

CURVE SHORTENING FLOWS IN WARPED PRODUCT MANIFOLDS

HENGYU ZHOU

(Communicated by Michael Wolf)

ABSTRACT. We study curve shortening flows in two types of warped product manifolds. These manifolds are $\mathbb{S}^1 \times N$ with two types of warped metrics where \mathbb{S}^1 is the unit circle in \mathbb{R}^2 and N is a closed Riemannian manifold. If the initial curve is a graph over \mathbb{S}^1 , then its curve shortening flow exists for all times and finally converges to a geodesic closed curve.

1. INTRODUCTION

1.1. Background. The curve shortening flow is an intriguing topic in the study of mean curvature flows. It is well known that some properties of curve shortening flows are unique and have no correspondences for mean curvature flows of surfaces in \mathbb{R}^3 . For instance, see the work of Gage-Hamilton [6], and Grayson [7], [8], [9]. On the other hand, curve shortening flows can also be viewed as one-dimensional mean curvature flows. An interesting example comes from the comparisons between the results of Wang [16] and Ma-Chen [12]. The former work described the graphical mean curvature flows in product manifolds with dimension greater than one. The latter one studied the curve shortening flows for a ramp curve¹ in product manifolds, which is a one-dimensional case for the former one.

In recent years, geometric properties of warped product manifolds were investigated by many authors; for instance, Montiel [13], Brendle [3] and Alias-Dajczer [1]. At the same time, there is growing interest in the interaction between geometric flows and warped product manifolds. For example, see Borisenko-Miquel [2] and Tran [14].

The main motivation here is to explore the connection between warped product manifolds and curve shortening flows.

1.2. Main results. Now we describe the main concepts we will work with. Let \mathbb{S}^1 denote the conical circle with the induced metric dr^2 . The curve shortening flow $F_t(\gamma_0) : \mathbb{S}^1 \times [0, t_0) \rightarrow M$ is a smooth solution of the following quasilinear equation:

$$(1.1) \quad \frac{d\gamma}{dt} = \vec{H}; \quad \gamma(., 0) = \gamma_0,$$

where \vec{H} is the mean curvature vector of $F_t(\gamma_0)$ in the Riemannian manifold M and γ_0 is a closed smooth curve.

Received by the editors November 17, 2015 and, in revised form, November 6, 2016.

2010 *Mathematics Subject Classification*. Primary 53C44.

¹It is a graphical curve over \mathbb{S}^1 in the product manifold $\mathbb{S}^1 \times N$.

Throughout this note, (N, g) is a fixed closed Riemannian manifold with a Riemannian metric g and (\mathbb{S}^1, dr^2) is the unit circle.

We deal with the following two types of warped product manifolds.

Definition 1.1. We call M_L a left warped product manifold if it is $\mathbb{S}^1 \times N$ with the metric $g_L = g + \psi^2(x)dr^2$ where $\psi : N \rightarrow \mathbb{R}$ is a smooth positive function.

We call M_R a right warped product manifold if it is $\mathbb{S}^1 \times N$ with the metric $g_R = \phi^2(r)g + dr^2$ where $\phi : \mathbb{S}^1 \rightarrow \mathbb{R}$ is a smooth positive function.

Remark 1.2. Here we only require that ψ and ϕ are smooth and positive. Since S^1 and N are compact, these assumptions are sufficient for the preservation of the graphical property along curve shortening flows in our setting (see Lemma 3.4 and Lemma 5.3).

The main objectives we are interested in are graphical curves and their angle functions.

Definition 1.3. Suppose γ is a closed smooth curve in M_R or M_L . Let T denote a unit tangent vector of γ .

- (1) In M_L , $\Theta_L = \langle T, \partial_r \rangle_{g_L}$ is called the angle function of γ .
- (2) In M_R , $\Theta_R = \langle T, \partial_r \rangle_{g_R}$ is called the angle function of γ .

In addition, γ is called a graphical curve in M_L or M_R if it is parameterized by $(r, f(r))$ in $\mathbb{S}^1 \times N$ where $r \in \mathbb{S}^1$ and $f : \mathbb{S}^1 \rightarrow N$ is a smooth map.

We have the following important observation. If γ_0 is a graphical curve in M_L or M_R , then its corresponding angle function Θ_L or Θ_R never vanishes respectively. Thus we can always assume Θ_L and Θ_R are positive for graphical curves.

The main results are stated as follows.

Theorem 1.4. *Let γ_0 be a graphical curve in a left warped product manifold M_L . Then its curve shortening flow $F_t(\gamma_0)$ in M_L exists for all time and converges smoothly to a totally geodesic curve.*

Theorem 1.5. *Let γ_0 be a graphical curve in a right warped product manifold M_R . Then its curve shortening flow $F_t(\gamma_0)$ in M_R exists for all time and converges smoothly to a totally geodesic curve.*

Remark 1.6. In fact, our proof for the two results above also works for the initial conditions that $\Theta_L > 0$ or $\Theta_R > 0$. In this sense, we generalize Ma-Chen's result [12] into the warped product manifolds. See Remark 3.3.

1.3. The plan of this note. Without confusion, we use Θ to denote Θ_L in M_L or Θ_R in M_R . The process to establish Theorem 1.4 and Theorem 1.5 is summarized as follows. First we establish the evolution equation of Θ along curve shortening flows in Lemma 3.2 and Lemma 5.2. According to the evolution equation of Θ , we obtain that the condition $\Theta > 0$ will be preserved if the curve shortening flows exist and $\frac{d}{dt}\Theta \geq \Delta\Theta + \frac{1}{2}|A|^2\Theta - C$ which is explained in Lemma 3.4 and Lemma 5.3. Finally following Wang's ideas in [16] we exclude the finite singularity of curve shortening flows under the setting in Theorem 1.4 and Theorem 1.5. The corresponding convergence part is from Theorem D of Ma-Chen [12]. It states that on any Riemannian manifold if the curve shortening flow for closed curves exists for all times and is not homotopic to a point, then it will converge smoothly to a totally geodesic curve.

This note is organized as follows. In Section 2 we record some notation and preliminary results. In Section 3 we study the evolution equation of Θ_L along curve shortening flows in M_L . In Section 4 we prove Theorem 1.4. In Section 5 we establish the evolution of Θ_R along curve shortening flows in M_R . In Section 6, we explain some minor modifications to obtain Theorem 1.5 based on Section 4.

