_n\dots, \\ d = 0, b_1b_2\dots b_{k-1}d_k\dots,$$

with $b_k < d_k$. Since these decimal expansions are reduced, there is $n > k$ such that $b_n < 9$.

Define $h: (0, 1) \rightarrow (c, d) \cap S_{(a_n)}$ by

$$h(0, c_1c_2c_3\dots) = 0, b_1b_2\dots b_k\dots b_{n-1}99a_1c_1a_2c_2a_3c_3\dots$$

Note that h is well-defined because $b_n < 9$ implies that $c < h(x)$ and $b_k < d_k$ implies that $h(x) < d$. By Remark 2.7(a) h is an injection since each $h(x)$ has a unique decimal expansion.

Finally,

$$\begin{aligned} f_{\mathcal{A}} \circ h(0, c_1c_2c_3\dots) &= f_{\mathcal{A}}(0, b_1b_2\dots b_k\dots b_{n-1}99a_1c_1a_2c_2a_3c_3\dots) \\ &= f_{(a_n)}(0, b_1b_2\dots b_k\dots b_{n-1}99a_1c_1a_2c_2a_3c_3\dots) \\ &= 0, c_1c_2c_3\dots \end{aligned}$$

□

3. A VECTOR SPACE OF FUNCTIONS EVERYWHERE LIKE $g: (0, 1) \rightarrow V$

We denote by $\mathcal{P}(\mathcal{A})$ the set of all subsets of \mathcal{A} . Since the cardinality of \mathcal{A} is \mathfrak{c} , then the cardinality of $\mathcal{P}(\mathcal{A})$ is $2^{\mathfrak{c}}$.

Definition 3.1. Let $\omega \in \mathcal{P}(\mathcal{A})$. We denote by

- (1) S_{ω} the set $\bigcup_{(a_n) \in \omega} S_{(a_n)}$;
- (2) χ_{ω} the characteristic function of S_{ω} ; and

(3) $\chi_{(a_n)}$ the characteristic function of $S_{(a_n)}$.

Lemma 3.2. *Let V be a vector space over a field F and let $g: (0, 1) \rightarrow V$ be a non-null function. Then $(g \circ f_{\mathcal{A}})\chi_{\omega}: (0, 1) \rightarrow V$ is everywhere like g .*

Proof. Let $(c, d) \subset (0, 1)$. For $(a_n) \in \omega$ consider the injection $h: (0, 1) \rightarrow (c, d) \cap S_{(a_n)}$ given by Lemma 2.9. Then $f_{\mathcal{A}} \circ h = id: (0, 1) \rightarrow (0, 1)$ and

$$((g \circ f_{\mathcal{A}})\chi_{\omega}) \circ h(x) = g \circ f_{\mathcal{A}}(h(x))\chi_{\omega}(h(x)) = g(x).$$

□

Theorem 3.3. *Let V be a vector space over a field F and let $g: (0, 1) \rightarrow V$ be a non-null function. Then*

$$\text{span}\{(g \circ f_{\mathcal{A}})\chi_{\omega} : \omega \in \mathcal{P}(\mathcal{A})\}$$

has dimension greater than \mathfrak{c}^+ and it is formed by functions everywhere like a non-null multiple of g , with the exception of the null function.

Proof. Notice that

$$g \circ f_{\mathcal{A}}(x)\chi_{\omega}(x) = \sum_{(a_n) \in \omega} g \circ f_{\mathcal{A}}(x)\chi_{(a_n)}(x),$$

for every $\omega \in \mathcal{P}(\mathcal{A})$. Let

$$l(x) = b_1 g \circ f_{\mathcal{A}}(x)\chi_{\omega_1}(x) + \dots + b_s g \circ f_{\mathcal{A}}(x)\chi_{\omega_s}(x) \neq 0$$

be a non-null linear combination. It is clear that we may write

$$l(x) = \sum_{(a_n) \in \mathcal{A}} \alpha_{(a_n)} g \circ f_{\mathcal{A}}(x)\chi_{(a_n)}(x)$$

with $\alpha_{(a_n)} \in \mathbb{R}$. Since $l(x) \neq 0$, there is an $\alpha_{(d_n)} \neq 0$. Thus, for $(c, d) \subset (0, 1)$ we may consider the injection $h: (0, 1) \rightarrow (c, d) \cap S_{(d_n)}$ given by Lemma 2.9 to obtain $l(h(x)) = \alpha_{(d_n)} g(x)$. Therefore the linear span of the set $\{(g \circ f_{\mathcal{A}})\chi_{\omega} : \omega \in \mathcal{P}(\mathcal{A})\}$ contains, with the exception of the null function, only functions everywhere like a non-null multiple of g .

