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A NOTE ON PARTIALLY HYPERBOLIC SYSTEMS WITH MOSTLY EXPANDING CENTERS

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Abstract. We show the existence and finiteness of SRB (Physical) measures for partially hyperbolic diffeomorphism f with dominated splitting $TM = E^u \oplus E^{cu} \oplus E^{cs}$, such that (f, E^{cu}) has the $\mathcal{G}^+$ property and (f, E^{cs}) has the $\mathcal{G}^-$ property.

1. Introduction

One topic in dynamical systems is to find invariant measures with the observable property. Sinai, Ruelle and Bowen succeeded in getting these kinds of invariant measures in the 1970s [6, 7, 9, 10] for uniformly hyperbolic systems, called *Sinai-Ruelle-Bowen* (*SRB*) or *physical measures*. A natural question arises: Are there some SRB/physical measures for systems beyond the uniform hyperbolicity?

Let M be a finite dimensional compact Riemannian manifold and $f : M \to M$ a diffeomorphism of class C^{1+}, meaning that f is C^1 and Df is Hölder continuous. In this paper, we say that an f-invariant measure μ is an SRB measure if it has positive Lyapunov exponents μ-a.e. and the conditional measures along *Pesin unstable manifolds* are absolutely continuous w.r.t. the Lebesgue measures induced in the corresponding manifolds. Generally, an invariant measure μ is said to be physical if the *basin of* μ (w.r.t. f),

$$B(\mu, f) = \left\{ x \in M : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \xrightarrow{weak^*} \mu \text{ as } n \to +\infty \right\}$$

has positive Lebesgue measure.

In this work, we say that f is a *partially hyperbolic* diffeomorphism if there exist $\lambda \in (0, 1)$ and a Df-invariant continuous splitting $TM = E^u \oplus E^c$ such that

- E^u *dominates* E^c: $\|Df|E^c(x)\| \cdot \|Df^{-1}|E^u(f(x))\| \leq \lambda$ for every $x \in M$. We rewrite $E^u \oplus_{\succ} E^c$ to denote this domination.
- E^u is *uniformly expanding*: $\|Df^{-1}|E^u(x)\| \leq \lambda$ for every $x \in M$.

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We shall mainly deal with the partially hyperbolic diffeomorphisms of the following type: $TM = E^u \oplus_\succ E^{cu} \oplus_\succ E^{cs}$, where E^{cu} dominates E^{cs} and E^u dominates E^{cu} and is uniformly expanding.

It is known that there exist strong unstable manifolds $W^u(x)$, $x \in M$ tangent to the distribution E^u. They are backward contracting with rate λ in the sense that: For every $x \in M$ we have $d(f^{-n}(x), f^{-n}(y)) \leq \lambda^n d(x,y)$, for every $n \geq 1, y \in W^u(x)$.

As an important candidate of SRB/physical measures, *Gibbs u-state* was introduced and studied in [8] by Pesin and Sinai for partially hyperbolic systems. Let f be a partially hyperbolic diffeomorphism; we say an f-invariant measure μ is a Gibbs u-state if the conditional measures of μ along *strong* unstable manifolds are absolutely continuous w.r.t. the Lebesgue measures along these manifolds.

Denote by $G^u(f)$ the set of all the Gibbs u-states of partially hyperbolic diffeomorphism f. Given a Df-invariant sub-bundle E, we say (f,E) satisfies

— the $\mathcal{G}^+$ *property* if all the Lyapunov exponents of (f,μ) along E are positive for every $\mu \in G^u(f)$;

— the $\mathcal{G}^-$ *property* if all the Lyapunov exponents of (f,μ) along E are negative for every $\mu \in G^u(f)$.

The main result of this paper is the following:

Theorem A. *Let f be a C^{1+} partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_\succ E^{cu} \oplus_\succ E^{cs}$. Assume that (f, E^{cu}) satisfies the $\mathcal{G}^+$ property and (f, E^{cs}) satisfies the $\mathcal{G}^-$ property. Then there exist finitely many ergodic SRB measures, which are physical measures whose basins cover a full Lebesgue measure subset of M.*

We also prove a related result.

Theorem B. *Let f be a C^{1+} partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_\succ E^{cu} \oplus_\succ E^{cs}$. Assume that (f, E^{cs}) satisfies the $\mathcal{G}^-$ property, and*

$$\int \log \mathrm{m}(Df|E^{cu}) d\mu > 0, \ \forall \mu \in G^u(f).$$

Then there exist finitely many ergodic SRB measures, which are physical measures whose basins cover a full Lebesgue measure subset of M.

Remark 1.1. If (f, E^{cu}) satisfies the $\mathcal{G}^+$ property, then we say f is *mostly expanding* along E^{cu}. Similarly, we say f is *mostly contracting* along E^{cs} when (f, E^{cs}) satisfies the $\mathcal{G}^-$ property.

Since the conditions of the above theorems are C^1-robust (see [12, Theorem B] for $\mathcal{G}^+$ and $\mathcal{G}^-$properties), we get the following: For f as in Theorem A or Theorem B, there is a C^1 open neighborhood $\mathcal{V}(f)$ of f such that every C^{1+} diffeomorphism $g \in \mathcal{V}(f)$ has finitely many physical measures whose basins cover a full Lebesgue measure subset of M.

Let us recall some related results.

- In [5], the existence and finiteness of SRB (physical) measures were achieved by Bonatti-Viana for partially hyperbolic diffeomorphisms with splitting $TM = E^u \oplus_\succ E^c$, such that (f, E^c) has the $\mathcal{G}^-$ property.
- Recently, Andersson and Vasquez [3, Theorem C] showed the existence of finitely many physical measures for C^{1+} partially hyperbolic systems with

splitting $TM = E^u \oplus_{\succ} E^c \oplus_{\succ} E^s$, such that (f, E^c) has the $\mathcal{G}^+$ property. More precisely, they proved that such systems are *weakly non-uniformly expanding* along E^c (see definition in [2] or [3]), then they obtained the desired result by using [2, Theorem A].