2. ONE-DIMENSIONAL RIEMANNIAN MANIFOLDS

We list some notation for later use. Assume M is a Riemannian manifold and γ is a closed smooth curve in M . Suppose $F_t(\gamma)$ is the curve shortening flow of γ in (1.1). Let $\{r\}$ be a local coordinate of γ . Then the unit tangent vector T of $F_t(\gamma)$ can be written as

$$(2.1) \quad T = \frac{(F_t)_*(\frac{\partial}{\partial r})}{\langle(F_t)_*(\frac{\partial}{\partial r}), (F_t)_*(\frac{\partial}{\partial r})\rangle^{\frac{1}{2}}}.$$

Notice that T is the only base in the tangent space of $F_t(\gamma)$. The following facts are easily verified:

- (1) The mean curvature vector of $F_t(\gamma)$ is given by $\vec{H} = \bar{\nabla}_T T$ where $\bar{\nabla}$ is the covariant derivative of M .
- (2) Assume $\{e_\alpha\}$ is a basis in the normal bundle of $F_t(\gamma)$. Then the second fundamental form is $A(T, T) = \langle \bar{\nabla}_T T, e_\alpha \rangle e_\alpha$. In addition

$$(2.2) \quad |A|^2 = \langle \bar{\nabla}_T T, e_\alpha \rangle^2 = |\vec{H}|^2.$$

- (3) The covariant derivative of $F_t(\gamma)$ with respect to the induced metric is denoted by ∇ . The Laplace operator of $F_t(\gamma)$ with respect to the induced metric is denoted by Δ . Since $\nabla_T T = 0$,

$$(2.3) \quad \Delta\eta = \nabla_T \nabla_T \eta - \nabla_{\nabla_T T} \eta = T(T(\eta));$$

for any smooth function η on $F_t(\gamma)$.

Theorem 2.1 (Lemma 4 in Ma-Chen [12]). *Suppose γ is a closed smooth curve in Riemannian manifold M and $F_t(\gamma)$ is its curve shortening flow on $[0, t_0]$. On $F_t(\gamma)$, the mean curvature vector \vec{H} and its unit tangent vector T satisfy*

$$(2.4) \quad \bar{\nabla}_{\vec{H}} T - \bar{\nabla}_T \vec{H} = |A|^2 T;$$

where $\bar{\nabla}$ denotes the covariant derivative of M .

3. LEFT WARPED PRODUCT MANIFOLD M_L

In this section, we work in the left warped product manifold M_L . In Definition 1.1 we define M_L as $\mathbb{S}^1 \times N$ with the metric $g_L = g + \psi^2(x)dr^2$.

First, we demonstrate a technique result in Proposition 3.1. Then we derive the evolution equation of Θ_L along curve shortening flows in M_L . It gives us some estimates in Lemma 3.2, which imply that graphical curves in M_L stay as graphs if their curve shortening flows exist.

We abbreviate \langle , \rangle_{g_L} by \langle , \rangle . The covariant derivative of M_L is denoted by $\bar{\nabla}$. $D(\log \psi)$ is the gradient of $\log \psi$ with respect to the Riemannian manifold (N, g) . ∂_r is the abbreviation of $\frac{\partial}{\partial r}$.

Proposition 3.1. *Given any two tangent vector fields X, Y in M_L , we have*

$$(3.1) \quad \langle Y, \bar{\nabla}_X \partial_r \rangle = \langle X, D(\log \psi) \rangle \langle Y, \partial_r \rangle - \langle X, \partial_r \rangle \langle Y, D(\log \psi) \rangle.$$

Proof. Suppose $\{x_1, \dots, x_n\}$ is a local coordinate of (N, g) . This generates a new coordinate $\{x_0 = r, x_1, \dots, x_n\}$ in M_L . Greek indices $\alpha, \beta, \gamma, \delta$ range from 0 to n and Latin i, j, k, l from 1 to n . Define

$$(3.2) \quad g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_g, \quad g_{L,\alpha\beta} = \langle \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \rangle;$$

$$(3.3) \quad (g^{ij}) = (g_{ij})^{-1}, \quad (g_L^{\alpha\beta}) = (g_{L,\alpha\beta})^{-1}.$$

Hence $D(\log \psi) = g^{kl} \frac{\partial \log \psi}{\partial x_l} \frac{\partial}{\partial x_k}$. An obvious fact is $g_L^{kl} = g^{kl}$, $g_{L,kl} = g_{kl}$ because $g_L = g + \psi^2 dr^2$. Direct computations yield that the Christoffel symbols take the form $\Gamma_{00}^0 = 0$, $\Gamma_{00}^i = -g^{ik} \psi \frac{\partial \psi}{\partial x_k}$, $\Gamma_{0k}^i = 0$ and $\Gamma_{0k}^0 = \frac{\partial \log \psi}{\partial x_k}$. Hence we get that

$$(3.4) \quad \bar{\nabla}_{\partial_r} \partial_r = -\psi D\psi, \quad \bar{\nabla}_{\partial_r} \frac{\partial}{\partial x_i} = \bar{\nabla}_{\frac{\partial}{\partial x_i}} \partial_r = \frac{\partial \log \psi}{\partial x_i} \partial_r.$$

A vector field X can be written as

$$(3.5) \quad X = \langle X, \frac{\partial}{\partial x_l} \rangle g^{kl} \frac{\partial}{\partial x_k} + \langle X, \partial_r \rangle \frac{1}{\psi^2} \partial_r.$$

Combining (3.4) with (3.5), we get

$$\begin{aligned} \langle \bar{\nabla}_X \partial_r, Y \rangle &= \langle \langle X, \frac{\partial}{\partial x_l} \rangle g^{kl} \bar{\nabla}_{\frac{\partial}{\partial x_k}} \partial_r + \langle X, \partial_r \rangle \frac{1}{\psi^2} \bar{\nabla}_{\partial_r} \partial_r, Y \rangle \\ &= \langle \langle X, g^{lk} \frac{\partial \log \psi}{\partial x_k} \frac{\partial}{\partial x_l} \rangle \partial_r - \frac{1}{\psi} \langle X, \partial_r \rangle D\psi, Y \rangle \\ &= \langle X, D(\log \psi) \rangle \langle \partial_r, Y \rangle - \langle X, \partial_r \rangle \langle D(\log \psi), Y \rangle. \end{aligned}$$

We complete the proof. \square

Lemma 3.2. *Let γ_0 be a closed and smooth curve in M_L . Suppose $F_t(\gamma_0)$ is the curve shortening flow of γ_0 on time interval $[0, t_0)$. Then the angle function Θ_L along $F_t(\gamma_0)$ satisfies*

$$\frac{d}{dt} \Theta_L = \Delta \Theta_L + |A|^2 \Theta_L + 2 \langle \vec{H}, D(\log \psi) \rangle \Theta_L - 2 \langle \nabla \Theta_L, T \rangle \langle T, D(\log \psi) \rangle.$$

Here Δ , ∇ denote the Laplacian and the covariant derivative of $F_t(\gamma_0)$ respectively. $|A|$ is the norm of the second fundamental form.