Notice that $g \circ f$ is non-null over each $S_{(a_n)}$. In fact, by Lemma 2.9, for every $(c, d) \subset (0, 1)$ there is an injection $h: (0, 1) \rightarrow (c, d) \cap S_{(a_n)}$ such that $f \circ h = id: (0, 1) \rightarrow (0, 1)$ and therefore $g \circ f \circ h = g: (0, 1) \rightarrow V$ and g is non-null. Besides, by Remark 2.7(e) the sets $S_{(a_n)}$, $(a_n) \in \mathcal{A}$, are disjoint subsets of $(0, 1)$. Since $\#\mathcal{A} = \mathfrak{c}$, it follows from Theorem 2.3 that the linear span of $\{(g \circ f)\chi_{\omega} : \omega \in \mathcal{P}(\mathcal{A})\}$ has dimension greater than \mathfrak{c}^+ . □

Remark 3.4. If the cardinality of the field F is smaller than $2^{\mathfrak{c}}$, then

$$\#\text{span}(\{(g \circ f_{\mathcal{A}})\chi_{\omega} : \omega \in \mathcal{P}(\mathcal{A})\}) = 2^{\mathfrak{c}},$$

because $\#\mathcal{P}(\mathcal{A}) = 2^{\mathfrak{c}}$ and the functions $g \circ f \chi_{\omega}$, for $\omega \in \mathcal{P}(\mathcal{A})$, are distinct. Since this vector space has cardinality greater than the cardinality of the field, its dimension coincides with the cardinality, which is $2^{\mathfrak{c}}$. In this case we do not need Theorem 2.3 to ensure that the dimension is bigger than \mathfrak{c}^+ .

Let us recall an important consequence of Theorem 3.3: The vector space is formed by functions that are, actually, fractals.

Definition 3.5. Let V be a vector space. A function $l: (0, 1) \rightarrow V$ is a *fractal function* if for every interval $(c, d) \subset (0, 1)$ there is an injection $h: (0, 1) \rightarrow (c, d)$ such that $l \circ h = l$.

This would be the first time in which the algebraic genericity of fractal functions is ever studied. However, in the past, some authors studied the algebraic genericity of functions of a similar nature, such as Peano curves (see [1]). The following lemma will be needed for our purposes.

Lemma 3.6. *For every $(c, d) \subset (0, 1)$ there is an injection $h: (0, 1) \rightarrow (c, d)$ such that $f \circ h = f: (0, 1) \rightarrow (0, 1)$ and $\chi_\omega \circ h = \chi_\omega: (0, 1) \rightarrow \{0, 1\}$, for every $\omega \in \mathcal{P}(\mathcal{A})$.*

Proof. As in the beginning of the proof of Lemma 2.9, consider the reduced decimal expansions of c and d :

$$c = 0, b_1b_2 \dots b_{k-1}b_k \dots b_n \dots,$$

$$d = 0, b_1b_2 \dots b_{k-1}d_k \dots,$$

where $b_k < d_k$ and $n > k$ is such that $b_n < 9$. Define $h: (0, 1) \rightarrow (c, d)$ by

$$h(0, c_1c_2c_3 \dots) = 0, b_1b_2 \dots b_k \dots b_{n-1}99c_1c_2c_3 \dots$$

Note that h is well-defined because $b_n < 9$ implies that $c < h(x)$ and $b_k < d_k$ implies that $h(x) < d$. Moreover h is injective since

$$h(x) = 0, b_1b_2 \dots b_k \dots b_{n-1}99 + \frac{x}{10^{(n+1)}},$$

for every $x \in (0, 1)$.

It is easy to see that, for each $(a_n) \in \mathcal{A}$, $x \in S_{(a_n)}$ if, and only if, $h(x) \in S_{(a_n)}$. Thus for $x \notin \bigcup_{(a_n) \in \mathcal{A}} S_{(a_n)}$, we have $h(x) \notin \bigcup_{(a_n) \in \mathcal{A}} S_{(a_n)}$ and so

$$f(h(x)) = 0, 5 = f(x).$$

Now, if $x \in S_{(a_n)}$, for some $(a_n) \in \mathcal{A}$, we may write $x = 0, d_1 \dots d_s 99a_1e_1a_2e_2 \dots$ and so

$$h(x) = 0, b_1b_2 \dots b_{k-1}99d_1 \dots d_s 99a_1e_1a_2e_2 \dots$$

Hence

$$f(h(x)) = 0, e_1e_2 \dots = f(x).$$

The assertion about χ_ω is proved analogously. □

Theorem 3.7. *Let V be a vector space over a field F and let $g: (0, 1) \rightarrow V$ be a non-null function. Then*

$$\text{span}\{(g \circ f_{\mathcal{A}})\chi_\omega : \omega \in \mathcal{P}(\mathcal{A})\}$$

is also a vector space of fractal functions.

Proof. Let $l \in \text{span}\{(g \circ f_{\mathcal{A}})\chi_\omega : \omega \in \mathcal{P}(\mathcal{A})\}$. Then l is of the form

$$l(x) = b_1g \circ f_{\mathcal{A}}(x)\chi_{\omega_1}(x) + \dots + b_s g \circ f_{\mathcal{A}}(x)\chi_{\omega_s}(x).$$

Considering the injection $h: (0, 1) \rightarrow (c, d)$ of Lemma 3.6 we have $l(h(x)) = l(x)$. □

4. A FRACTAL ALGEBRA

Let $K[x]$ be the polynomial ring over the field K and let (x) be the ideal generated by x in $K[x]$.

