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KNOT CONTACT HOMOLOGY DETECTS CABLED, COMPOSITE, AND TORUS KNOTS

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Abstract. Knot contact homology is an invariant of knots derived from Legendrian contact homology which has numerous connections to the knot group. We use basic properties of knot groups to prove that knot contact homology detects every torus knot. Further, if the knot contact homology of a knot is isomorphic to that of a cable (respectively composite) knot, then the knot is a cable (respectively composite).

1. Introduction

Associated to a knot $K \subset \mathbb{R}^3$, knot contact homology is a combinatorial invariant which arises from constructions in contact and symplectic geometry [Ng05a, Ng05b, Ng08, EENS13]. More specifically, the conormal bundle Λ_K of K in the unit cotangent bundle $ST^*\mathbb{R}^3$ is a Legendrian submanifold, and knot contact homology comes from the Legendrian contact homology of Λ_K. This invariant is able to distinguish mirrors and mutant pairs, determines the Alexander polynomial, and detects the unknot [Ng08]. In fact, it is still open as to whether knot contact homology is a complete knot invariant.[1]

In this paper, we will show that knot contact homology detects each of the torus knots as well as being a cable or composite. Here, we will work with a version of the fully noncommutative degree zero knot contact homology with $U = 1$, which we denote by $\widetilde{HC}_0(K)$; for comparison with other appearances in the literature, this is denoted by $H_0^{\text{contact}}(K)$ in [CELN16] and $\widetilde{HC}_0(K)|_{U=1}$ in [Ng14].

Theorem 1. *Fix p, q relatively prime integers. Let K be an oriented knot in $\mathbb{R}^3$ and let $T_{p,q}$ denote the (p,q)-torus knot. If $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(T_{p,q})$, then K is isotopic to $T_{p,q}$.*

The version of knot contact homology we work with is actually an invariant of framed knots, where the above framings are the Seifert framings. We will prove a detection statement for knots with arbitrary framing in Theorem 10 below.

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[1]Since the original appearance of this note, Shende has shown that the conormal bundle is a complete invariant [She16]. An enhanced version of knot contact homology was then shown to be a complete invariant by Ekholm, Ng, and Shende [ENS16]. Further, degree zero knot contact homology is isomorphic to degree zero string homology, an object defined by Cieliebak, Ekholm, Latschev, Ng in [CELN16]. It is shown there that a variant of degree zero string homology is a complete invariant.

Using Theorem 1 we are able to obtain the following.

Theorem 2. *If Λ_K is Legendrian isotopic to $\Lambda_{T_{p,q}}$ in $ST^*\mathbb{R}^3$, then K is (smoothly) isotopic to $T_{p,q}$ or its mirror. Further, if the Legendrian isotopy sends meridian to meridian and longitude to longitude, then K is isotopic to $T_{p,q}$.*

While the latter statement follows immediately from Theorem 1, there will be a bit more to show for an arbitrary Legendrian isotopy.

As discussed, Theorem 2 has recently been generalized to show that the conormal lift detects all knots as above. However, we still include a proof for the special case of torus knots due to its simplicity.

Using similar techniques, we will also show a similar result for certain satellite knots.

Theorem 3. *Let K be an oriented knot in $\mathbb{R}^3$ and let $C_{p,q}(J)$ denote the (p,q)-cable of a knot J. If $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(C_{p,q}(J))$, then K is isotopic to a (p',q')-cable of a knot J' where $pq = p'q'$. On the other hand, if $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(J)$ and J is composite, then so is K.*

In the next section, we review a theorem of Cieliebak, Ekholm, Latschev, and Ng which relates $\widetilde{HC}_0$ to the knot group [CELN16] and is a key component of our proof. From there, we will prove Theorem 1 and Theorem 3. In Section 3, we will prove a slight generalization of Theorem 1 for arbitrary framed knots as well as Theorem 2.

