

## EXAMPLES OF CENTER CYCLICITY BOUNDS USING THE REDUCED BAUTIN DEPTH

ISAAC A. GARCÍA

(Communicated by Yingfei Yi)

ABSTRACT. There is a method for bounding the cyclicity of non-degenerate monodromic singularities of polynomial planar families of vector fields  $\mathcal{X}_\lambda$  which can work even in the case that the Poincaré first return map has associated a non-radical Bautin ideal  $\mathcal{B}$ . The method is based on the stabilization of the integral closures of an ascending chain of polynomial ideals in the ring of polynomials in the parameters  $\lambda$  of the family that stabilizes at  $\mathcal{B}$ . In this work we use computational algebra methods to provide an explicit example in which the classical procedure to find the Bautin depth of  $\mathcal{B}$  fails but the new approach is successful.

### 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

This work deals with real planar polynomial differential families  $\dot{x} = -y + P(x, y; \lambda)$ ,  $\dot{y} = x + Q(x, y; \lambda)$  being  $P, Q \in \mathbb{R}[x, y]$  polynomials without independent and linear terms depending also polynomially on  $\lambda \in \mathbb{R}^p$ , the finite number of parameters of the family. As usual we consider  $\mathcal{X}_\lambda = (-y + P(x, y; \lambda))\partial_x + (x + Q(x, y; \lambda))\partial_y$ , the associated vector field to the family. Under these hypotheses, the full family  $\mathcal{X}_\lambda$  possesses a non-degenerate isolated monodromic singularity at the origin (nearby trajectories of  $\mathcal{X}_\lambda$  rotate about it) having associated non-zero pure imaginary eigenvalues. Hence  $(0, 0)$  is either a center or a focus.

Throughout this work we focus on the local sixteenth Hilbert problem at the origin of  $\mathcal{X}_\lambda$ , or equivalently, we want to study the *cyclicity*  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0))$  of the origin in family  $\mathcal{X}_\lambda$ . Roughly speaking, the cyclicity  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0))$  is defined as the maximum number of small amplitude *limit cycles* (isolated periodic orbits) that can bifurcate from  $(0, 0)$  in family  $\mathcal{X}_\lambda$  as  $\lambda$  varies.

Let  $\Sigma$  be a transversal section to the flow of  $\mathcal{X}_\lambda$  in a neighborhood of the origin parameterized by  $h \in [0, \hat{h})$  with endpoint  $h = 0$  at the origin. Then we define, as usual, the *displacement map*  $d : \Sigma \times \mathbb{R}^p \rightarrow \Sigma$  via  $d(h; \lambda) = \Pi(h; \lambda) - h$ , where  $\Pi$  denotes the *Poincaré first return map* which is well defined on  $\Sigma$ . Since  $d(h; \lambda)$  is analytic at  $h = 0$ , we can expand in convergent power series  $d(h; \lambda) = \sum_{i \geq 3} v_i(\lambda)h^i$  about  $h = 0$ . The coefficients  $v_i$  in this Taylor series are called *Poincaré–Liapunov quantities*, and it can be seen that  $v_i \in \mathbb{R}[\lambda]$ . The reader can consult the book [12] for a detailed explanation about this construction.

---

Received by the editors March 3, 2016 and, in revised form, October 27, 2016.

2010 *Mathematics Subject Classification*. Primary 37G15, 37G10, 34C07.

*Key words and phrases*. Center, polynomial vector fields, Bautin ideal, cyclicity, limit cycle.

The author was partially supported by MINECO grant number MTM2014-53703-P and by CIRIT grant number 2014 SGR 1204.

Let  $\mathcal{B}$  be the *Bautin ideal* associated to the singularity at the origin of  $\mathcal{X}_\lambda$ . Thus  $\mathcal{B}$  is the ideal in the polynomial ring  $\mathbb{R}[\lambda]$  generated by all the Poincaré–Liapunov quantities. For each  $j \in \mathbb{N}$  with  $j \geq 3$ , we define  $\mathcal{B}_j = \langle v_3(\lambda), \dots, v_j(\lambda) \rangle$ , the ideal in  $\mathbb{R}[\lambda]$  generated by the first  $j-2$  Poincaré–Liapunov quantities. It is well-known that  $v_{2j} \in \mathcal{B}_{2j-1}$  for any subindex. Since  $\mathcal{B}$  is Noetherian, Hilbert’s basis theorem asserts that it is finitely generated; see for example [4]. Hence there is  $k \in \mathbb{N}$ , the smallest positive integer such that the ascending chain of ideals  $\mathcal{B}_3 \subseteq \mathcal{B}_5 \subseteq \dots \subseteq \mathcal{B}_k = \mathcal{B}$  stabilizes. The paper [8] refers to  $k$  as the *Bautin index* of  $\mathcal{B}$ .