Remark 3.3. A special case is that M_L is a product manifold $S^1 \times N$ and the angle function Θ_L satisfies $\frac{d}{dt} \Theta_L = \Delta \Theta_L + |A|^2 \Theta_L$. This is indeed Proposition 13 of Ma-Chen [12].

Proof. According to Proposition 3.1, we observe that $\langle T, \bar{\nabla}_T \partial_r \rangle \equiv 0$. Recall that $\vec{H} = \bar{\nabla}_T T$. We compute

$$\begin{aligned} \langle \nabla \Theta_L, T \rangle &= T \langle \nabla \Theta_L, \partial_r \rangle \\ &= \langle \bar{\nabla}_T T, \partial_r \rangle + \langle T, \bar{\nabla}_T \partial_r \rangle \\ (3.6) \quad &= \langle \vec{H}, \partial_r \rangle. \end{aligned}$$

Next we compute the t -derivative of Θ_L as follows:

$$\begin{aligned} \frac{d}{dt} \Theta_L &= \langle \bar{\nabla}_{\vec{H}} T, \partial_r \rangle + \langle T, \bar{\nabla}_{\vec{H}} \partial_r \rangle \\ (3.7) \quad &= \langle \bar{\nabla}_T \vec{H}, \partial_r \rangle + |A|^2 \Theta_L + \langle \vec{H}, D(\log \psi) \rangle \Theta_L - \langle T, D(\log \psi) \rangle \langle \vec{H}, \partial_r \rangle. \end{aligned}$$

In the second identity we apply Theorem 2.1 and Proposition 3.1. On the other hand,

$$\begin{aligned}\Delta\Theta_L &= T(T\langle T, \partial_r \rangle) \quad \text{by (2.3)} \\ &= T\langle \vec{H}, \partial_r \rangle + T\langle T, \bar{\nabla}_T \partial_r \rangle \\ (3.8) \quad &= \langle \bar{\nabla}_T \vec{H}, \partial_r \rangle + \langle T, D(\log \psi) \rangle \langle \nabla \Theta_L, T \rangle - \langle \vec{H}, D(\log \psi) \rangle \Theta_L\end{aligned}$$

where in the last step we apply Proposition 3.1. The lemma follows from combining (3.6), (3.7) with (3.8). \square

The evolution equation of Θ_L is explored in the following estimation.

Lemma 3.4. *Assume $\Theta_L > 0$ on a graphical curve γ_0 in M_L . Suppose its curve shortening flow $F_t(\gamma_0)$ exists on $[0, t_0)$ where $t_0 < \infty$. Then*

- (1) *For any $t \in [0, t_0)$, $F_t(\gamma_0)$ is graphical and*

$$(3.9) \quad \Theta_L \geq e^{-C_L t} \min_{p \in \gamma_0} \Theta_L(p) > 0$$

where $C_L = \max_{q \in N} |D(\log \psi)(q)|^2$ and $|D(\log \psi)|$ is the norm of $D(\log \psi)$ with respect to (N, g) .

- (2) *For $t \in [0, t_0)$, Θ_L satisfies*

$$(3.10) \quad \frac{d}{dt} \Theta_L \geq \Delta \Theta_L + \frac{|A|^2}{2} \Theta_L - C_L(t_0, \gamma_0);$$

where $C_L(t_0, \gamma_0) = 4C_L(1 + \max_{q \in N} |\psi^2(q)| \frac{e^{C_L t_0}}{\min_{p \in \gamma_0} \Theta_L(p)})$.

Proof. To prove (1), we only have to demonstrate (3.9). Suppose (3.9) is true and $F_t(\gamma_0)$ loses its graphical property at some time t ; then Θ_L has to be vanishing. It is a contradiction by (3.9).

Since $\Theta_L > 0$ on γ_0 , we can assume $\Theta_L > 0$ on $F_t(\gamma_0)$ for all $t \in (0, t_1]$ with $t_1 \leq t_0$. We complete the following perfect square:

$$\begin{aligned}|A|^2 \Theta_L + 2\langle \vec{H}, D(\log \psi) \rangle \Theta_L &\geq |A|^2 \Theta_L - 2|A||D \log \psi|\Theta_L \\ &\geq (|A| - |D(\log \psi)|)^2 \Theta_L - |D(\log \psi)|^2 \Theta_L \\ (3.11) \quad &\geq -C_L \Theta_L;\end{aligned}$$

where $C_L = \max_{q \in N} |D(\log \psi)(q)|^2$. Here we also apply the estimate

$$2\langle \vec{H}, D(\log \psi) \rangle \geq -2|\vec{H}||D(\log \psi)| = -2|A||D(\log \psi)|.$$

Combining Lemma 3.2 with (3.11), we get for all $t \leq t_1$

$$(3.12) \quad \frac{d}{dt} \Theta_L \geq \Delta \Theta_L - C_L \Theta_L - 2\langle \nabla \Theta_L, T \rangle \langle T, D \log \psi \rangle.$$

The maximum principle (Theorem 4.4 in [4]) implies that for $t \in [0, t_1)$

$$\Theta_L \geq e^{-C_L t} \min_{p \in \gamma_0} \Theta_L(p) \geq e^{-C_L t_0} \min_{p \in \gamma_0} \Theta_L(p) > 0.$$

Repeating the above process and letting $t_1 = t_0$, we establish (1).