The allowance of positive Lyapunov exponents along the central direction push us to deal with the systems here in a different way than [5]. We would like to mention that during the proof of Theorem A, we prove that the $\mathcal{G}^+$ property implies the *non-uniformly expanding* (NUE for short, see [1] for this term) property for some iterate, not just the *weakly* non-uniformly expanding as in [3]. With this improvement, we do not need to construct the *Young-tower*, and can dispose of the weaker center direction E^{cs} more freely.

On the existence of SRB measures for two non-hyperbolic bundles, we recall the following conjecture proposed by Alves, Bonatti and Viana in [1, Section 6]:

Suppose f is a C^{1+} diffeomorphism with the dominated splitting $TM = E^{cu} \oplus_{\succ} E^{cs}$, does there exist some SRB /physical measures when E^{cu} is NUE and E^{cs} is NUC (non-uniformly contracting[1])?

In [1], the authors proved this conjecture with the extra assumption that the system admitted the so-called *simultaneous hyperbolic times* (see [1, Proposition 6.5]). It remains to be a challenging open problem. The present work gives some partial results along this way.

2. Gibbs u-states and Gibbs cu-states

Denote by $\mathrm{m}(A) = \inf_{\|v\|=1} \|Av\| = \|A^{-1}\|^{-1}$ the *mini-norm* of linear isomorphism A. Let Leb represent the Lebesgue measure of M. Given a sub-manifold $\gamma \subset M$ we use Leb_γ to denote the normalized Lebesgue measure on γ induced by the restriction of the Riemannian structure to γ.

We say D is a *strong unstable disk* if it is a disk contained in some strong unstable manifold with the same dimension.

Given $f \in C^{1+}$, an f-invariant probability μ and a Df-invariant sub-bundle E, define

$$\mathcal{L}_E(f, \mu) := \int \log \mathrm{m}(Df|E) d\mu.$$

Throughout this section, let f be a C^{1+} diffeomorphism on M.

We will use the following properties of Gibbs u-states (see [4, Subsection 11.2]).

Proposition 2.1. *Let f be a partially hyperbolic diffeomorphism, then*

1. *$G^u(f)$ is non-empty, convex and compact relative to the weak$_*$ topology in the space of probabilities.*
2. *Every ergodic component of any $\mu \in G^u(f)$ is a Gibbs u-state also.*
3. *There exists a set $\mathcal{U}(f)$ intersecting every strong unstable disk on a full Lebesgue measure, such that for any $x \in \mathcal{U}(f)$ every accumulation point of $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}$ is a Gibbs u-state.*

Let f be a partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_{\succ} E^{cu} \oplus_{\succ} E^{cs}$; we say an invariant measure μ is a *Gibbs cu-state* of f if μ has at least $\dim(E^u \oplus E^{cu})$ positive Lyapunov exponents μ-a.e., and the conditional measures along the Pesin unstable manifolds tangent to $E^u \oplus_{\succ} E^{cu}$ are μ-a.e. absolutely continuous w.r.t. Lebesgue measures of these manifolds.

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[1] See [1, Page 391] for this notion.

Denote by $G^{cu}(f)$ the set of all the Gibbs cu-states of f. A folklore result says that $G^{cu}(f) \subset G^u(f)$.

Similar to the Gibbs u-states, the Gibbs cu-states have the next property related to the ergodic components. A proof can be found in [11, Lemma 2.4].

Proposition 2.2. *Let f be a partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_{\succ} E^{cu} \oplus_{\succ} E^{cs}$ and μ an f-invariant measure. Then μ is a Gibbs cu-state if and only if every ergodic component of μ is a Gibbs cu-state for f.*

The next lemma follows directly by definitions, hence we omit the proof here.

Lemma 2.3. *Let f be a partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_{\succ} E^{cu} \oplus_{\succ} E^{cs}$. Given $m \in \mathbb{N}$, if $\mu \in G^u(f^m)$* (resp. $G^{cu}(f^m)$)*, then the measure $\mu_0 = \frac{1}{m}\sum_{i=0}^{m-1} f_*^i\mu$ is contained in $G^u(f)$* (resp. $G^{cu}(f)$)*. Moreover, if μ is f^m-ergodic, then μ_0 is f-ergodic.*

Lemma 2.4. *Let f be a partially hyperbolic diffeomorphism. Let E be a continuous Df-invariant sub-bundle. Then (f, E) satisfies the $\mathcal{G}^+$* (resp. $\mathcal{G}^-$) *property if and only if (f^m, E) satisfies the $\mathcal{G}^+$* (resp. $\mathcal{G}^-$) *property for any $m \in \mathbb{N}$.*

Proof. We prove the $\mathcal{G}^+$ case here, the $\mathcal{G}^-$ case can be deduced analogously. One need only to prove the " only if " part for any given $m \in \mathbb{N}$. By item (2) of Proposition 2.1, it suffices to show that for any f^m-ergodic measure μ, the Lyapunov exponents are positive along E w.r.t. μ. Consider the f-invariant measure $\mu_0 = \frac{1}{m}\sum_{i=0}^{m-1} f_*^i\mu$, then μ-a.e. x is in a full μ_0 measure set, and by Lemma 2.3 we know $\mu_0 \in G^u(f)$. Thus, by definition, for μ_0-a.e. x,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{m}(Df^n|E(x)) > 0.$$

Consequently, for μ-a.e. x we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{m}(Df^{nm}|E(x)) = m \lim_{n\to\infty} \frac{1}{nm} \log \mathrm{m}(Df^{nm}|E(x)) > 0.$$

□

3. $\mathcal{G}^+$ property and non-uniform expansion

Let f be a diffeomorphism and E a continuous Df-invariant sub-bundle. Some results about the relations between $\mathcal{G}^+$ property and NUE property are provided in this section, similar results hold for $\mathcal{G}^-$ and NUC.

The following theorem asserts that $\mathcal{G}^+$ implies NUE for some iterate.

Theorem 3.1. *If (f, E) satisfies the $\mathcal{G}^+$ property, then there exist $N \in \mathbb{N}$, $\alpha > 0$ such that*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \mathrm{m}(Df^N|E(f^{iN}x)) > \alpha, \textit{ for Leb-a.e. } x \in M.$$

The proof of Theorem 3.1 can be deduced by Proposition 3.3 and Proposition 3.2 directly.