Following [12] we recall here the notion of minimal basis of  $\mathcal{B}$  with respect to the ordered Poincaré–Liapunov quantities (and call it throughout this work the minimal basis formed by Poincaré–Liapunov quantities, omitting, for simplicity, any mention to the order). Now we give this definition and see from it that we attain uniqueness of the cardinality of this minimal basis.

**Definition 1.** The basis  $B = \{v_{j_1}(\lambda), \dots, v_{j_m}(\lambda)\}$  of  $\mathcal{B}$  with order  $3 \leq j_1 < \dots < j_m$  is called a *minimal basis formed by Poincaré–Liapunov quantities* when the following properties hold:

- (i)  $v_i(\lambda) \equiv 0$  for  $3 \leq i \leq j_1 - 1$  and  $v_{j_1}(\lambda) \not\equiv 0$ .
- (ii) For  $i \geq j_1 + 1$ , if  $v_i(\lambda) \notin \mathcal{B}_{i-1}$ , then  $v_i(\lambda) \in B$ .

The cardinality  $m$  of a minimal basis formed by Poincaré–Liapunov quantities of  $\mathcal{B}$  is termed the *Bautin depth* of  $\mathcal{B}$  in [10].

Hopf bifurcations at foci are easy to analyze. In case the origin is a focus of  $\mathcal{X}_{\lambda^\dagger}$  there is some integer  $\ell \geq 1$  (called the order of the focus) such that  $v_3(\lambda^\dagger) = \dots = v_{2\ell}(\lambda^\dagger) = 0$ , but  $v_{2\ell+1}(\lambda^\dagger) \neq 0$ . In that case  $d(h; \lambda^\dagger) = \sum_{i \geq 2\ell+1} v_i(\lambda^\dagger) h^i \sim h^{2\ell+1}$ , and therefore, for  $\|\lambda - \lambda^\dagger\| \ll 1$ , the cyclicity  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \leq \ell - 1$ ; see Proposition 6.1.2 in [12].

On the contrary, Hopf bifurcations from a center are harder to analyze and, from now on, we will focus on this kind of bifurcation. By definition,  $\mathcal{X}_{\lambda^*}$  possesses a center at the origin when  $d(h; \lambda^*) \equiv 0$  or, equivalently, when  $v_i(\lambda^*) = 0$  for all  $i \in \mathbb{N}$ . In the seminal paper [1] it is shown that, for parameters  $\|\lambda - \lambda^*\| \ll 1$ , we can express  $d(h, \lambda) = \sum_{j=1}^m v_{i_j}(\lambda)[1 + \psi_j(h, \lambda)]h^{i_j}$ , where  $m$  is the Bautin depth of  $\mathcal{B}$ ,  $\{v_{i_1}, \dots, v_{i_m}\}$  is a minimal basis of  $\mathcal{B}$ , and  $\psi_j$  are certain analytic functions; see also [12, 13]. From here it is not difficult to prove (see Lemma 6.1.6 and Theorem 6.1.7 of [12]) that  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \leq m - 1$  when  $\|\lambda - \lambda^*\| \ll 1$ .

*Remark 2.* Usually, the monodromic singularity is unfolded with the more general family  $\dot{\mathcal{X}}_\lambda = (-y + \lambda_1 x + P(x, y; \lambda))\partial_x + (x + \lambda_1 y + Q(x, y; \lambda))\partial_y$  that permits a strong focus at the origin when the additional parameter  $\lambda_1 \neq 0$ . Then one more limit cycle can bifurcate from the origin, and the above focus and center cyclicity bounds become now  $\text{Cyc}(\dot{\mathcal{X}}_\lambda, (0, 0)) \leq \ell$  and  $\text{Cyc}(\dot{\mathcal{X}}_\lambda, (0, 0)) \leq m$ , respectively.

For a field  $\mathbb{K}$  we denote by  $\mathbf{V}_{\mathbb{K}}(\mathcal{I}) \subset \mathbb{K}^p$  the affine variety associated to a polynomial ideal  $\mathcal{I}$  in  $\mathbb{K}[\mathbf{x}]$  with  $\mathbf{x} \in \mathbb{K}^p$ . Therefore  $\mathbf{V}_{\mathbb{K}}(\mathcal{I})$  is the set of common zeros in  $\mathbb{K}^p$  of all elements of  $\mathcal{I}$ . Also, the radical of  $\mathcal{I}$  is defined as the ideal  $\sqrt{\mathcal{I}} = \{r \in R : r^n \in \mathcal{I} \text{ for some } n \in \mathbb{N}\}$ . Finally,  $\mathcal{I}$  is called radical if  $\mathcal{I} = \sqrt{\mathcal{I}}$ . See [4] for more details.