In (3.6) we prove $\langle \nabla \Theta_L, T \rangle = \langle \vec{H}, \partial_r \rangle$. Using the Cauchy inequality, we compute

$$(3.13) \quad 2\langle \nabla \Theta_L, T \rangle \langle T, D(\log \psi) \rangle = 2\langle \vec{H}, \partial_r \rangle \langle T, D(\log \psi) \rangle \\ \geq -2|A||D(\log \psi)||\psi|$$

$$(3.14) \quad \geq -\frac{|A|^2}{4}\Theta_L - 4\frac{|D(\log \psi)|^2|\psi|^2}{\Theta_L}.$$

Here we use $\langle \partial_r, \partial_r \rangle = \psi^2$ and $|A| = |\vec{H}|$. On the other hand,

$$(3.15) \quad 2\langle \vec{H}, D(\log \psi) \rangle \Theta_L \geq -2|A||D(\log \psi)|\Theta_L \\ \geq -\frac{|A|^2}{4}\Theta_L - 4|D(\log \psi)|^2\Theta_L.$$

With (3.14), (3.15) and Lemma 3.2, we obtain

$$(3.16) \quad \frac{d}{dt}\Theta_L \geq \Delta\Theta_L + \frac{|A|^2}{2}\Theta_L - 4|D(\log \psi)|^2\Theta_L - 4\frac{|D(\log \psi)|^2|\psi|^2}{\Theta_L}.$$

According to (1), for any $t \in [0, t_0]$ we have $1 \geq \Theta_L \geq e^{-C_L t_0} \min_{p \in \gamma_0} \Theta_L(p)$. Hence (2) follows from

$$-4|D(\log \psi)|^2\Theta_L - 4\frac{|D(\log \psi)|^2|\psi|^2}{\Theta_L} \geq -4|D(\log \psi)|^2(1 + \frac{|\psi|^2 e^{C_L t_0}}{\min_{p \in \gamma_0} \Theta_L(p)}).$$

We complete the proof. \square

4. THE PROOF OF THEOREM 1.4

There are at least two ways to derive long time existence of mean curvature flows. The first way is to estimate directly the upper bound of the second fundamental form in terms of the evolution equation of geometric quantities (angle functions). See [5], [2] and [12]. Another way is to prove the Gaussian density of any point along mean curvature flows in space-time space is 1. White [17] showed that such mean curvature flows exist for all times.

Unfortunately, it seems that the first method fails in our case. In Wang's work [16] to derive the regularity of mean curvature flows, a key ingredient is to apply a certain evolution equation of $*\Omega$ and demonstrate the Gaussian density² of corresponding mean curvature flows is always 1. After comparisons, we find that $*\Omega$ in [16] plays a similar role as that of Θ_L in M_L . Therefore we choose to follow the method in [16].

4.1. The Gaussian density. First we recall some facts about Gaussian density. We isometrically embed M_L into a Euclidean space \mathbb{R}^{n_0} for sufficiently large n_0 . Suppose γ_0 is a graphical curve in Theorem 1.4, and $F_t(\gamma_0)$ is the curve shortening flow of γ_0 existing smoothly on $[0, t_0)$. Without confusion, let $F_t(\gamma_0)$ also denote its coordinate function in \mathbb{R}^{n_0} . The curve shortening flow equation in terms of $F_t(\gamma_0)$ becomes

$$(4.1) \quad \frac{d}{dt}F_t(\gamma_0) = \vec{H} = \tilde{H} + E;$$

²Also see Section 4 in [16] and Proposition 5.2 in [15].

where \vec{H} is the mean curvature vector of $F_t(\gamma_0)$ in M_L and \tilde{H} is the mean curvature vector of $F_t(\gamma_0)$ in \mathbb{R}^{n_0} . As for E , we have

$$E = -(\bar{\nabla}_T T)^\perp = \vec{H} - \tilde{H}$$

where T is the unit tangent vector of $F_t(\gamma_0)$ and \perp denotes the projection into the normal bundle of M (not $F_t(\gamma_0)$). Provided M is compact and T is a unit vector, E should always be uniformly bounded independent of the position of $F_t(\gamma_0)$.

A main tool to detect a possible singularity at (y_0, t_0) is the backward heat kernel ρ_{y_0, t_0} at (y_0, t_0) , which was proposed by Huisken [10] as follows:

$$(4.2) \quad \rho_{y_0, t_0}(y, t) = \frac{1}{\sqrt{4\pi(t_0 - t)}} \exp\left(-\frac{|y - y_0|}{4(t_0 - t)}\right).$$

Let ρ_{y_0, t_0} be the abbreviation of $\rho_{y_0, t_0}(F_t(\gamma_0), t)$. With direct computations, along $F_t(\gamma_0)$ ρ_{y_0, t_0} satisfies that

$$(4.3) \quad \begin{aligned} \frac{d\rho_{y_0, t_0}}{dt} &= -\Delta \rho_{y_0, t_0} - \rho_{y_0, t_0} \left(\frac{|(F_t(\gamma_0) - y_0)^\perp|^2}{4(t - t_0)^2} \right. \\ &\quad \left. + \frac{\langle F_t(\gamma_0) - y_0, \tilde{H} \rangle}{(t_0 - t)} + \frac{\langle (F_t(\gamma_0) - y_0), E \rangle}{2(t - t_0)} \right); \end{aligned}$$

where $(F_t(\gamma_0) - y_0)^\perp$ is the normal component of $F_t(\gamma_0) - y_0$ in the normal bundle of $F_t(\gamma_0)$, and Δ is the Laplace operator of $F_t(\gamma_0)$.

By the results of Huisken [10] and White [17], for the curve shortening flow $F_t(\gamma_0)$

$$\lim_{t \rightarrow t_0, t < t_0} \int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t$$

exists and is finite. The above limit is called the *Gaussian density* at (y_0, t_0) . Here $d\mu_t$ is the length element of $F_t(\gamma_0)$. White [17] obtained that if this limit is 1, the curve shortening flow exists smoothly in a neighborhood of (y_0, t_0) in the space-time space $\mathbb{R}^{n_0} \times [0, \infty)$.

4.2. The proof of Theorem 1.4.

Our proof is divided into three steps.

Lemma 4.1. *Assume γ_0 is a graphical curve in M_L and $F_t(\gamma_0)$ exists smoothly on $[0, t_0)$ for $t_0 < \infty$. Then*

$$\int_0^{t_0} \int_{F_t(\gamma_0)} |A|^2 \rho_{y_0, t_0} d\mu_t dt < \infty$$

where A is the second fundamental form of $F_t(\gamma_0)$ in M_L .