Proposition 3.2. *If $\mathcal{L}_E(f, \mu) > 0$ for every $\mu \in G^u(f)$, then there is $\sigma > 0$ such that* Leb*-a.e. x is non-uniformly expanding along E with rate σ, i.e.,*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \mathrm{m}(Df|E(f^i x)) > \sigma.$$

Proposition 3.3. *If (f, E) satisfies the $\mathcal{G}^+$ property, then there exist $N \in \mathbb{N}$, $\alpha > 0$ such that $\mathcal{L}_E(f^N, \mu) > \alpha$ for every $\mu \in G^u(f^N)$.*

Proof of Proposition 3.2. By the compactness of $G^u(f)$ (item (1) of Proposition 2.1), we know that there exists $\sigma > 0$ such that

$$\mathcal{L}_E(f, \mu) > \sigma, \ \forall \mu \in G^u(f). \tag{3.1}$$

Recall the full Lebesgue measure subset $\mathcal{U}(f)$ of M as in item (3) of Proposition 2.1. We will show that every point $x \in \mathcal{U}(f)$ is non-uniformly expanding along E with rate σ. If not, there exists $x_0 \in \mathcal{U}(f)$ and a subsequence $\{n_k\}$ satisfying

$$\lim_{k\to+\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \log \mathrm{m}\left(Df|E(f^i(x_0))\right) \le \sigma.$$

Up to replacing $\{n_k\}$ by some subsequence, we may assume that

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i(x_0)} \xrightarrow{weak^*} \mu_0 \in G^u(f) \text{ as } k \to \infty.$$

Thus,

$$\mathcal{L}_E(f, \mu_0) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \log \mathrm{m}(Df|E(f^i(x_0))) \le \sigma.$$

This contradicts (3.1). The proof is complete. □

To prove Proposition 3.3, we need the lemmas below. The proof of the next lemma can be found in [3, Lemma 4.1] or [12, Proposition 5.5].

Lemma 3.4. *If (f, E) satisfies the $\mathcal{G}^+$ property, then there exist $\alpha > 0$ and $L \in \mathbb{N}$ such that $\mathcal{L}_E(f^L, \mu) > \alpha$ for every $\mu \in G^u(f)$.*

Lemma 3.5. *Given $L \in \mathbb{N}$, $\alpha_1 > 0$ and $\alpha_2 \in (0, \alpha_1/L)$, then there exists N as a multiple of L such that for any ergodic measure μ of f , if $\mathcal{L}_E(f^L, \mu) > \alpha_1$, then for μ-a.e. x,*

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{i=0}^{n-1} \log \mathrm{m}(Df^N|E(f^{iN}(x))) > \alpha_2.$$

Proof. Put $C = \max_{x\in M} \max\{|\log \mathrm{m}(Df(x))|, |\log \mathrm{m}(Df^{-1}(x))|\}$, $k = [\frac{2CL}{\alpha_1-\alpha_2 L}] + 1$. Take $N = kL$. Since μ is f-ergodic and f^L-invariant, we consider the finite ergodic components ν_i (for f^L), $0 \le i \le L-1$,[2] with $f_*\nu_i = \nu_{i+1}$, $\nu_L = \nu_0$, and $\mu = \frac{1}{L}\sum_{i=0}^{L-1} \nu_i$. Then, by assumption we have

$$\frac{1}{L} \sum_{i=0}^{L-1} \mathcal{L}_E(f^L, \nu_i) = \mathcal{L}_E(f^L, \mu) > \alpha_1.$$

Thus, there exists $0 \le j \le L-1$ such that $\mathcal{L}_E(f^L, \nu_j) > \alpha_1$. Observe that the union of the basins $B(\nu_i, f^L)$, $0 \le i \le L-1$ has full μ measure and note that $f(B(\nu_i, f^L)) = B(\nu_{i+1}, f^L)$ for any i by identifying $\nu_i = \nu_{i+mL}$. Therefore, for

[2]These measures may be counted more than once.

μ-a.e. x there exists $0 \le k_0 \le L-1$ such that $f^{k_0}(x) \in B(\nu_j, f^L)$, by Birkhoff's ergodic theorem, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{\ell=0}^{n-1} \log \mathrm{m}(Df^L|E(f^{L\ell+k_0}(x))) = \mathcal{L}_E(f^L,\nu_j) > \alpha_1. \tag{3.2}$$

For every $\ell \in \mathbb{N}$, the chain rule yields that

$$\mathrm{m}(Df^{kL}|E(f^{Lk\ell+k_0}(x))) \ge \prod_{i=0}^{k-1} \mathrm{m}(Df^L|E(f^{L(k\ell+i)+k_0}(x)).$$

Combining (3.2) with the above observation, we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{\ell=0}^{n-1} \log \mathrm{m}(Df^{kL}|E(f^{kL\ell+k_0}(x))) > k\alpha_1. \tag{3.3}$$

By the choices of C, N and using chain rule again, for every $\ell \in \mathbb{N}$, we have

$$\begin{aligned} \log \mathrm{m}(Df^N|E(f^{\ell N}(x))) \ &\ge\ \log \mathrm{m}(Df^{-k_0}|E(f^{(\ell+1)N+k_0}(x))) \\ &+\ \log \mathrm{m}(Df^N|E(f^{\ell N+k_0}(x))) \\ &+\ \log \mathrm{m}(Df^{k_0}|E(f^{\ell N}(x))) \\ &\ge\ \log \mathrm{m}(Df^N|E(f^{\ell N+k_0}(x))) - 2LC. \end{aligned}$$

Consequently, by the choice of k and (3.3), we get

$$\liminf_{n\to\infty} \frac{1}{nN}\sum_{\ell=0}^{n-1} \log \mathrm{m}(Df^N|E(f^{N\ell}(x))) > \frac{k\alpha_1 - 2CL}{kL} \ge \alpha_2.$$

Since μ is an f^N-invariant measure, by Birkhoff's ergodic theorem the above "$\liminf$" can be changed into "$\lim$", which completes the proof. $\square$

Based on Lemma 3.4 and Lemma 3.5, we can give the proof of Proposition 3.3.