Theorem 4.1. *Let K be a field and $g: [0, 1] \rightarrow K$. Consider the set of functions*

$$W = \left\{ \sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) : p_{(a_n)}(x) \in (x) \right\}.$$

- (1) W is an algebra with the usual product and sum of functions.
- (2) If $g: [0, 1] \rightarrow K$ is non-null, then for every non-null $l \in W$, there exists $0 \neq p(x) \in (x)$ such that l is everywhere like $p \circ g$.
- (3) Every $l \in W$ is fractal.

Proof.

(1) Let $\sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x), \sum_{(a_n) \in A} q_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \in W$.

Notice that

$$\begin{aligned} & \sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \pm \sum_{(a_n) \in A} q_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \\ &= \sum_{(a_n) \in A} (p_{(a_n)} \circ g \circ f(x) \pm q_{(a_n)} \circ g \circ f(x))\chi_{(a_n)}(x) \\ &= \sum_{(a_n) \in A} (p_{(a_n)} \pm q_{(a_n)}) \circ g \circ f(x)\chi_{(a_n)}(x) \in W. \end{aligned}$$

Notice also that

$$\begin{aligned} & \sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \times \sum_{(a_n) \in A} q_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \\ &= \sum_{(a_n) \in A} (p_{(a_n)} \circ g \circ f(x))(q_{(a_n)} \circ g \circ f(x))\chi_{(a_n)}(x) \\ &= \sum_{(a_n) \in A} (p_{(a_n)} \times q_{(a_n)}) \circ g \circ f(x)\chi_{(a_n)}(x) \in W. \end{aligned}$$

- (2) If $l = \sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \neq 0$, then there is $p_{(d_n)} \circ g \circ f(x) \neq 0$. Therefore $p_{(d_n)} \neq 0$. Let $(c, d) \subset (0, 1)$ and pick an injection $h: (0, 1) \rightarrow (c, d) \cap S(d_n)$ of Lemma 2.9 to obtain $l \circ h = p_{(d_n)} \circ g: (0, 1) \rightarrow K$. Therefore l is everywhere like $p_{(d_n)} \circ g$.
- (3) Let $l = \sum_{(a_n) \in A} p_{(a_n)} \circ g \circ f(x)\chi_{(a_n)}(x) \in W$. Let $(c, d) \subset (0, 1)$ and choose an injection $h: (0, 1) \rightarrow (c, d)$ of Lemma 3.6. Notice that $l \circ h = l: (0, 1) \rightarrow K$. □

Corollary 4.2. *If K is an algebraically closed field and $g: [0, 1] \rightarrow K$ is surjective, then every function $l \in W$ from Theorem 4.1 is everywhere surjective.*

Proof. If $0 \neq p(x) \in (x)$, then $p(x) - c$ has a root for each $c \in K$. Let y be this root. Now since $g(x)$ is surjective, there is a $z \in (0, 1)$ such that $g(z) = y$ and $p(g(z)) = c$. Therefore $p \circ g: (0, 1) \rightarrow K$ is surjective.

By item 2 in Theorem 4.1, if $l \in W$ and $l \neq 0$, then l is everywhere like $p \circ g: (0, 1) \rightarrow K$, for a non-null $p(x) \in (x)$. Thus, l is an everywhere surjective function. □

Corollary 4.3. *The set of everywhere surjective functions (over an algebraically closed field) is algebrable.*

Remark 4.4. Let us recall that Corollary 4.3 was already obtained in [3, 5]. However, the proof obtained here is shorter (although not at all simpler) than the ones from [3, 5]. A simple way of obtaining this Corollary 4.3 (for finite algebrability), and without passing through fractal functions, would be to take any $N \in \mathbb{N}$ and then consider the set $S_N = \{f^N, f^{N+1}, \dots, f^{2N-1}\}$, in which f is any previously fixed everywhere surjective function in $\mathbb{C}^{\mathbb{C}}$. S_N has cardinality N , and it can be seen, without much difficulty, that if $\mathfrak{A}(S_N)$ stands for the algebra generated by the set S_N , then, for every $h \in S_N$, $h \notin \mathfrak{A}(S_N \setminus \{h\})$. Also, every non-zero element of $\mathfrak{A}(S_N)$ is everywhere surjective. Indeed, any function $0 \neq g \in \mathfrak{A}(S_N)$ can be written as $g = \sum_{i=1}^k \lambda_i h^{n_i}$, where the n_i 's are all integers bigger than or equal to N . Now take $\emptyset \neq U \subset \mathbb{C}$ any open subset of \mathbb{C} and any $w \in \mathbb{C}$. Let $d \in \mathbb{C}$ be a zero of the polynomial $\sum_{j=1}^n a_j z^{k_j} - w$. Since $f \in ES$, there is $c \in U$ so that $f(c) = d$. Thus, $g(c) = w$. Therefore we have obtained N -algebrability of ES . The linear independence of the functions in S_N is easy to show.

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