Proof. Recall that

$$\frac{d}{dt} d\mu_t = -|\tilde{H}|^2 d\mu_t = -\langle \tilde{H}, \tilde{H} + E \rangle d\mu_t;$$

and $1 \geq \Theta_L \geq e^{-C_L t_0} \min_{p \in \gamma_0} \Theta_L(p) = C_0 > 0$ by Lemma 3.4. With (4.3) and (2) in Lemma 3.4, we compute

$$(4.4) \quad \begin{aligned} & \frac{d}{dt}((1 - \Theta_L)\rho_{y_0, t_0} d\mu_t) \\ & \leq (\Delta(1 - \Theta_L)\rho_{y_0, t_0} - (1 - \Theta_L)\Delta\rho_{y_0, t_0})d\mu_t - C_0 \frac{|A|^2}{2}\rho_{y_0, t_0} d\mu_t \\ & + C_L(t_0, \gamma_0)\rho_{y_0, t_0} d\mu_t \end{aligned}$$

$$(4.5) \quad - (1 - \Theta_L)\rho_{y_0, t_0} \left(\frac{|F_t^\perp(\gamma_0) - y_0|^2}{4(t - t_0)^2} + \frac{\langle F_{t_0}(\gamma_0) - y_0, \tilde{H} \rangle}{(t_0 - t)} \right)$$

$$(4.6) \quad + \frac{\langle (F_t(\gamma_0) - y_0), E \rangle}{2(t - t_0)} + |\tilde{H}|^2 + \langle \tilde{H}, E \rangle).$$

We complete the perfect square in the last term of the equation above and obtain

$$(4.5) + (4.6) = -(1 - \Theta_L)\rho_{y_0, t_0} \left| \frac{F_t^\perp(\gamma_0) - y_0}{2(t - t_0)} + \tilde{H} + \frac{E}{2} \right|^2 + (1 - \Theta_L)\rho_{y_0, t_0} \frac{|E|^2}{4}.$$

Since M_L is compact in \mathbb{R}^{n_0} , E is uniformly bounded. Notice that $0 \leq (1 - \Theta_L) \leq 1$ and $\int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t$ is finite for $t \leq t_0$. Therefore, the above equation implies that

$$\int_{F_t(\gamma_0)} [(4.5) + (4.6)] d\mu_t \leq \int_{F_t(\gamma_0)} (1 - \Theta_L)\rho_{y_0, t_0} \frac{|E|^2}{4} d\mu_t = C < \infty.$$

Therefore the integral of (4.4) becomes

$$\begin{aligned} & \frac{d}{dt} \int_{F_t(\gamma_0)} ((1 - \Theta_L)\rho_{y_0, t_0} d\mu_t) \leq \int_{F_t(\gamma_0)} (\Delta(1 - \Theta_L)\rho_{y_0, t_0} - (1 - \Theta_L)\Delta\rho_{y_0, t_0})d\mu_t \\ & - C_0 \int_{F_t(\gamma_0)} \frac{|A|^2}{2}\rho_{y_0, t_0} d\mu_t + C_L(t_0, \gamma_0) \int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t + C. \end{aligned}$$

By integrating by part the first term above vanishes. It is not hard to verify the following facts for all $t \leq t_0$: (1) $\int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t$ and $\int_{F_t(\gamma_0)} d\mu_t$ are finite; (2) $\int_{F_t(\gamma_0)} (1 - \Theta_L)\rho_{y_0, t_0} d\mu_t$ is finite. We take the integral with respect to t and complete the proof. \square

We denote $F_s^\lambda(\gamma_0) = \lambda(F_t(\gamma_0) - y_0)$ for $-s = \lambda^2(t - t_0)$ where $-s \in (-\lambda^2 t_0, 0]$.

Lemma 4.2. *Assume γ_0 is a graphical curve in M_L and $F_t(\gamma_0)$ exists smoothly on $[0, t_0]$. Then there exists a sequence $\{s_j, \lambda_j\}_{j=1}^\infty$ such that $s_j \in [1, 2]$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$ and for any compact set $K \in \mathbb{R}^{n_0}$,*

$$(4.7) \quad \lim_{j \rightarrow \infty} \int_{F_{s_j}^{\lambda_j}(\gamma_0) \cap K} |\tilde{A}_j|^2 d\mu_{s_j}^{\lambda_j} = 0$$

where \tilde{A}_j is the second fundamental form of $F_{s_j}^{\lambda_j}(\gamma_0)$ in \mathbb{R}^{n_0} , and $d\mu_{s_j}^{\lambda_j}$ is the length element of $F_{s_j}^{\lambda_j}(\gamma_0)$.

Proof. According to Lemma 4.1, there exists a sequence $\{\lambda_j\}$ going to infinity such that

$$(4.8) \quad \lim_{j \rightarrow \infty} \int_{t_0 - \frac{2}{\lambda_j^2}}^{t_0 - \frac{1}{\lambda_j^2}} |A|^2 \rho_{y_0, t_0} d\mu_t dt = 0.$$

It is not hard to see that for any $p \in \gamma_0$, λ, s

$$(4.9) \quad \lambda A(F_s^\lambda(p)) = A(F_t(p)), \quad \lambda \tilde{A}(F_s^\lambda(p)) = \tilde{A}(F_t(p)),$$

$$(4.10) \quad \rho_{y_0, t_0} d\mu_t = \rho_{0,0} d\mu_s^\lambda,$$

where $\rho_{0,0} = \rho_{0,0}(F_s^\lambda(\gamma_0), s)$ and $t = t_0 - \frac{s}{\lambda^2}$.