2. Knot contact homology and the knot group

The starting point for Theorem 1 is a result in [CELN16] which identifies $\widetilde{HC}_0$ with a certain subring of the group ring of the knot group. We first set our notation before stating their theorem. Since degree zero knot contact homology detects the unknot [Ng08, Proposition 5.10], we will assume throughout that all knots are oriented and non-trivial to simplify the discussion. (See Remark 5 for a discussion of the unknot.) Note that the orientation induces canonical representatives for the Seifert framing and the meridian, λ_K and μ_K, respectively, in the knot group $\pi_K = \pi_1(S^3 \setminus K)$. We let $\hat{\pi}_K$ denote the peripheral subgroup of π_K, i.e., the subgroup generated by μ_K and λ_K. Of course, $\mathbb{Z}[\pi_K]$ contains $\mathbb{Z}[\hat{\pi}_K] \cong \mathbb{Z}[\mu_K^{\pm 1}, \lambda_K^{\pm 1}]$ as a subring. Finally, let $\mathfrak{R}_K$ be the subring of $\mathbb{Z}[\pi_K]$ generated by $\mathbb{Z}[\hat{\pi}_K]$ and $\{\gamma - \mu_K \gamma \mid \gamma \in \pi_K\}$.

Degree zero knot contact homology, $\widetilde{HC}_0(K)$, takes the form of a ring equipped with an embedding of $\mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$ into $\widetilde{HC}_0(K)$. An isomorphism between $\widetilde{HC}_0(K)$ and $\widetilde{HC}_0(K')$ is a ring isomorphism which restricts to the identity on the subrings $\mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$. We are now ready to state a powerful relationship between $\widetilde{HC}_0(K)$ and π_K.

Theorem 4 (Cieliebak-Ekholm-Latschev-Ng, [CELN16])**.** *Let K be a non-trivial, oriented knot. Then, there is an isomorphism between $\widetilde{HC}_0(K)$ and $\mathfrak{R}_K$ which sends μ, λ to μ_K, λ_K, respectively.*

The subring $\mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$ is identified with $\mathbb{Z}[H_1(\Lambda_K)]$, by sending μ and λ to μ_K and λ_K, respectively. A Legendrian isotopy $\Psi : \Lambda_K \to \Lambda_{K'}$ induces a ring isomorphism ψ between knot contact homologies which sends μ_K (respectively, λ_K) to $\mu_{K'}$ (respectively, $\lambda_{K'}$). More precisely, $\psi : \mathbb{Z}[\hat{\pi}_K] \to \mathbb{Z}[\hat{\pi}_{K'}]$ is the map induced by Ψ on homology. Note that if a Legendrian isotopy does not carry μ_K to $\mu_{K'}$ and λ_K to $\lambda_{K'}$, then it does not induce an isomorphism between $\widetilde{HC}_0(K)$ and $\widetilde{HC}_0(K')$ in the above sense. (See Section 3 for results in this more general setting.)

Theorems 1 and 3 will follow easily from Theorem 4, when we combine this with the fact that knot groups are locally indicable, which follows from [HS85, Lemma 2], and the classical result of Higman that group rings of locally indicable groups have no zero-divisors [Hig40].

Remark 5. The degree zero knot contact homology of the unknot is given by $\mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]/(1-\mu)(1-\lambda)$. It is pointed out in [CELN16] that since $\widetilde{HC}_0(U)$ has zero-divisors and $\widetilde{HC}_0(K)$ has no zero-divisors for non-trivial K by Theorem 4, we obtain an alternate proof that degree zero knot contact homology detects the unknot. Note that this does not depend on any knowledge of the peripheral structure on K.

Proof of Theorem 1. As discussed, degree zero knot contact homology detects the unknot, so we suppose throughout that K is non-trivial. Let $T = T_{p,q}$ and suppose that $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(T)$. First, we show that K must be equivalent to $T_{p',q'}$ where $p'q' = pq$. By Theorem 4, we have a ring isomorphism ψ between $\mathfrak{R}_T$ and $\mathfrak{R}_K$ which sends μ_T, λ_T to μ_K, λ_K, respectively. We will focus in particular on the elements $\phi_K = \mu_K^{pq}\lambda_K$ and $\phi_T = \mu_T^{pq}\lambda_T$, which are identified via ψ.