The *center problem* associated to the origin of family  $\mathcal{X}_\lambda$  consists of determining the *center variety*  $\mathbf{V}_{\mathbb{R}}(\mathcal{B})$ . This means that  $\mathcal{X}_{\lambda^*}$  has a center at the origin if and only if  $\lambda^* \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ . We recall here that it may happen that  $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) = \mathbf{V}_{\mathbb{R}}(\mathcal{B}_r)$  for some integer  $r$ , but  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) \neq \mathbf{V}_{\mathbb{C}}(\mathcal{B}_r)$ .

The computation of the Bautin depth  $m$  of  $\mathcal{B}$  is, in general, a difficult task. Here we present a method that allows us to determine  $m$  in radical Bautin ideals under some conditions. It is proved in [7] for some kind of monodromic nilpotent singularities, but it can also be applied to our framework in the context of non-degenerate singularities.

**Theorem 3** ([7]). *Assume that the ideal  $\mathcal{B}_r \subseteq \mathcal{B}$ , has a minimal basis formed by Poincaré–Liapunov quantities of cardinality  $m$ . Suppose that  $\mathcal{B}_r$  is radical and that the equality of complex varieties  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) = \mathbf{V}_{\mathbb{C}}(\mathcal{B}_r)$  holds. Then  $\mathcal{B} = \mathcal{B}_r$  and, in particular,  $m$  is the Bautin depth of  $\mathcal{B}$ .*

Of course, Theorem 3 cannot be applied when  $\mathcal{B}$  is non-radical. In this case, an upper bound of  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0))$  not based on the computation of the Bautin depth can be established following the ideas of the work [8]. Before stating the results of [8] we need the following definition; see the books [9] and [14] for more details.

**Definition 4.** Let  $R$  be an arbitrary Noetherian ring and  $\mathcal{I}$  an ideal in  $R$ . An element  $r \in R$  is said to be integral over  $\mathcal{I}$  if there exist  $n \in \mathbb{N}$  and elements  $b_i \in \mathcal{I}^i$  such that  $r^n + b_1 r^{n-1} + b_2 r^{n-2} + \cdots + b_n = 0$ . The set of all elements of  $R$  that are integral over  $\mathcal{I}$  is called the integral closure of  $\mathcal{I}$  and is denoted by  $\bar{\mathcal{I}}$ . It follows that  $\bar{\mathcal{I}}$  is an ideal such that  $\mathcal{I} \subseteq \bar{\mathcal{I}} \subseteq \sqrt{\mathcal{I}}$ .

In [8], the authors introduce the so-called *reduced Bautin index*  $\bar{k}$  of  $\mathcal{B}$  defined as the smallest integer such that the integral closure  $\bar{\mathcal{B}}_{\bar{k}}$  of  $\mathcal{B}_{\bar{k}}$  is just  $\bar{\mathcal{B}}$ , the integral closure of the Bautin ideal  $\mathcal{B}$ . More specifically, [8] considers the ascending chain of integral closures  $\bar{\mathcal{B}}_3 \subseteq \bar{\mathcal{B}}_5 \subseteq \cdots \subseteq \bar{\mathcal{B}}_{\bar{k}} = \bar{\mathcal{B}}$ . The deep result in [8] concludes that  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \leq \bar{k}$ . Notice that  $\bar{k} \leq k$  since  $\mathcal{B}_j \subseteq \bar{\mathcal{B}}_j$  for any  $j \in \mathbb{N}$ .

Finally, in [5] the *reduced Bautin depth*  $\kappa \in \mathbb{N}$  of the Bautin ideal  $\mathcal{B}$  is defined as the cardinality of a minimal basis formed by Poincaré–Liapunov quantities of the ideal  $\mathcal{B}_{\bar{k}}$  where  $\bar{k}$  is the reduced Bautin index of  $\mathcal{B}$ . Also in [5] it is proved that the cyclicity bound  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \leq \kappa - 1$ . Observe that this is the best cyclicity bound since  $\kappa \leq \bar{k}$  and also  $\kappa \leq m$ .

The reduced Bautin depth  $\kappa$  is also difficult to obtain in general. From the computational point of view, in [5] is proved the following result to determine  $\kappa$  in some specific cases.