According to (4.8), for each j there exists s_j such that

$$\frac{1}{\lambda_j^2} \int_{F_{t_j}(\gamma_0)} |A|^2 \rho_{y_0, q_0} d\mu_{t_j} = \int_{F_{s_j}^{\lambda_j}(\gamma_0)} |A_j|^2 \rho_{0,0} d\mu_{s_j}^{\lambda_j};$$

and

$$(4.11) \quad \lim_{j \rightarrow \infty} \int_{F_{s_j}^{\lambda_j}(\gamma_0)} |A_j|^2 \rho_{0,0} d\mu_{s_j}^{\lambda_j} = 0,$$

where $t_j = t_0 - \frac{s_j}{\lambda_j^2}$. We analyze it more carefully by considering:

$$\rho_{0,0}(F_{s_j}^{\lambda_j}(\gamma_0), s_j) = \frac{1}{\sqrt{4\pi s_j}} \exp\left(-\frac{|F_{s_j}^{\lambda_j}(\gamma_0)|^2}{4\pi s_j}\right).$$

Fix $R > 0$ and let $B^R(0)$ be the ball centered at 0 with radius R in \mathbb{R}^{n_0} . Since $s_j \in [1, 2]$, we get

$$\int_{F_{s_j}^{\lambda_j}(\gamma_0)} |A_j|^2 \rho_{0,0} d\mu_{s_j}^{\lambda_j} \geq \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{R^2}{4\pi}\right) \int_{F_{s_j}^{\lambda_j}(\gamma_0)} |A_j|^2 d\mu_{s_j}^{\lambda_j}.$$

For any fixed compact set $K \subset R^N$ (4.11) leads to

$$\int_{F_{s_j}^{\lambda_j}(\gamma_0) \cap K} |A_j|^2 d\mu_{s_j}^{\lambda_j} \rightarrow 0.$$

Notice that $\tilde{A} - A$ is the component of \tilde{A} in the normal bundle of the left warped product manifold M_L in \mathbb{R}^{n_0} . It is uniformly bounded since M_L is compact. On the other hand, by (4.9) we have

$$(4.12) \quad |\tilde{A}_j - A_j| = \frac{|\tilde{A} - A|}{\lambda_j} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

With (4.12) we obtain

$$\int_{F_{s_j}^{\lambda_j}(\gamma_0) \cap K} |\tilde{A}_j|^2 d\mu_{s_j}^{\lambda_j} \rightarrow 0.$$

We complete the proof. \square

Lemma 4.3. *Assume γ_0 is a graphical curve in M_L and $F_t(\gamma_0)$ is its curve shortening flow. Let $t_0 > 0$ be any finite time. Then for any (y_0, t_0) ,*

$$(4.13) \quad \lim_{t \rightarrow t_0, t < t_0} \int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t = 1.$$

That is, the curve shortening flow $F_t(\gamma_0)$ exists for all time.

Proof. We take λ_j and s_j as in Lemma 4.2,

$$(4.14) \quad \begin{aligned} \lim_{t \rightarrow t_0} \int_{F_t(\gamma_0)} \rho_{y_0, t_0} d\mu_t &= \lim_{j \rightarrow \infty} \int_{F_{t_0 - \frac{s_j}{\lambda_j^2}}(\gamma_0)} \rho_{y_0, t_0} d\mu_{t_0 - \frac{s_j}{\lambda_j^2}} \\ &= \lim_{j \rightarrow \infty} \int_{F_{s_j}^{\lambda_j}(\gamma_0)} \rho_{0, 0} d\mu_{s_j}^{\lambda_j}. \end{aligned}$$

We may assume that the origin is a limit point of $F_{s_j}^{\lambda_j}(\gamma_0)$ otherwise the limit above is 1 and nothing needs to be proven.

Recall that in Lemma 3.4 we conclude that $\Theta_L \geq C > 0$ on $[0, t_0]$. On the other hand, $F_t(\gamma_0)$ is the graph of f_t in M_L . And Θ_L is written as

$$(4.15) \quad \Theta_L = \frac{1}{\sqrt{1 + \psi^2(x)|df_t|^2}}.$$

Therefore we conclude that df_t is uniformly bounded on $[0, t_0]$.

Denote $f_{t - \frac{s_j}{\lambda_j^2}}$ by f_j . Therefore $F_{s_j}^{\lambda_j}(\gamma)$ is the graph of $\tilde{f}_j = \lambda_j f_j$ which is defined on $\lambda_j \mathbb{S}^1 \subset \mathbb{R}^{n_0}$. It is not hard to see that $d\tilde{f}_j$ is also uniformly bounded. Now our assumptions on $F_{s_j}^{\lambda_j}$ imply $\lim_{j \rightarrow \infty} \tilde{f}_j(0) = 0$. Therefore we may assume $\tilde{f}_j \rightarrow \tilde{f}_\infty$ in C^α on compact sets, where \tilde{f}_∞ is a map on \mathbb{R}^1 . On the other hand, with similar computations as in equation (29) on page 30 of [11], we have

$$(4.16) \quad |\tilde{A}_j| \leq |\nabla d\tilde{f}_j| \leq (1 + |d\tilde{f}_j|^2)^{\frac{3}{2}} |\tilde{A}_j|.$$

Here ∇ is the covariant derivative of $F_{s_j}^{\lambda_j}$. From Lemma 4.2, $\tilde{f}_j \rightarrow \tilde{f}_\infty$ in $C^\alpha \cap W_{loc}^{1,2}$ and the second derivative of \tilde{f}_∞ is vanishing. Then f_∞ is a one-dimensional linear map. This implies that $F_{s_j}^{\lambda_j}(\gamma_0) \rightarrow F_{-1}^\infty$ are Radon measures and F_{-1}^∞ is the graph of a linear map. Finally

$$\lim_{j \rightarrow \infty} \int_{F_{s_j}^{\lambda_j}(\gamma_0)} \rho_{0, 0} d\mu_{s_j}^{\lambda_j} = \int_{F_{-1}^\infty} \rho_{0, 0} d\mu_{-1}^\infty = 1.$$

Our lemma follows from (4.14). □

Now we are ready to conclude Theorem 1.4.

Proof. According to Lemma 4.3 and White's regularity results [17] (y_0, t_0) is a regular point of the curve shortening flows $F_t(\gamma_0)$ in M_L . Therefore, for a graphical γ_0 the curve shortening flow $F_t(\gamma_0)$ in M_L exists for all time. Theorem 1.4 follows from Theorem D in Ma-Chen [12]. It says that as long as a curve shortening flow exists for all time, it will converge smoothly to a geodesic curve. □

5. RIGHT WARPED PRODUCT MANIFOLD M_R

In this section, we study curve shortening flows in the right warped product manifold M_R . In Definition 1.1 we define M_R as $\mathbb{S}^1 \times N$ with the metric $g_R = \phi^2(r)g + dr^2$.

In this section, we denote $\langle \cdot, \cdot \rangle_{g_R}$ by $\langle \cdot, \cdot \rangle$. Let $\bar{\nabla}$ be the covariant derivative of M_R . $(\log \phi)', (\log \phi)''$ are the first derivative and second derivative of $\log \phi$ respectively.