Recall that π_T has non-trivial center, isomorphic to $\langle \phi_T \rangle$, since this is the fiber slope coming from the Seifert structure on the exterior of T. On the other hand, by work of Burde and Zieschang [BZ66], if K is not a torus knot, then π_K has trivial center. Therefore, if K is not isotopic to $T_{p',q'}$ with $pq = p'q'$, we have that ϕ_K is not central in π_K. Thus, there exists $\gamma \in \pi_K$ such that $\gamma\phi_K \neq \phi_K\gamma$. However, since $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(T_{p,q})$, we have that ϕ_K is central in $\mathfrak{R}_K$. Thus, $\phi_K(\gamma - \mu_K\gamma) = (\gamma - \mu_K\gamma)\phi_K$. In $\mathbb{Z}[\pi_K]$, we rearrange to obtain

$$\begin{aligned} \phi_K\gamma - \gamma\phi_K &= \phi_K\mu_K\gamma - \mu_K\gamma\phi_K \\ &= \mu_K(\phi_K\gamma - \gamma\phi_K), \end{aligned}$$

where the second equality comes from the fact that μ_K and ϕ_K commute, being elements of $\hat{\pi}_K \cong \mathbb{Z}^2$. Again, we rearrange to obtain, in $\mathbb{Z}[\pi_K]$, the equality $(\mu_K - 1)(\phi_K\gamma - \gamma\phi_K) = 0$. Note that $\mu_K \neq 1$, since $\mathbb{Z}[\pi_K]$, as an abelian group, is freely generated by the elements of π_K. Consequently, $\mathbb{Z}[\pi_K]$ has zero-divisors, since $\phi_K\gamma \neq \gamma\phi_K$. As discussed above, π_K is locally indicable, and therefore $\mathbb{Z}[\pi_K]$ has no zero-divisors. This is a contradiction. It therefore follows that $K = T_{p',q'}$, where $p'q' = pq$. In particular, we point out that K is not isotopic to $T_{-p,q}$.

It remains to show that degree zero knot contact homology distinguishes $T_{p',q'}$ from $T_{p,q}$ where $p'q' = pq$, but $\{p,q\} \neq \{p',q'\}$ or $\{-p',-q'\}$. If $T_{p,q}$ and $T_{p',q'}$ had isomorphic degree zero knot contact homology, then they would have the same augmentation polynomial $\widetilde{A}_K(\mu,\lambda)$ by [Ng08]. Since $\widetilde{A}_K(\mu^{-1},\lambda) = \widetilde{A}_{-K}(\mu,\lambda)$ [Ng08], where $-K$ denotes the mirror of K, we may assume that we have two torus knots $T_{p,q}, T_{p',q'}$ with $1 < p < q$ and $1 < p' < q'$ with the same augmentation polynomials. In [Cor13, Corollary 1.6], Cornwell shows:

$$\widetilde{A}_{T_{p,q}}(\mu,\lambda) = (1-\mu)(\lambda\mu^{(p-1)q} + (-1)^p)\prod_{n=1}^{p-1}(\lambda^n\mu^{(n-1)pq} - 1). \tag{2.1}$$

Recall that the augmentation polynomial is well defined up to multiplication by units in $\mathbb{Z}[\mu^{\pm},\lambda^{\pm}]$. We denote this relation by $\doteq$. As normalized above, we have that $\widetilde{A}_{T_{p,q}}(\mu,\lambda)$ has no negative powers of μ or λ and has constant term ± 1. Therefore, $\widetilde{A}_{T_{p,q}}(\mu,\lambda) = \pm\widetilde{A}_{T_{p',q'}}(\mu,\lambda)$ with the given normalizations. Note that in the above polynomial on the right-hand side, the smallest power of μ greater than 1 appearing with λ^1 occurs in a single term and has exponent $(p-1)q$, and similarly for $T_{p',q'}$. Since $pq = p'q'$, by assumption, we see $q = q'$ and thus $p = p'$ as well. $\square$

In order to detect cables and composite knots, we recall the result of Simon [Sim76], generalizing the characterization of torus knots in terms of centralizers; this states that if there exists a non-trivial element $v \in \hat{\pi}_K$ and $g \in \pi_K \setminus \hat{\pi}_K$ with $vg = gv$, then K is either a cable or a composite knot. In fact, the proof in [Sim76] yields:

Theorem 6. *Suppose a non-trivial element $v \in \hat{\pi}_K$ commutes with $g \in \pi_K \setminus \hat{\pi}_K$. Then, either:*

(1) *K is composite and v is a power of μ, or*
(2) *K is a (p,q)-cable and v is a power of $\mu^{pq}\lambda$.*

With this we are now able to prove Theorem 3.

Proof of Theorem 3. For notational simplicity, we give only the argument for cabled knots. In light of Theorem 6, it will be clear that the same argument applies for composite knots as well.