Proof of Proposition 3.3. Since (f,E) satisfies the $\mathcal{G}^+$ property, by Lemma 3.4, there are $L \in \mathbb{N}$ and $\alpha_1 > 0$ such that $\mathcal{L}_E(f^L,\mu) > \alpha_1$ for every $\mu \in G^u(f)$.

For L, α_1 and some $\alpha_2 \in (0, \alpha_1/L)$, by Lemma 3.5, we can find $N \in \mathbb{N}$ such that for any f-ergodic $\mu \in G^u(f)$, and for μ-a.e. x, we have

$$\lim_{n\to\infty} \frac{1}{nN}\sum_{i=0}^{n-1} \log \mathrm{m}(Df^N|E(f^{iN}(x))) > \alpha_2 > 0.$$

For any f^N-ergodic measure $\tilde\mu \in G^u(f^N)$, let $\mu = \frac{1}{N}\sum_{i=0}^{N-1} f^i_*\tilde\mu$. By Lemma 2.3, we know μ is an f-ergodic measure contained in $G^u(f)$. Observe that from the construction of μ, we have that a full $\tilde\mu$ measure set is contained in a full μ measure set. By the Birkhoff's ergodic theorem for f^N and $\tilde\mu$, we conclude that

$$\mathcal{L}_E(f^N,\tilde\mu) = \int \log \mathrm{m}(Df^N|E)d\tilde\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \mathrm{m}(Df^N|E(f^{iN}(x))) > N\alpha_2 > 0$$

for $\tilde\mu$-a.e. x. Then using item (2) of Proposition 2.2, we know $\mathcal{L}_E(f^N,\mu) > N\alpha_2$ for every $\mu \in G^u(f^N)$. Now we complete the proof by taking $\alpha = N\alpha_2$. $\square$

4. Uniform properties under non-uniform expansion

Assume that f is a C^{1+} diffeomorphism with dominated splitting $TM = F \oplus_{\succ} E^{cs}$.

Let δ_0 be sufficiently small so that the inverse of the exponential map is well defined on the δ_0-neighborhood of every point.

Consider the following Grassmannian manifold:

$$G^{\dim F}(M) = \{V : V \text{ is a sub-space of } T_xM \text{ with dimension } \dim F, x \in M\}.$$

The Riemannian metric on M induces a distance "dist" on $G^{\dim F}(M)$.

We introduce some notation.

Cone field. For $a > 0$, define the cone field $\mathcal{C}_a^F(x)$, $x \in M$ by

$$\mathcal{C}_a^F(x) = \{\nu = \nu_1 + \nu_2 \in T_xM : \nu_1 \in E^{cs}(x), \nu_2 \in F(x) \text{ and } \|\nu_1\| \leq a\|\nu_2\|\}.$$

A C^1 sub-manifold D of M is *tangent to* $\mathcal{C}_a^F$ if it has dimension $\dim F$ and $T_xD \subset \mathcal{C}_a^F(x)$ for any $x \in D$.

NUE set. For $\sigma > 0$,

$$\mathcal{A}_\sigma = \left\{ x : \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \mathrm{m}(Df|F(f^i(x))) > \sigma > 0 \right\}.$$

Iterated measures. For a disk D, denote by $\mu_{n,D} = (1/n)\sum_{i=0}^{n-1} f_*^i \mathrm{Leb}_D$ for each $n \in \mathbb{N}$.

Balls inside sub-manifolds. For a sub-manifold D and small $r > 0$, denote by $B_D(x, r) := \{y \in D : d_D(x, y) \leq r\}$ the closed ball of radius r around x inside D.

Hölder curvature. Given $\xi \in (0, 1)$ and a C^1 disk D tangent to $\mathcal{C}_a^F$, we define the Hölder curvature of D as $H_\xi(D) = \inf\{C > 0 : TD \text{ is } (C, \xi)\text{-Hölder}\}$. Here we say TD is (C, ξ)-Hölder if

$$\mathit{dist}(T_xD, T_yD) \leq C d_D(x, y)^\xi, \text{ for all } y \in B(x, \delta_0) \cap D \text{ and } x \in D.$$

Theorem 4.1. *For any $\sigma > 0$, $\xi \in (0, 1)$ and $\eta > 0$, there exist constants $\lambda \in (0, 1)$, $a > 0$, $r > 0$ and $\beta > 0$, such that for any disk D tangent to $\mathcal{C}_a^F$ satisfying $H_\xi(f^n(D)) \leq \eta$ for all sufficiently large n, if $\mathrm{Leb}_D(D \cap \mathcal{A}_\sigma) \geq 1/2$, then for any accumulation point μ of $\{\mu_{n,D}\}$, there exist $z \in M$, a neighborhood $U(\mu) := U_z$ of z and a C^1 local chart $\Psi_z : U_z \to \mathbb{R}^{\dim M}$, $\Psi_z(z) = 0$ satisfying*

$$\mathrm{D}\Psi_z(z)(F(z)) = \{0\} \times \mathbb{R}^{\dim F}, \ \mathrm{D}\Psi_z(z)(E^{cs}(z)) = \mathbb{R}^{\dim E^{cs}} \times \{0\} \tag{4.1}$$

and there exists a family $\mathcal{L}_\infty(\mu)$ of disjoint disks tangent to $\mathcal{C}_a^F$ contained in $U(\mu)$, such that

1. *$d_{f^{-i}\gamma}(f^{-i}(x), f^{-i}(y)) \leq \lambda^i d_\gamma(x, y)$ for every $x, y \in \gamma$ and each $\gamma \in \mathcal{L}_\infty(\mu)$;*
2. *$\mu(L_\infty(\mu)) \geq \beta$, where $L_\infty(\mu)$ is the union of disks of $\mathcal{L}_\infty(\mu)$;*
3. *$U(\mu)$ contains a ball of radius $r/4$ around z, and the diameter of γ is contained in $(r/2, 2r)$ for every $\gamma \in \mathcal{L}_\infty(\mu)$.*

For proving Theorem 4.1, we need to choose (auxiliary) constants by several subsections.