Proposition 5.1. *Given two tangent vector fields X, Y in M_R , we have*

$$(5.1) \quad \langle Y, \bar{\nabla}_X \partial_r \rangle = (\log \phi)'(\langle X, Y \rangle - \langle X, \partial_r \rangle \langle Y, \partial_r \rangle).$$

Proof. A well-known fact is that $\phi(r)\partial_r$ is a conformal vector field (see Montiel [13]), i.e., $\bar{\nabla}_X(\phi(r)\partial_r) = \phi'(r)X$ for any smooth vector field X . Then

$$\begin{aligned} \langle Y, \bar{\nabla}_X \partial_r \rangle &= \langle Y, \bar{\nabla}_X \left(\frac{\phi(r)\partial_r}{\phi(r)} \right) \rangle \\ &= \frac{\phi'(r)}{\phi(r)} (\langle X, Y \rangle - \langle Y, \partial_r \rangle \langle X, \partial_r \rangle). \end{aligned}$$

□

With this result we obtain the evolution equation of Θ_R .

Lemma 5.2. *Let γ_0 be a closed smooth curve in M_R . Suppose $F_t(\gamma_0)$ is the curve shortening flow of γ_0 on $[0, t_0]$. Then the angle function Θ_R satisfies*

$$(5.2) \quad \frac{d}{dt}\Theta_R = \Delta\Theta_R + |A|^2\Theta_R + 2(\log\phi)'\Theta_R\langle\nabla\Theta_R, T\rangle - (\log\phi)''\Theta_R(1 - \Theta_R^2).$$

Here Δ , ∇ denote the Laplacian and the covariant derivative of $F_t(\gamma_0)$ respectively. $|A|$ is the norm of the second fundamental form.

Proof. Recall that $\Theta_R = \langle T, \partial_r \rangle$. From Proposition 5.1, we observe a symmetry property as follows. For any two tangent vector fields X, Y ,

$$(5.3) \quad \langle X, \bar{\nabla}_Y \partial_r \rangle = \langle Y, \bar{\nabla}_X \partial_r \rangle.$$

Applying Theorem 2.1 and (5.3), we compute $\frac{d}{dt}\Theta_R$ as

$$\begin{aligned} \frac{d}{dt}\Theta_R &= \langle \bar{\nabla}_{\vec{H}} T, \partial_r \rangle + \langle T, \bar{\nabla}_{\vec{H}} \partial_r \rangle \\ (5.4) \quad &= \langle \bar{\nabla}_T \vec{H}, \partial_r \rangle + |A|^2\Theta_R + \langle \vec{H}, \bar{\nabla}_T \partial_r \rangle. \end{aligned}$$

On the other hand, by (2.3) we have

$$\begin{aligned} \Delta\Theta_R &= T(T\langle T, \partial_r \rangle) \\ &= T\langle \vec{H}, \partial_r \rangle + T\langle T, \bar{\nabla}_T \partial_r \rangle \\ &= \langle \bar{\nabla}_T \vec{H}, \partial_r \rangle + \langle \vec{H}, \bar{\nabla}_T \partial_r \rangle + T((\log\phi)'(1 - \Theta_R^2)) \\ &= \langle \bar{\nabla}_T \vec{H}, \partial_r \rangle + \langle \vec{H}, \bar{\nabla}_T \partial_r \rangle + (\log\phi)''\Theta_R(1 - \Theta_R^2) \\ (5.5) \quad &- 2(\log\phi)'\Theta_R\langle\nabla\Theta_R, T\rangle \end{aligned}$$

because $\langle T, \bar{\nabla}_T \partial_r \rangle = (\log\phi)'(1 - \Theta_R^2)$ by Proposition 5.1. The lemma follows from combining (5.4) with (5.5). □

We have the following estimations for Θ_R .

Lemma 5.3. *Assume $\Theta_R > 0$ on a graphical curve γ_0 in M_R . If its curve shortening flow $F_t(\gamma_0)$ exists on $[0, t_0)$ where $t_0 < \infty$,*

- (1) *For any $t \in [0, t_0)$, $F_t(\gamma_0)$ are graphical curves and*

$$(5.6) \quad \Theta_R \geq e^{-C_R t} \min_{p \in \gamma_0} \Theta_R(p) > 0$$

where $C_R = \max_{r \in \mathbb{S}^1} |(\log\phi)''(r)|$.

- (2) *For $t \in [0, t_0)$, Θ_R satisfies the following inequality:*

$$(5.7) \quad \frac{d}{dt}\Theta_R \geq \Delta\Theta_R + \frac{|A|^2}{2}\Theta_R - C_R(\phi);$$

where $C_R(\phi) = \max_{r \in \mathbb{S}^1} \{4((\log\phi)')^2 + |(\log\phi)''|\}(r)$.

Proof. With the same reason as in Lemma 3.4, it is only sufficient to prove (5.6) to conclude (1).

Since $\Theta_R > 0$ on γ_0 , we can assume $\Theta_R > 0$ on $F_t(\gamma_0)$ for some $t_1 \in (0, t_0]$. It is easy to see that for $t \in (0, t_1)$,

$$(5.8) \quad -(\log \phi)''\Theta_R(1 - \Theta_R^2) \geq -|\log \phi''|\Theta_R \geq -C_R\Theta_R$$

since $0 < \Theta_R \leq 1$ for $t \in [0, t_1]$. Here $C_R = \max_{r \in \mathbb{S}^1} |(\log \phi)''(r)|$. According to Lemma 5.2, for $t \in [0, t_1]$ we have

$$(5.9) \quad \frac{d}{dt}\Theta_R \geq \Delta\Theta_R - C_R\Theta_R + 2(\log \phi)'\Theta_R^2 \langle \nabla\Theta_R, T \rangle.$$

By the maximum principle (Theorem 4.4 in [4]) we get

$$\Theta_R \geq e^{-C_R t} \min_{p \in \gamma_0} \Theta_R(p) \geq e^{-C_R t_0} \min_{p \in \gamma_0} \Theta_R(p) > 0$$

for $t \in [0, t_1]$. Repeating the above process, we can choose $t_1 = t_0$ and therefore obtain (1).