As before, we may assume that K is a non-trivial knot. Suppose that K is not a cabled knot but $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(C_{p,q}(J))$ for some knot J. Let $g \in \pi_{C_{p,q}(J)}$ be a non-peripheral element which commutes with $\mu^{pq}\lambda$. In $\widetilde{HC}_0(C_{p,q}(J))$, we see that $\mu^{pq}\lambda$ commutes with $(1-\mu)g$. Let $\psi : \widetilde{HC}_0(C_{p,q}(J)) \to \widetilde{HC}_0(K)$ denote the isomorphism. Note that we can write $\psi((1-\mu)g)$ as $z + (1-\mu)w$, where $z \in \mathbb{Z}[\hat{\pi}_K]$ (possibly zero) and $w = \sum_{i=1}^n a_i w_i$, where a_i are non-zero integers and w_i are distinct elements in $\pi_K \setminus \hat{\pi}_K$; indeed, if $\psi((1-\mu)g) = z \in \mathbb{Z}[\hat{\pi}_K]$, this would contradict the injectivity of ψ as $\psi : \mathbb{Z}[\hat{\pi}_{C_{p,q}(J)}] \to \mathbb{Z}[\hat{\pi}_K]$ is an isomorphism. Of course z and $\mu^{pq}\lambda$ commute, so we see that $\mu^{pq}\lambda$ and $(1-\mu)w$ commute in $\widetilde{HC}_0(K)$ and thus in $\mathbb{Z}[\pi_K]$. As in the proof of Theorem 1, since $(1-\mu) \neq 0$, we have

$$\sum_{i=1}^n a_i(\mu^{pq}\lambda w_i - w_i\mu^{pq}\lambda) = 0.$$

Note, in particular, that for each i, $\mu^{pq}\lambda w_i = w_j\mu^{pq}\lambda$ for some j. It follows that there exists $k > 0$ such that $(\mu^{pq}\lambda)^k w_1 = w_1(\mu^{pq}\lambda)^k$. Since w_1 is not peripheral, it now follows from Theorem 6 that K is a (p',q')-cable with $p'q' = pq$. $\square$

It is an interesting problem to try to determine if $\widetilde{HC}_0(K) \cong \widetilde{HC}_0(C_{p,q}(J))$ implies that K must in fact be a (p,q)-cable of a knot J' with $\widetilde{HC}_0(J') \cong \widetilde{HC}_0(J)$. A similar question exists for composite knots.

Remark 7. In fact, the proofs of Theorems 1 and 3 use a weaker property than local indicability of the knot group. The proofs only use that left-multiplication by $(1-\mu)$ is an injection on the group ring of any knot group. That this property holds can be seen by a similar argument as at the end of the proof of Theorem 3.

3. Framed knot contact homology and further detection

In this section, we will work with a more general notion of an isomorphism of knot contact homologies. We will say that a ring isomorphism $\psi : \widetilde{HC}_0(K) \to \widetilde{HC}_0(K')$ is a *weak isomorphism*, if $\psi|_{\hat{\pi}_K}$ induces an isomorphism from $\hat{\pi}_K$ to $\hat{\pi}_{K'}$. For example, a Legendrian isotopy from Λ_K to $\Lambda_{K'}$ induces a weak isomorphism from $\widetilde{HC}_0(K)$ to $\widetilde{HC}_0(K')$; it will only be a true isomorphism of knot contact homologies if the isotopy takes meridian to meridian and longitude to longitude. For notation, given an isomorphism $\psi : \hat{\pi}_K \to \hat{\pi}_{K'}$, by simply dropping the subscripts K and K', we obtain an isomorphism $\psi : \mathbb{Z}[\mu^{\pm}, \lambda^{\pm 1}] \to \mathbb{Z}[\mu^{\pm}, \lambda^{\pm 1}]$. We will call ψ *orientation-preserving* if the isomorphism is in $SL_2(\mathbb{Z})$, when represented in the standard meridian-longitude bases. We say a weak isomorphism of knot contact homologies is orientation-preserving if the restriction to $\hat{\pi}_K$ is orientation-preserving. A priori, a weak isomorphism need not be orientation-preserving. Finally, if ψ is a weak isomorphism, we note that it follows from the definition of the augmentation polynomial [Ng08] that $\widetilde{A}_K(\psi(\mu), \psi(\lambda)) = \widetilde{A}_{K'}(\mu, \lambda)$, when these polynomials are defined.