**Theorem 5** ([5]). *Assume that the ideal  $\mathcal{B}_r \subseteq \mathcal{B}$  has a minimal basis formed by Poincaré–Liapunov quantities of cardinality  $\kappa$ . Suppose that the integral closure  $\bar{\mathcal{B}}_r$  is radical and that the equality of complex varieties  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) = \mathbf{V}_{\mathbb{C}}(\mathcal{B}_r)$  holds. Then  $\kappa$  is the reduced Bautin depth of  $\mathcal{B}$ .*

The aim of this paper is to see how Theorem 3 and Theorem 5 complement each other because it is sometimes possible to use one but not the other. Thus we will provide an explicit example in the forthcoming Theorem 10 in which  $\mathcal{B}_r$  is not radical, but its integral closure  $\bar{\mathcal{B}}_r$  is radical. Therefore Theorem 3 cannot be used, but Theorem 5 can be applied. This example will show the usefulness of the theory presented in [5]. We want to mention here that, although there are two implementations for computing the integral closure of an ideal (`normalI` in the library `reesclos.lib` of SINGULAR and `integralClosure` in the `IntegralClosure` package of MACAULAY2), they are not optimized to perform computations with a large number of generators and variables. This has led us to change our strategy as

explained in §3 in order to achieve the desired example. But before that, §2 is devoted to obtaining the cyclicity of the polynomial normal form of a non-degenerate weak focus.

2. THE CYCLICITY OF THE POLYNOMIAL NORMAL FORM

Let us consider an analytic vector field  $\mathcal{Y} = (-y + \dots)\partial_x + (x + \dots)\partial_y$  having a non-degenerate weak focus at the origin. In [2] is shown the existence of a smooth and non-flat transformation  $\Phi(x, y) = (x + \dots, y + \dots)$  that brings the considered field  $\mathcal{Y}$  to the Birkhoff normal form  $\Phi_*\mathcal{Y} = (-y + S_1(r^2)x - S_2(r^2)y)\partial_x + (x + S_2(r^2)x + S_1(r^2)y)\partial_y$  where  $S_1$  and  $S_2$  are smooth functions of  $r^2 = x^2 + y^2$ .

The next result about the cyclicity of polynomial fields  $\mathcal{Y}$  of degree  $\max\{2m + 1, \deg S_2\}$  can be obtained by applying Theorem 3.

**Theorem 6.** *Let us consider a polynomial family*

$$(1) \quad \begin{aligned} \dot{x} &= -y + S_1(r^2; \lambda)x - S_2(r^2; \lambda)y, \\ \dot{y} &= x + S_2(r^2; \lambda)x + S_1(r^2; \lambda)y, \end{aligned}$$

where  $S_j \in \mathbb{R}[r^2]$  are polynomials on  $r^2 = x^2 + y^2$  without an independent term. Take  $S_2$  arbitrary and  $S_1(r^2; \lambda) = \sum_{i=1}^m \lambda_i r^{2i}$ . Then the Bautin ideal  $\mathcal{B}$  has a minimal basis formed by Poincaré–Liapunov quantities given by  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ . In particular, the Bautin depth of  $\mathcal{B}$  is  $m$ , and the cyclicity is  $\text{Cycl}(\mathcal{X}_\lambda, (0, 0)) = m - 1$ .

*Proof.* Let  $v_j \equiv \tilde{v}_j \pmod{\mathcal{B}_{j-1}}$ ; that is,  $\tilde{v}_j$  will denote the remainder of  $v_j$  upon division by a Gröbner basis of the ideal  $\mathcal{B}_{j-1}$ . It is easy to check that  $\tilde{v}_{2j} \equiv 0$  for any  $j \in \mathbb{N}$ ,  $\tilde{v}_{2i+1} = \lambda_i$  for any  $i = 1, \dots, m$ . Also we know that (1) has a center at the origin if and only if  $S_1 \equiv 0$ . Therefore  $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) = \mathbf{V}_{\mathbb{R}}(\mathcal{B}_{2m+1})$  where a minimal basis of  $\mathcal{B}_{2m+1}$  is

$$\{\tilde{v}_3, \tilde{v}_5, \dots, \tilde{v}_{2m+1}\} = \{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

But it remains to prove that actually  $\tilde{v}_{2i+1} \equiv 0$  when  $i > m$ , or equivalently that  $\mathcal{B} = \mathcal{B}_{2m+1}$ .

First we observe that  $\mathcal{B}_{2m+1}$  is a monomial ideal since it is generated by monomials. It is well known (see [4]) that monomial ideals are radical if and only if they are generated by square-free monomials. Since this is our case, we obtain that  $\mathcal{B}_{2m+1}$  is a radical ideal in the ring  $\mathbb{R}[\lambda]$ .

On the other hand,  $v_j \in \mathbb{R}[\lambda]$  where  $\lambda = (\lambda_1, \dots, \lambda_m, \nu)$  is  $\nu \in \mathbb{R}^n$ , the parameters associated to the coefficients of  $S_2$ . We know that any center at the origin of (1) with arbitrary  $\lambda^* = (0, \dots, 0, \nu^*) \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$  is of the form  $\dot{x} = -y(1 + S_2(r^2; \lambda^*))$ ,  $\dot{y} = x(1 + S_2(r^2; \lambda^*))$ ; hence it is time-reversible, i.e., invariant under the discrete symmetry  $(x, y, t) \mapsto (x, -y, -t)$ . For the strata of time-reversible centers, in [6] it is proved that  $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) = \mathbf{V}_{\mathbb{R}}(\mathcal{B}_{2m+1})$  implies the equality  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) = \mathbf{V}_{\mathbb{C}}(\mathcal{B}_{2m+1})$  of complex varieties.