By Proposition 5.1, we have

$$\langle \nabla\Theta_R, T \rangle = T \langle T, \partial_r \rangle = \langle \vec{H}, \partial_r \rangle + (\log \phi)'(1 - \Theta_R^2).$$

Using the Cauchy inequality on $-2(\log \phi)'\Theta_R^2 \langle \nabla\Theta, T \rangle$ and $1 \geq \Theta_R > 0$, we get

$$\begin{aligned} -2(\log \phi)'\Theta_R^2 \langle \nabla\Theta, T \rangle &= -2(\log \phi)'\Theta_R^2 \langle \vec{H}, \partial_r \rangle - 2((\log \phi)')^2\Theta_R^2(1 - \Theta_R^2) \\ &\geq -2(\log \phi)'|A|\Theta_R - 2((\log \phi)')^2\Theta_R \\ (5.10) \quad &\geq -\frac{|A|^2\Theta_R}{2} - 4((\log \phi)')^2. \end{aligned}$$

Here we use $|\vec{H}| = |A|$ since $F_t(\gamma_0)$ are curves. Plugging (5.8) and (5.10) into (5.2), we obtain

$$(5.11) \quad \frac{d}{dt}\Theta_R \geq \Delta\Theta_R + |A|^2\Theta_R - \frac{|A|^2\Theta_R}{2} - 4((\log \phi)')^2 - |(\log \phi)''|.$$

Let $C_R(\phi)$ denote $\max_{r \in \mathbb{S}^1} \{4((\log \phi)')^2 + |(\log \phi)''|\}(r)$; we get (2). We complete the proof. \square

6. THE PROOF OF THEOREM 1.5

The proof of Theorem 1.5 is very similar to that of Theorem 1.4 with minor modifications. We list the key ingredients in the proof. Their derivations are skipped and can be easily written according to Section 4. First, Lemma 5.3 will imply

Lemma 6.1. *Assume γ_0 is a graphical curve in M_R and $F_t(\gamma_0)$ exists smoothly on $[0, t_0)$ for $t_0 < \infty$. Then*

$$\int_0^{t_0} \int_{F_t(\gamma_0)} |A|^2 \rho_{y_0, t_0} d\mu_t dt < \infty$$

where A is the second fundamental form of $F_t(\gamma_0)$ in M_R .

Then Lemma 6.1 indicates the following result.

Lemma 6.2. *Assume γ_0 is a graphical curve in M_R and $F_t(\gamma_0)$ exists smoothly on $[0, t_0)$. Then there exists $\{s_j, \lambda_j\}_{j=1}^\infty$ such that $s_j \in [1, 2]$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$ and for any compact set $K \in \mathbb{R}^{n_0}$,*

$$(6.1) \quad \lim_{j \rightarrow \infty} \int_{F_{s_j}^{\lambda_j}(\gamma_0) \cap K} |\tilde{A}_j|^2 d\mu_{s_j}^{\lambda_j} = 0$$

where \tilde{A}_j is the second fundamental form of $F_{s_j}^{\lambda_j}(\gamma_0)$ in \mathbb{R}^{n_0} , and $d\mu_{s_j}^{\lambda_j}$ is the length element of $F_{s_j}^{\lambda_j}(\gamma_0)$.

The way to choose s_j, λ_j is the same as in Section 4. One difference is that equation (4.15) is replaced by $\Theta_R = \frac{1}{\sqrt{1+|df_t|^2(\phi(r))^{-2}}}$. Here $|df|$ is the norm of df with respect to the Riemannian manifold (N, g) . The conclusion that df is uniformly bounded still works under the setting of Theorem 1.5. Then we establish Theorem 1.5 with a similar derivation as in Section 4.

ACKNOWLEDGMENT

The author wishes to thank his advisors, Professor Zheng Huang and Professor Yunping Jiang, at the Graduate Center, CUNY for their warm discussions and encouragement during this project. The author would also like to express his sincere gratitude to the referee for numerous corrections, many constructive suggestions and helpful comments.

REFERENCES

- [1] Luis J. Alías and Marcos Dajczer, *Constant mean curvature hypersurfaces in warped product spaces*, Proc. Edinb. Math. Soc. (2) **50** (2007), no. 3, 511–526, DOI 10.1017/S0013091505001069. MR2360513
- [2] Alexander A. Borisenko and Vicente Miquel, *Mean curvature flow of graphs in warped products*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4551–4587, DOI 10.1090/S0002-9947-2012-05425-0. MR2922601
- [3] Simon Brendle, *Constant mean curvature surfaces in warped product manifolds*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 247–269, DOI 10.1007/s10240-012-0047-5. MR3090261
- [4] Bennett Chow and Dan Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004. MR2061425
- [5] Klaus Ecker and Gerhard Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569, DOI 10.1007/BF01232278. MR1117150
- [6] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), no. 1, 69–96. MR840401
- [7] Matthew A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), no. 2, 285–314. MR906392
- [8] Matthew A. Grayson, *A short note on the evolution of a surface by its mean curvature*, Duke Math. J. **58** (1989), no. 3, 555–558, DOI 10.1215/S0012-7094-89-05825-0. MR1016434
- [9] Matthew A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), no. 1, 71–111, DOI 10.2307/1971486. MR979601
- [10] Gerhard Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299. MR1030675
- [11] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, Preprint, 1995.
- [12] Li Ma and Dezhong Chen, *Curve shortening in a Riemannian manifold*, Ann. Mat. Pura Appl. (4) **186** (2007), no. 4, 663–684, DOI 10.1007/s10231-006-0025-y. MR2317784
- [13] Sebastián Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), no. 2, 711–748, DOI 10.1512/iumj.1999.48.1562. MR1722814
- [14] Hung Tran, *Harnack estimates for Ricci flow on a warped product*, J. Geom. Anal. **26** (2016), no. 3, 1838–1862, DOI 10.1007/s12220-015-9610-x. MR3511460

- [15] Mu-Tao Wang, *Mean curvature flow of surfaces in Einstein four-manifolds*, J. Differential Geom. **57** (2001), no. 2, 301–338. MR1879229
- [16] Mu-Tao Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, Invent. Math. **148** (2002), no. 3, 525–543, DOI 10.1007/s002220100201. MR1908059
- [17] Brian White, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519, DOI 10.4007/annals.2005.161.1487. MR2180405

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, NO. 135, XINGANG XI ROAD,
GUANGZHOU, 510275, PEOPLE'S REPUBLIC OF CHINA

E-mail address: hyzhou84@yahoo.com