4.1. **Choose λ and the auxiliary constant σ_0.** Given $\sigma > 0$, let $\sigma_0 = \exp(\sigma/4)$ and $\lambda = \sigma_0^{-\frac{1}{2}}$.

4.2. **Choose a and auxiliary constants r_0, C_1, ρ, κ.** By continuity, we may choose $a = a(f,\sigma) > 0$ and $0 < r_0 = r_0(f,\sigma) \le \delta_0$ sufficiently small so that

$$\mathrm{m}(Df|\widetilde{F}(y)) \ge {\sigma_0}^{-1/2}\mathrm{m}(Df|F(x)),\ \forall d(x,y) \le r_0, \forall \text{ sub-space } \widetilde{F}(y) \subset \mathcal{C}_a^F(y).$$

Recall that we say n is a σ_0-*hyperbolic time* for x if $\prod_{k=n-j}^{n-1} \mathrm{m}(Df|F(f^k(x))) \ge \sigma_0^j$ for every $1 \le j \le n$. By Pliss's lemma (see [1, Lemma 3.1]), there is a constant ρ related to f, σ and σ_0 such that if n is large enough, and $\prod_{k=0}^{n-1} \mathrm{m}(Df|F(f^k(x))) \ge e^{\sigma n}$, then there are at least ρn σ_0-hyperbolic times in the time interval $[0,n]$.

Property. There exists a constant $C_1 > 0$ such that for any D tangent to $\mathcal{C}_a^F$ with $H_\xi(f^n(D)) \le \eta$ for all sufficiently large n, if $x \in D \cap \mathcal{A}_\sigma$ and n is a sufficiently large σ_0-hyperbolic time for x, then

$$d_{f^i(D)}(f^i(y), f^i(z)) \le {\sigma_0}^{-(n-i)/2} d_{f^n(D)}(f^n(y), f^n(z)), \tag{4.2}$$

$$\frac{1}{C_1} \le \frac{|\mathrm{Det} Df^n|T_y D|}{|\mathrm{Det} Df^n|T_z D|} \le C_1, \tag{4.3}$$

whenever $f^n(y), f^n(z)$ are contained in $B_{f^n(D)}(f^n(x), r_0)$ (see [1, Lemma 2.7, Proposition 2.8]).

By reducing r_0 if necessary, we restate the following covering lemma from [1, Lemma 3.4].

Lemma 4.2. *There exists $\kappa > 0$ such that for any $\delta \in (0, r_0]$, for any C^1-embedded sub-manifold Δ tangent to $\mathcal{C}_a^F$, for any finite Borel measure $\mathcal{P}$ on Δ, if $O \subset \Delta$ is measurable,* Closure(O) *is compact and far away from the δ-neighborhood of $\partial\Delta$, then there exists a finite subset $Q \subset O$ such that the balls $\{B_\Delta(x, \delta/4), x \in Q\}$ are pairwise disjoint, and*

$$\mathcal{P}(O \cap B) \ge \kappa \mathcal{P}(B),\ \textit{ where } B = \bigcup_{x \in Q} B_\Delta(x, \delta/4).$$

4.3. **Choose r, β and auxiliary constants r_1, K, β_0.** By the uniformity of Hölder curvature, we may choose constant $r_1 < r_0/4$ such that for any disk Δ satisfying $H_\xi(\Delta) \le \eta$ and $B_\Delta(x, r_0/4) \subset \Delta$ for some $x \in \Delta$, then

$$\frac{\mathrm{Leb}_\Delta(B_\Delta(x, r_0/4) \setminus B_\Delta(x, r_0/4 - r_1))}{\mathrm{Leb}_\Delta(B_\Delta(x, r_0/4))} \le \frac{\kappa\rho}{24 C_1}. \tag{4.4}$$

Claim 4.3. There exists $r < r_1/2$ with following properties: If B is a disk tangent to F with radius $r/2$ around x, let $U_x = \bigcup_{y \in B} \{\exp_y(v) : v \perp T_y B, \|v\| < r/2\}$, then

- U_x contains a ball of radius $r/4$ with center x;
- for every disk γ tangent to $\mathcal{C}_a^F$, if it crosses U_x, then $\gamma \cap U_x$ has diameter between $r/2$ and $2r$.

This is because F is Hölder continuous by domination (see e.g. [2, Proposition 6.2]), and disks γ tangent to the cone field with a fixed width a. The claim gives the constant r we needed.

We say that U_x as constructed above is an $r/2$-cylinder around x, which can be equipped with a local chart Ψ_x satisfying Property (4.1).

As M is compact, we can take K as the minimum number of $r/4$-balls needed to cover M. Now we fix

$$\beta_0 = \frac{\kappa\rho}{12},\ \beta = \frac{\beta_0}{2K} = \frac{\kappa\rho}{24K}. \tag{4.5}$$

4.4. **Sketch of the proof of Theorem 4.1.** The main ideas of the proof go back to the previous work of [1], here we make some modifications to obtain the uniformity. Thus, a sketched proof is sufficient.

Let D be a disk as in the assumption of the theorem. Since $\mathrm{Leb}_D(D\cap\mathcal{A}_\sigma) \geq 1/2$, we can take a sub-disk D_0 in the interior of Closure(D) such that $\mathrm{Leb}_D(D_0\cap\mathcal{A}_\sigma) \geq 1/3$. Define

$$\mathcal{A}_n = \left\{ x \in D_0 \cap \mathcal{A}_\sigma : \sum_{\ell=1}^{m-1} \log \mathrm{m}(Df|F(f^\ell x)) \geq \frac{\sigma}{2} m,\ \forall m > n \right\}.$$

Thus, $D_0 \cap \mathcal{A}_\sigma = \bigcup_{n\geq 0} \mathcal{A}_n$, together with property (4.2) and assumption of Hölder curvature control ensures that there exists $m_0 \in \mathbb{N}$ so that:

— $\mathrm{Leb}_D(\mathcal{A}_n) \geq \mathrm{Leb}_D(D_0 \cap \mathcal{A}_\sigma)/2$ for every $n \geq m_0$.
— For every $x \in \mathcal{H}_n$ and $n \geq m_0$, we have $d(f^n(x), \partial(f^n(D))) > r_0$, where $\mathcal{H}_n = \{x \in D_0 \cap \mathcal{A}_\sigma : n \text{ is a } \sigma_0\text{-hyperbolic time for } x\}$.
— $H_\xi(f^n(D)) \leq \eta$ for every $n \geq m_0$.