With this in mind, Theorem 2 is a direct corollary of the following detection result for torus knots in the setting of weak isomorphisms.

Proposition 8. *Suppose there exists a weak isomorphism $\psi : \widetilde{HC}_0(T_{p,q}) \to \widetilde{HC}_0(K)$. Then, K is isotopic to $T_{p,q}$ or its mirror. If ψ is orientation-preserving, then K is isotopic to $T_{p,q}$.*

Proof. Let $T = T_{p,q}$. Since ψ is a weak isomorphism, we have that $\widetilde{A}_K(\mu, \lambda) = \widetilde{A}_T(\psi(\mu), \psi(\lambda))$. Since ϕ_T is central in $\mathfrak{R}_T$, $\psi(\phi_T) = \omega_K$ is central in $\mathfrak{R}_K$. We no longer can say that $\psi(\phi_T) = \mu_K^{pq}\lambda_K$, but we do still have that $\omega_K \in \hat{\pi}_K$. The same argument as in Theorem 1 shows that ω_K is central in π_K, and thus K is a torus knot T' and ω_K must equal $\phi_{T'}^{\pm 1}$. Indeed, the only elements in $\pi_{T'}$ with this property are of the form $\phi_{T'}^n$, and ω_K cannot be a different power since ψ induces an isomorphism of the peripheral subgroups. The following lemma now completes the proof. ☐

Lemma 9. *Fix two torus knots, T and T', and an isomorphism $\psi : \hat{\pi}_T \to \hat{\pi}_{T'}$ of their peripheral subgroups which takes ϕ_T to $\phi_{T'}^{\pm 1}$. If $\widetilde{A}_{T'}(\mu, \lambda) = \widetilde{A}_T(\psi(\mu), \psi(\lambda))$, then T is isotopic to T' or its mirror. Further, if ψ is orientation-preserving, then T and T' are isotopic.*

Proof. We first consider the case that $T = T_{p,q}$ and $T' = T_{p',q'}$ are both positive torus knots with $1 < p < q$ and $1 < p' < q'$. We rewrite $\widetilde{A}_T$ in terms of μ and $\phi = \mu^{pq}\lambda$. We have

$$\widetilde{A}_T(\mu, \phi) \doteq (1-\mu)(\phi\mu^{-q} + (-1)^p) \prod_{n=1}^{p-1} (\phi^n \mu^{-pq} - 1), \tag{3.1}$$

and similarly for $\widetilde{A}_{T'}$. By assumption, we have $\widetilde{A}_{T'}(\mu,\phi) = \widetilde{A}_T(\psi(\mu),\phi^\epsilon)$, where $\epsilon \in \{-1,1\}$. Since μ_T, ϕ_T form a basis for $\hat{\pi}_T$ (and similarly for T'), we must have that $\psi(\mu_T) = \mu_{T'}^\eta \phi_{T'}^k$ for some $\eta \in \{-1,1\}$ and $k \in \mathbb{Z}$. We thus have

$$\begin{aligned} (3.2) \qquad & (1-\mu)(\phi\mu^{-q'} + (-1)^{p'}) \prod_{n=1}^{p'-1} (\phi^n \mu^{-p'q'} - 1) \\ & \doteq (1-\mu^\eta\phi^k)(\phi^\epsilon \mu^{-q\eta}\phi^{-qk} + (-1)^p) \prod_{n=1}^{p-1} (\phi^{\epsilon n}\mu^{-pq\eta}\phi^{-kpq} - 1). \end{aligned}$$

Setting $\phi = 1$, we have

$$(1-\mu)(\mu^{-q'} + (-1)^{p'}) \prod_{n=1}^{p'-1} (\mu^{-p'q'} - 1) \doteq (1-\mu^\eta)(\mu^{-q\eta} + (-1)^p) \prod_{n=1}^{p-1} (\mu^{-pq\eta} - 1),$$

which after multiplication by units, gives

$$(3.3) \qquad (1-\mu)(\mu^{q'} + (-1)^{p'}) \prod_{n=1}^{p'-1} (\mu^{p'q'} - 1) \doteq (1-\mu)(\mu^q + (-1)^p) \prod_{n=1}^{p-1} (\mu^{pq} - 1).$$

Neither of the polynomials in (3.3) has a negative power of μ. Further, the constant term is ± 1 on both sides. Therefore, the coefficients for μ^i must agree up to sign for all i. The next term for each polynomial is $\pm\mu$, so we compare the subsequent lowest-degree terms. These are $\pm\mu^{q'}$ and $\pm\mu^q$, respectively, and thus we must have $q = q'$. Since the total degrees of the two polynomials in (3.3) must also agree, we see that $p' = p$ as well.