From Theorem 3 we can conclude that  $\mathcal{B} = \mathcal{B}_{2m+1}$  and that the Bautin depth of  $\mathcal{B}$  is just  $m$ , hence  $\text{Cycl}(\mathcal{X}_\lambda, (0, 0)) \leq m - 1$ .

Since additionally the generators  $\lambda_i$  with  $i = 1, \dots, m$  are independent they can be chosen freely so that we can take them as  $|\lambda_1| \ll |\lambda_2| \ll \dots \ll |\lambda_m| \ll 1$  alternating signs as  $\lambda_j \lambda_{j+1} < 0$  for  $j = 1, \dots, m - 1$ . Then using standard arguments of bifurcation theory we deduce that  $\text{Cycl}(\mathcal{X}_\lambda, (0, 0)) = m - 1$ . □

## 3. CONSTRUCTION OF THE EXAMPLES

We consider the polynomial Birkhoff normal form

$$(2) \quad \begin{aligned} \dot{x} &= -y + S_1(r^2; \lambda)x - S_2(r^2; \lambda)y, \\ \dot{y} &= x + S_2(r^2; \lambda)x + S_1(r^2; \lambda)y, \end{aligned}$$

where  $S_j \in \mathbb{R}[r^2]$  are polynomials on  $r^2 = x^2 + y^2$  without independent term. Now we take  $S_1(r^2; \lambda) = \sum_{i=1}^n \mu_i(\lambda)r^{2i}$  with polynomial coefficients  $\mu_i \in \mathbb{R}[\lambda]$  and  $\lambda \in \mathbb{R}^p$  for  $i = 1, \dots, n$ . From the proof of Theorem 6 it follows that  $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) = \mathbf{V}_{\mathbb{R}}(\mathcal{B}_{2n+1})$  where  $\mathcal{B}_{2n+1} = \langle v_3(\lambda), v_5(\lambda), \dots, v_{2n+1}(\lambda) \rangle = \langle \mu_1(\lambda), \mu_2(\lambda), \dots, \mu_n(\lambda) \rangle$  and also that  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) = \mathbf{V}_{\mathbb{C}}(\mathcal{B}_{2n+1})$ .

Now we select the polynomials  $\mu_i(\lambda)$  such that  $\sqrt{\mathcal{B}_{2n+1}} = \bar{\mathcal{B}}_{2n+1}$ . This step is the most technical, and we must be careful in this selection.

**Definition 7.** Given two ideals  $\mathcal{J} \subset \mathcal{I}$ ,  $\mathcal{J}$  is said to be a *reduction* of  $\mathcal{I}$  if there exists a non-negative integer  $n$  such that  $\mathcal{I}^{n+1} = \mathcal{J}\mathcal{I}^n$ . It is known that if  $\mathcal{I}$  is finitely generated, then  $\mathcal{J}$  is a reduction of  $\mathcal{I}$  if and only if  $\bar{\mathcal{J}} = \bar{\mathcal{I}}$ ; see [9].

*Remark 8.* Recall that in a polynomial ring in at least three variables there exist prime ideals (non-maximal and not of height one) which can have very many generators. So we take  $p \geq 3$  and we construct  $\mathcal{P}$ , a prime ideal in  $\mathbb{R}[\lambda]$ , with  $\lambda \in \mathbb{R}^p$ , generated minimally by at least  $p + 2$  generators. By a theorem of Katz [11] (see also Theorem 8.7.3 of the book [9]),  $\mathcal{P}$  has a reduction ideal  $\mathcal{J}$  generated by at most  $p + 1$  generators. By the generator count,  $\mathcal{J} \neq \mathcal{P}$ , but the integral closure is  $\bar{\mathcal{J}} = \bar{\mathcal{P}} = \mathcal{P}$ , where the first equality follows from the characterization of reductions and the last because prime ideals are integrally closed. Therefore we conclude that  $\bar{\mathcal{J}}$  is a radical ideal since prime ideals are radical.

Here we describe a way to get such prime ideals  $\mathcal{P}$ . We will construct  $\mathcal{P}$  as the kernel of a polynomial map from the polynomial ring  $\mathbb{R}[\lambda]$  to the polynomial ring  $\mathbb{R}[\nu]$  where  $\nu \in \mathbb{R}^s$  and  $1 \leq s \leq p - 2$ . Thus we take  $\mathcal{P} = \ker f$ , a prime ideal in  $\mathbb{R}[\lambda]$ .