Therefore, $B_{f^n(D)}(f^n(x), r_0/4)$ is a well-defined ball inside $f^n(D)$ for any $x \in \mathcal{H}_n$ whenever $n \geq m_0$. By Lemma 4.2, for each $n \geq m_0$, by taking $\Delta = f^n(D)$, $\mathcal{P} = f^n_*\mathrm{Leb}_D$, $\delta = r_0$ and $O = f^n(\mathcal{H}_n)$, there exists a finite number of pairwise disjoint balls $\{B_{f^n(D)}(x_{n,i}, r_0/4) : x_{n,i} \in f^n(\mathcal{H}_n), 1 \leq i \leq k(n)\}$ contained in $f^n(D)$, satisfying

$$f^n_*\mathrm{Leb}_D \left(\bigcup_{1\leq i\leq k(n)} B_{f^n(D)}(x_{n,i}, r_0/4) \cap f^n(\mathcal{H}_n) \right) \geq \kappa \mathrm{Leb}_D(\mathcal{H}_n). \tag{4.6}$$

Given $\eta > 0$ small enough, for each $\ell \geq m_0$ and $n > m_0$ we define

$$B_{\ell,\eta} = \bigcup_{i=1}^{k(\ell)} B_{f^\ell(D)}(x_{\ell,i}, r_0/4-\eta),\ \nu_{n,\eta} = \frac{1}{n}\sum_{\ell=m_0}^{n-1} f^\ell_*\mathrm{Leb}_D|B_{\ell,\eta},\ B_{\ell,0} = B_\ell,\ \nu_{n,0} = \nu_n.$$

Claim 4.4. There exists some integer $n_0 \geq m_0$, such that $\nu_n(M) \geq \beta_0$ for every $n \geq n_0$.

Now we prove the claim. By definition of ν_n and inequality (4.6), we obtain

$$\nu_n \left(\bigcup_{\ell=m_0}^{n-1} f^\ell(\mathcal{A}_\sigma \cap D) \right) \geq \nu_n \left(\bigcup_{\ell=m_0}^{n-1} (f^\ell(\mathcal{H}_\ell) \cap B_\ell) \right) \geq \frac{1}{n} \sum_{\ell=m_0}^{n-1} \kappa \mathrm{Leb}_D(\mathcal{H}_\ell). \tag{4.7}$$

Consider the product space $\{m_0, \ldots, n-1\} \times D$ admitting measure $\theta_n \times \mathrm{Leb}_D$, where θ_n is defined by $\theta_n(A) = card(A)/n$ for every $A \subset \{m_0, \ldots, n-1\}$. By the choice of m_0 and using Fubini's theorem, one gets

$$\begin{aligned}\frac{1}{n}\sum_{\ell=m_0}^{n-1}\mathrm{Leb}_D(\mathcal{H}_\ell) &= \int\left(\int \chi_{\mathcal{H}_\ell}(x)d\,\mathrm{Leb}_D(x)\right)d\theta_n(\ell)\\ &= \int\left(\int \chi_{\mathcal{H}_\ell}(x)d\theta_n(\ell)\right)d\,\mathrm{Leb}_D(x)\\ &\geq \mathrm{Leb}_D(\mathcal{A}_{m_0})\cdot\frac{\rho n-m_0}{n}\\ &\geq \frac{1}{6}\rho(1-\frac{m_0}{n\rho})\geq \rho/12, \text{ for } n \text{ large.}\end{aligned}$$

Therefore, we get the claim by (4.5)— the choice of β_0 .

Claim 4.5. $\nu_{n,r_1}(M)\geq \beta_0/2$ for every $n\geq n_0$.

Indeed, given $\ell\in\mathbb{N}$, $1\leq i\leq k(\ell)$, take $A=B_{f^\ell D}(x_{\ell,i},r_0/4)\setminus B_{f^\ell D}(x_{\ell,i},r_0/4-r_1)$, $B=B_{f^\ell D}(x_{\ell,i},r_0/4)$. By property (4.3) and the mean value theorem, we have

$$\begin{aligned}\frac{\mathrm{Leb}_{f^\ell D}(A)}{\mathrm{Leb}_{f^\ell D}(B)} &= \frac{\int_{f^{-\ell}A}\left|\mathrm{Det}Df^\ell|T_x(f^{-\ell}B)\right|d\mathrm{Leb}_D(x)}{\int_{f^{-\ell}B}\left|\mathrm{Det}Df^\ell|T_x(f^{-\ell}B)\right|d\mathrm{Leb}_D(x)}\\ &= \frac{\left|\mathrm{Det}Df^\ell|T_y(f^{-\ell}B)\right|\mathrm{Leb}_D(f^{-\ell}A)}{\left|\mathrm{Det}Df^\ell|T_z(f^{-\ell}B)\right|\mathrm{Leb}_D(f^{-\ell}B)}\\ &\geq \frac{1}{C_1}\frac{\mathrm{Leb}_D(f^{-\ell}A)}{\mathrm{Leb}_D(f^{-\ell}B)},\end{aligned}$$

where y,z are some points inside $f^{-\ell}(B)$. It follows from (4.4) that

$$f^\ell_*\mathrm{Leb}_D(A)\leq\frac{f^\ell_*\mathrm{Leb}_D(A)}{f^\ell_*\mathrm{Leb}_D(B)}=\frac{\mathrm{Leb}_D(f^{-\ell}A)}{\mathrm{Leb}_D(f^{-\ell}B)}\leq C_1\frac{\mathrm{Leb}_{f^\ell D}(A)}{\mathrm{Leb}_{f^\ell D}(B)}\leq\frac{\kappa\rho}{24},$$