By taking mirrors of both sides if necessary, it remains to consider the case when $T = T_{p,q}$ is a positive torus knot and $T' = T_{-p',q'}$ is a negative torus knot, where we write $1 < p < q$ and $1 < p' < q'$. Recall that $\widetilde{A}_{T'}(\mu,\lambda) \doteq \widetilde{A}_{T_{p',q'}}(\mu^{-1},\lambda)$. Since $\phi_{T'} = \mu_{T'}^{-p'q'}\lambda_{T'}$ and $\phi_{T_{p',q'}} = \mu_{T_{p',q'}}^{p'q'}\lambda_{T_{p',q'}}$, we have that $\widetilde{A}_{T'}(\mu,\phi) \doteq \widetilde{A}_{T_{p',q'}}(\mu^{-1},\phi)$. As before, we may write $\psi(\phi) = \phi^\epsilon$ and $\psi(\mu) = \mu^\eta\phi^k$. Thus, we obtain in analogy with (3.2)

$$\begin{aligned} (3.4) \qquad & (1-\mu^{-1})(\phi\mu^{q'} + (-1)^{p'}) \prod_{n=1}^{p'-1} (\phi^n \mu^{p'q'} - 1) \\ & \doteq (1-\mu^\eta\phi^k)(\phi^\epsilon \mu^{-q\eta}\phi^{-kq} + (-1)^p) \prod_{n=1}^{p-1} (\phi^{\epsilon n}\mu^{-pq\eta}\phi^{-kpq} - 1). \end{aligned}$$

Following the same argument as above, setting $\phi = 1$ implies that $p = p'$ and $q = q'$. Therefore, we see that T and T' are isotopic after mirroring.

It remains to prove that if ψ is orientation-preserving, then we cannot be in the case that T is a positive torus knot and T' is a negative torus knot. Continuing with the above case, set $\mu = 1$ in (3.4). We obtain

$$0 = (1-\phi^k)(\phi^{\epsilon-kq} + (-1)^p) \prod_{n=1}^{p-1} (\phi^{\epsilon n - kpq} - 1).$$

This is only possible if $k = 0$, since $q \geq 3$. Returning to (3.4), we set $\phi\mu^{pq} = 1$, and obtain

$$0 = (1 - \mu^{\eta})(\mu^{-\epsilon pq}\mu^{-q\eta} + (-1)^p)\prod_{n=1}^{p-1}(\mu^{-\epsilon npq}\mu^{-pq\eta} - 1).$$

This equality can only happen if $\eta = -\epsilon$, which implies that ψ is orientation-reversing. This is a contradiction. $\square$

Finally, we are able to establish a similar detection result for n-framed knot contact homology. Recall that the n-framed knot contact homology of a knot K (fully non-commutative, degree zero) is identical to that of $\widetilde{HC}_0(K)$ as a ring; the difference is that the embedding of $\mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$ into $\widetilde{HC}_0(K)$ sends μ to μ_K and λ to $\mu_K^n\lambda_K$. In this setting, 0-framed knot contact homology is the version we have been working with in this paper. As above, we will say that a weak isomorphism between degree zero framed knot contact homologies is a ring isomorphism which restricts to an isomorphism between the peripheral subgroups. Further, we do not require that the framings be the same.

Theorem 10. *Suppose that the n-framed degree zero knot contact homology of K is weakly isomorphic to the m-framed degree zero knot contact homology of $T_{p,q}$. Then K is isotopic to $T_{p,q}$ or its mirror. If the weak isomorphism is orientation-preserving, then K is isotopic to $T_{p,q}$.*

Proof. It is straightforward to verify that a weak isomorphism between framed degree zero knot contact homologies induces a weak isomorphism between the 0-framed degree zero knot contact homologies. The result now follows from Proposition 8. $\square$

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