Turning to our original problem, we let  $\mathcal{B}_{2n+1}$  be an ideal reduction of  $\ker f$ . Therefore we know that  $\bar{\mathcal{B}}_{2n+1}$  will be radical as desired. It only remains to find the generators (at most  $p + 1$  according with Katz's theorem) of  $\mathcal{B}_{2n+1}$ ; see Remark 9 for the construction of an explicit example. Additionally we have to check that  $\mathcal{B}_{2n+1} \neq \sqrt{\mathcal{B}_{2n+1}}$ ; hence we cannot apply Theorem 3 to get the Bautin depth  $m$  of  $\mathcal{B}$ . Let  $\kappa \leq n$  be the cardinality of a minimal basis of  $\mathcal{B}_{2n+1}$  formed by Poincaré–Liapunov quantities. Using Theorem 5 we get that  $\kappa$  is just the reduced Bautin depth of the Bautin ideal  $\mathcal{B}$ .

*Remark 9.* Using MACAULAY2 we construct the following example with  $p = 4$ . Take  $\lambda = (a, b, c, d) \in \mathbb{R}^4$  and let  $f : \mathbb{R}[\lambda] \rightarrow \mathbb{R}[a, b]$  be defined by  $a \mapsto 2 + a^3 - a^2b$ ,  $b \mapsto b^4 - a^2b^2$ ,  $c \mapsto a^3b^3$  and  $d \mapsto a^4$ . Then  $\ker f$  is (may not be minimally)

generated by 6 elements. More specifically,  $\mathcal{P} := \ker f = \langle p_1, \dots, p_6 \rangle$  with

$$\begin{aligned}
 p_1(\lambda) &= -c^4 + 3bc^2d^2 + b^3d^3 + c^2d^3, \\
 p_2(\lambda) &= 16bc - 32abc + 24a^2bc - 8a^3bc + a^4bc + 16cd - 32acd \\
 &\quad + 24a^2cd - 8a^3cd + a^4cd + 16c^2d - 16ac^2d + 4a^2c^2d + c^3d \\
 &\quad + 16b^2d^2 - 16ab^2d^2 + 4a^2b^2d^2 - b^2cd^2 - 3bcd^3 - cd^4, \\
 p_3(\lambda) &= -16c^2 + 32ac^2 - 24a^2c^2 + 8a^3c^2 - a^4c^2 + 16bd^2 - 32abd^2 \\
 &\quad + 24a^2bd^2 - 8a^3bd^2 + a^4bd^2 - 16bcd^2 + 16abcd^2 - 4a^2bcd^2 \\
 &\quad + bc^2d^2 - 16cd^3 + 16acd^3 - 4a^2cd^3 - 2b^2d^4 - bd^5, \\
 p_4(\lambda) &= 16c^2 - 32ac^2 + 24a^2c^2 - 8a^3c^2 + a^4c^2 + 16c^3 - 16ac^3 \\
 &\quad + 4a^2c^3 - c^4 + 16b^2d - 32ab^2d + 24a^2b^2d - 8a^3b^2d + a^4b^2d \\
 (3) \quad &\quad - 16bcd^2 + 16abcd^2 - 4a^2bcd^2 + 3bc^2d^2 + c^2d^3 - b^2d^4, \\
 p_5(\lambda) &= 64b^2 - 192ab^2 + 240a^2b^2 - 160a^3b^2 + 60a^4b^2 - 12a^5b^2 \\
 &\quad + a^6b^2 + 64bd - 192abd + 240a^2bd - 160a^3bd + 60a^4bd \\
 &\quad - 12a^5bd + a^6bd - 60bc^2d + 60abc^2d - 15a^2bc^2d + 4bc^3d \\
 &\quad - 4b^3d^2 + 4ab^3d^2 - a^2b^3d^2 + 4c^3d^2 + 52b^2d^3 - 52ab^2d^3 \\
 &\quad + 13a^2b^2d^3 - 12b^2cd^3 - 4bd^4 + 4abd^4 - a^2bd^4 - 16bcd^4 - 4cd^5, \\
 p_6(\lambda) &= 256 - 1024a + 1792a^2 - 1792a^3 + 1120a^4 - 448a^5 \\
 &\quad + 112a^6 - 16a^7 + a^8 + 256c - 768ac + 960a^2c - 640a^3c \\
 &\quad + 240a^4c - 48a^5c + 4a^6c + 208c^2 - 416ac^2 + 312a^2c^2 \\
 &\quad - 104a^3c^2 + 13a^4c^2 + 192bcd^2 - 192abcd^2 + 48a^2bcd^2 \\
 &\quad - 13bc^2d^2 - 32d^3 + 64ad^3 - 48a^2d^3 + 16a^3d^3 - 2a^4d^3 \\
 &\quad + 176cd^3 - 176acd^3 + 44a^2cd^3 - c^2d^3 + 28b^2d^4 + 16bd^5 + d^6.
 \end{aligned}$$