and hence, $f^\ell_*\mathrm{Leb}_D(B_\ell\setminus B_{\ell,r_1})\leq\kappa\rho/24.$ We then get the result by definitions of ν_n and ν_{n,r_1}. Define $B_\infty=\bigcap_{n\geq m_0}\overline{\bigcup_{\ell\geq n}B_\ell}$. Take subsequences if necessary, we may suppose $\mu_n\to\mu$, $\nu_n\to\nu$. By definition, $\mathrm{supp}(\nu)\subset B_\infty$. Therefore, by using the Ascoli-Arzela theorem, for every $y\in\mathrm{supp}(\nu)$, there exist points $x_j\in B_{\ell_j}$ converge to a point x for subsequence $\ell_j\to\infty$, and $B_{f^{\ell_j}D}(x_j,r_0/4)$ converges to some disk $B(x)$ containing y with center x of radius $r_0/4$ in C^1 topology. Denote by $\mathcal{B}_\infty$ the family of disks $B(x)$.

Let us assume $\nu_{n,r_1}\to\nu_{r_1}$(up to considering subsequences), whose support is contained in the union of the family of disks $\mathcal{B}_{\infty,r_1}$ obtained by removing r_1-neighborhood of the boundary from each disk of $\mathcal{B}_\infty$. From the first statement of Claim 4.3 and the choice of K, there exists an $r/2$-cylinder U_z around some point $z\in B(x)\in\mathcal{B}_\infty$ for some x such that $\nu_{r_1}(U_z)\geq\kappa\rho/24K$.

Let $L_\infty:=L_\infty(\mu)$ be the union of the intersections of U_z with all the disks from $\mathcal{B}_\infty$ that cross U_z, the family of these restricted disks is denoted by $\mathcal{L}_\infty$. Observe that by the second statement of Claim 4.3, for every $\gamma\in\mathcal{B}_{\infty,r_1}$ intersecting U_z, then its extension $\gamma'\in\mathcal{B}_\infty$ must cross U_z. Combining this observation with Claim 4.5 we obtain

$$\mu(L_\infty)\geq\nu_{r_1}(L_\infty)\geq\frac{\kappa\rho}{24K}.$$

Recalling (4.5), the proof of item (2) is complete.

Item (3) is a direct consequence of the construction of U_z.

Item (1) follows from the construction and property (4.2); see [1, Lemma 3.7] for details.

Now a sketch of the proof of Theorem 4.1 is finished.

5. The existence and finiteness of physical measures

5.1. The finiteness of ergodic Gibbs cu-states.

Theorem 5.1. *Let f be a C^{1+} partially hyperbolic diffeomorphism with splitting $TM = E^u \oplus_{\succ} E^{cu} \oplus_{\succ} E^{cs}$. If (f, E^{cs}) has the $\mathcal{G}^-$ property and $\mathcal{L}_{E^{cu}}(f,\mu) > 0$ for every $\mu \in G^u(f)$, then there exist at most finitely many ergodic Gibbs cu-states for f.*

We set $F = E^u \oplus E^{cu}$. As we have shown, one can choose $\xi, \eta > 0$ so that F is (ξ,η)-Hölder continuous (i.e. $dist(F(x), F(y)) \le \eta d(x,y)^\xi$ for every x, y). To prove this theorem, we need the following observation.

Lemma 5.2. *Under the assumption of Theorem* 5.1*, $G^{cu}(f)$ is non-empty. Moreover, there exist $\sigma > 0$ and constants r, β, λ depend only on f, σ as fixed in Theorem* 4.1*. For every ergodic $\mu \in G^{cu}(f)$, there exists a Pesin unstable disk D tangent to F such that $\mu = \lim_{n\to\infty} \mu_{n,D}$ and for* Leb*-a.e. $x \in D$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \mathrm{m}(Df|F(f^i(x)) > \sigma;$$

consequently, there are $U(\mu)$, $\mathcal{L}_\infty(\mu)$ as described in Theorem 4.1.

Proof. By the compactness of $G^u(f)$ (item (1) of Proposition 2.1), there is $\sigma_{cu} > 0$ so that $\mathcal{L}_{E^{cu}}(f,\mu) > \sigma_{cu}$ for every $\mu \in G^u(f)$. This implies that there is $\sigma > 0$ such that $\mathcal{L}_F(f,\mu) > \sigma$ for every $\mu \in G^u(f)$. By Proposition 3.2, we know $\mathcal{A}_\sigma$ has full Lebesgue measure of M. Therefore the existence of Gibbs cu-states is guaranteed by [1, Theorem 6.3].

Given an ergodic Gibbs cu-state μ, one can choose a Pesin unstable disk D tangent to F such that Leb-a.e. point of D is contained in $B(\mu, f)$. As a result, for any continuous function φ, by dominated convergence theorem, we have

$$\lim_{n\to\infty} \int \varphi d\mu_{n,D} = \int_D \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) d\mathrm{Leb}_D = \int \varphi d\mu.$$

Thus, $\mu_{n,D} \xrightarrow{weak*} \mu$ as $n \to \infty$. Moreover, we obtain that for Leb-a.e. $x \in D$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \mathrm{m}(Df|F(f^i(x))) = \mathcal{L}_F(f,\mu) > \sigma.$$

Certainly, $\mathrm{Leb}_D(D \cap \mathcal{A}_\sigma) \ge 1/2$ and $H_\xi(f^n(D)) \le \eta$ for every $n \in \mathbb{N}$. We can fix constants r, β, λ by Theorem 4.1 and, moreover, every accumulation of $\mu_{n,D}$ admits a cylinder U and the family of unstable disks $\mathcal{L}_\infty$ with properties item (1)–item (3) there, so does μ. $\square$

Proposition 5.3. *There exist $C_2 > 0$ such that for any ergodic $\mu \in G^{cu}(f)$, we have*

$$\frac{1}{C_2} \le \frac{d\mu_\gamma}{d\,\mathrm{Leb}_\gamma} \le C_2, \text{ for almost every } \gamma \in \mathcal{L}_\infty(\mu).$$