Katz’s theorem guarantees the existence of a reduction  $\mathcal{J}$  of  $\mathcal{P}$  generated by at most five elements. We will generate  $\mathcal{J}$  by taking into account that any five sufficiently general elements of  $\mathcal{P}$  will do it. So, we ask MACAULAY2 for random rational numbers  $\ell_{ij} \in \mathbb{Q}$  to stick as coefficients in the five linear combinations

$$\mu_i(\lambda) = \sum_{j=1}^6 \ell_{ij} p_j(\lambda), \quad i = 1, \dots, 5.$$

With the matrix choice

$$(4) \quad L = (\ell_{ij}) = \begin{pmatrix} 5/2 & 3/4 & 6/7 & 3/5 & 1/5 & 1/2 \\ 5/4 & 1 & 7/9 & 1/2 & 5/4 & 2 \\ 8 & 2 & 1 & 8/5 & 1 & 10/9 \\ 3/4 & 2/3 & 9/4 & 3/5 & 2 & 3/4 \\ 5/3 & 10/7 & 1/2 & 5/4 & 3 & 10/9 \end{pmatrix},$$

the ideal  $\mathcal{J} = \langle \mu_1, \dots, \mu_5 \rangle$  generated by those five general elements turns out to be a reduction of  $\mathcal{P}$  since MACAULAY2 computations show that  $\mathcal{P}\mathcal{J} = \mathcal{P}^2$  with  $\mathcal{J} \subsetneq \mathcal{P}$ .

Finally, using for example SINGULAR, we check that  $\sqrt{\mathcal{J}} \neq \mathcal{J}$ . A MACAULAY2 code is the following:

```
R=QQ[a,b,c,d];
f=map(R,R,2+a^3-a^2*b,b^4-a^2*b^2,a^3*b^3,a^4);
P=ker f;
M=transpose mingens P; m=rank target M; LL=random(R^5,R^m);
J=ideal flatten entries (LL*M);
J==P
P*J==P^2
```

Now we can explicitly show the following example.

**Theorem 10.** *Let us consider a polynomial Birkhoff normal form (2) with parameters  $\lambda = (a, b, c, d) \in \mathbb{R}^4$ ,  $S_2$  arbitrary and  $S_1(r^2; \lambda) = \sum_{i=1}^5 \mu_i(\lambda)r^{2i}$  with polynomial coefficients  $\mu_i \in \mathbb{R}[\lambda]$ . Take the linear combinations  $\mu_i(\lambda) = \sum_{j=1}^6 \ell_{ij} p_j(\lambda)$  with rational coefficients  $\ell_{ij}$  and polynomials  $p_j(\lambda)$  given in (4) and (3), respectively. Then, the Bautin ideal  $\mathcal{B}$  associated to the origin of (2) has reduced Bautin depth  $\kappa = 5$ .*

*Proof.* From the proof of Theorem 6 we have  $\mathcal{B}_{11} = \langle v_3, v_5, v_7, v_9, v_{11} \rangle = \langle \mu_1, \dots, \mu_5 \rangle$  and, since all the centers of (2) are time-reversible, from the results of [6], the equality  $\mathbf{V}_{\mathbb{C}}(\mathcal{B}) = \mathbf{V}_{\mathbb{C}}(\mathcal{B}_{11})$  holds. But  $\sqrt{\mathcal{B}_{11}} \neq \mathcal{B}_{11}$  so that the hypotheses of Theorem 3 fail, and we cannot determine the Bautin depth  $m$  of  $\mathcal{B}$  using it. We only know that  $m \geq 5$ . Nevertheless, the integral closure  $\bar{\mathcal{B}}_{11}$  of  $\mathcal{B}_{11}$  satisfies  $\sqrt{\bar{\mathcal{B}}_{11}} = \bar{\mathcal{B}}_{11}$ , and therefore using Theorem 5 we conclude that the reduced Bautin depth  $\kappa$  of  $\mathcal{B}$  is  $\kappa = 5$ , finishing the proof.  $\square$

*Remark 11.* Sometimes it is possible to calculate the cyclicity in each irreducible component of the center variety following an interesting approach based on ideas from [3]. We sketch this point of view. Let us assume that family  $\mathcal{X}_\lambda$  has a center at the origin for  $\lambda = \lambda^* \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ . Also we suppose that  $\text{rank}(J_{\lambda^*}^k) = k$  where  $J_{\lambda^*}^k$  is the Jacobian of the polynomial map  $\lambda \mapsto (v_{i_1}(\lambda), v_{i_2}(\lambda), \dots, v_{i_k}(\lambda))$  evaluated at  $\lambda = \lambda^*$ , where  $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \dots, v_{i_k}(\lambda)\}$  is a minimal basis of the ideal  $\mathcal{B}_r$  for some  $r \geq k$ . Then,  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \geq k - 1$ . If additionally,  $\lambda^* \in \mathbf{V}_{\mathbb{R}}(\mathcal{I}) \subset \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ , a component of the center variety of codimension  $k$ , then  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) = k - 1$ .

Particularizing to the family analyzed in Theorem 10, the cyclicity bound  $\text{Cyc}(\mathcal{X}_\lambda, (0, 0)) \leq 4$  is proved. We consider the map  $\lambda \mapsto (v_3(\lambda), v_5(\lambda), v_7(\lambda), v_9(\lambda), v_{11}(\lambda))$ . Then  $J_{\lambda^*}^5$  is a  $5 \times 4$  matrix; hence under the full rank condition only 3 local limit cycles can bifurcate taking into account only the linear parts at  $\lambda^*$  of the Poincaré-Liapunov quantities. Some particular cases are listed below:

- $\lambda^* = (2, b, 0, 0) \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$  with associated null matrix  $J_{\lambda^*}^5$ .
- If  $d^3 = 16 - 32a + 24a^2 - 8a^3 + a^4$ , then  $\lambda^* = (a, 0, 0, d) \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$  and  $\text{rank } J_{\lambda^*}^5 = 2$ .

#### ACKNOWLEDGEMENTS

The author wishes to extend his sincere thanks to Professor Irena Swanson from the Department of Mathematics at Reed College for the great kindness shown to him. She explained all the necessary algebraic background needed in order to explicitly find generators of polynomial ideals having radical integral closure.

## REFERENCES

- [1] N. N. Bautin, *On the number of limit cycles appearing with variation of the coefficients from an equilibrium state of the type of a focus or a center* (Russian), Mat. Sbornik N.S. **30(72)** (1952), 181–196. MR0045893
- [2] G. R. Belitskiĭ, *Smooth equivalence of germs of vector fields with one zero or a pair of purely imaginary eigenvalues* (Russian), Funktsional. Anal. i Prilozhen. **20** (1986), no. 4, 1–8, 96. MR878039
- [3] Colin Christopher, *Estimating limit cycle bifurcations from centers*, Differential equations with symbolic computation, Trends Math., Birkhäuser, Basel, 2005, pp. 23–35, DOI 10.1007/3-7643-7429-2.2. MR2187371
- [4] David Cox, John Little, and Donal O’Shea, *Ideals, varieties, and algorithms, An introduction to computational algebraic geometry and commutative algebra*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2007. MR2290010
- [5] Isaac A. García, *The cyclicity of polynomial centers via the reduced Bautin depth*, Proc. Amer. Math. Soc. **144** (2016), no. 6, 2473–2478, DOI 10.1090/proc/12896. MR3477063
- [6] Isaac A. García, Jaume Llibre, and Susanna Maza, *The Hopf cyclicity of the centers of a class of quintic polynomial vector fields*, J. Differential Equations **258** (2015), no. 6, 1990–2009, DOI 10.1016/j.jde.2014.11.018. MR3302528
- [7] Isaac A. García and Douglas S. Shafer, *Cyclicity of a class of polynomial nilpotent center singularities*, Discrete Contin. Dyn. Syst. **36** (2016), no. 5, 2497–2520, DOI 10.3934/dcds.2016.36.2497. MR3485407
- [8] H. Hauser, J.-J. Risler, and B. Teissier, *The reduced Bautin index of planar vector fields*, Duke Math. J. **100** (1999), no. 3, 425–445, DOI 10.1215/S0012-7094-99-10015-9. MR1719738
- [9] Craig Huneke and Irena Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR2266432
- [10] Yulij Ilyashenko and Sergei Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008. MR2363178
- [11] D. Katz, *Generating ideals up to projective equivalence*, Proc. Amer. Math. Soc. **120** (1994), no. 1, 79–83, DOI 10.2307/2160169. MR1176070
- [12] Valery G. Romanovski and Douglas S. Shafer, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser Boston, Inc., Boston, MA, 2009. MR2500203
- [13] Robert Roussarie, *Bifurcation of planar vector fields and Hilbert’s sixteenth problem*, Progress in Mathematics, vol. 164, Birkhäuser Verlag, Basel, 1998. MR1628014
- [14] Wolmer Vasconcelos, *Integral closure, Rees algebras, multiplicities, algorithms*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. Rees algebras, multiplicities, algorithms. MR2153889

DEPARTAMENT DE MATEMÀTICA, UNIVERSITAT DE LLEIDA, AVDA. JAUME II, 69, 25001 LLEIDA, SPAIN

*E-mail address:* `garcia@matematica.udl.cat`