Proof. Let $\{\rho_\gamma, \gamma \in \mathcal{L}_\infty(\mu)\}$ be the family of density functions of the conditional measures of μ along $\mathcal{L}_\infty(\mu)$. By [11, Proposition 2.1], for every $x, y \in \gamma$ and almost every $\gamma \in \mathcal{L}_\infty(\mu)$, we have

$$\frac{\rho_\gamma(y)}{\rho_\gamma(x)} = \prod_{i=1}^{\infty} \frac{|\mathrm{Det} Df|F(f^{-i}x)|}{|\mathrm{Det} Df|F(f^{-i}y)|}.$$

Since $f \in C^{1+}$ and F is Hölder continuous, there are $C_u > 0$ and $\eta_u > 0$ such that $|\mathrm{Det} Df|F(x)|$ is (C_u, η_u)-Hölder continuous w.r.t. x. Therefore, for every $m \geq 1$, one gets

$$\begin{aligned}
\log \prod_{i=1}^{m} \frac{|\mathrm{Det} Df|F(f^{-i}x)|}{|\mathrm{Det} Df|F(f^{-i}y)|} &\leq \sum_{i=1}^{m} \left|\log|\mathrm{Det} Df|F(f^{-i}x)| - \log|\mathrm{Det} Df|F(f^{-i}y)|\right| \\
&\leq \sum_{i=1}^{m} C_u d_{f^{-i}\gamma}(f^{-i}x, f^{-i}y)^{\eta_u} \\
&\leq \sum_{i=1}^{m} C_u \lambda^{i\eta_u} d_\gamma(x,y)^{\eta_u} \quad \text{(item (3) of Theorem 4.1)} \\
&\leq \sum_{i=1}^{m} C_u \lambda^{i\eta_u} {r_0}^{\eta_u} \leq \frac{C_u \lambda^{\eta_u} r_0^{\eta_u}}{1-\lambda^{\eta_u}}.
\end{aligned}$$

This implies that the densities are Hölder continuous functions bounded away from zero and infinity by some constant C_2 depending only on f and σ. □

The following result is a slightly generalized version of [1, Lemma 6.1].

Proposition 5.4. *Let A be a set on which there is a measurable partition $\mathcal{P}^u$ formed by disjoint local unstable manifolds. Given any invariant measure μ, if there is a measure ν which is not necessarily invariant, such that*

- *$\nu \leq \mu$[3] and $\nu(A) > 0$,*
- *the conditional measures of ν along these local unstable manifolds are absolutely continuous w.r.t. Lebesgue measures along these manifolds,*

then there exist some ergodic components of μ to be Gibbs cu-states admitting positive measure on A.

Proof of Theorem 5.1. The existence of ergodic Gibbs *cu*-states has been obtained by Lemma 5.2; it remains to prove the finiteness. By contradiction, we suppose that μ_n is a sequence of different ergodic Gibbs *cu*-states that converge to an f-invariant measure μ. We then construct open neighborhoods U_n and families $\mathcal{L}_\infty^n$ associated to μ_n, respectively, as in Theorem 4.1. Furthermore, we can assume that for every n, for any disk $\gamma \in \mathcal{L}_\infty^n$, Leb_γ-a.e. point is contained in $B(\mu_n, f)$ by the ergodicity and Proposition 5.3. Up to considering subsequences, we may prove $\overline{U}_n$ converges to a (compact) cylinder $\mathcal{C}$ with the following properties (see [11, Lemma 3.1–3.3] for details):

- let $\mathcal{L}_\infty^\infty$ be the family of disks contained in $\mathcal{C}$ which are accumulations of disks from $\mathcal{L}_\infty^n$, if we denote by L_∞^∞ the union of this family, then $\mu(L_\infty^\infty) \geq \beta$;
- the conditional measures of μ along disks of $\mathcal{L}_\infty^\infty$ are absolutely continuous w.r.t. Lebesgue measures along these disks.

[3]This means that $\nu(B) \leq \mu(B)$ for every measurable set B.

It follows from Proposition 5.4 that one can find an ergodic component μ_{ac} of μ to be a Gibbs *cu*-state with $\mu_{ac}(L_\infty^\infty) > 0$. Also, μ_{ac} has negative Lyapunov exponents along E^{cs} by assumption. Therefore, we can pick a disk $\gamma \in \mathcal{L}_\infty^\infty$ such that Leb_γ-a.e. point is in $B(\mu_{ac}, f)$ and admits a local Pesin stable manifold. Thus, one can find a subset of γ with positive Lebesgue measure such that their local Pesin stable manifolds exhibit uniform size. Since γ is an accumulation of disks from $\mathcal{L}_\infty^n$, these local Pesin stable manifolds intersect γ_n with a positive Lebesgue measure subset for sufficiently large n, using the absolute continuity of Pesin stable foliation. Observe that a stable manifold can only lie in the same basin, we have $\mu_n = \mu_{ac}$ for sufficiently large n, a contradiction. So the proof is finished. $\square$

5.2. **Basin covering property and the finiteness of physical measures.** In this subsection, we will give the proof of Theorem B, and then complete the proof of Theorem A by using Theorem B and previous results.

Under the assumption of Theorem B, as we have seen in Lemma 5.2, $\mathcal{A}_\sigma$ has full Lebesgue measure. Let $\mathcal{N}$ be a subset of $\mathcal{A}_\sigma$ with positive Lebesgue measure, such that we have following result:

Proposition 5.5. *There exists a disk D intersecting $\mathcal{N}$ with a positive Lebesgue measure, and some ergodic Gibbs cu-state μ_* such that* $\mathrm{Leb}_D(\mathcal{N} \cap B(\mu_*, f)) > 0$.

Proof. By taking an open neighborhood of a density point of $\mathcal{N}$ and foliate it by disks tangent to the F direction cone field, we may find a small C^2 disk D so that $\mathrm{Leb}_D(\mathcal{N} \cap D) \geq 1/2$. We then choose some $\xi, \eta > 0$ such that $H_\xi(f^n D) \leq \eta$ for all n large enough, guaranteed by [1, Corollary 2.4]. Replacing $\mathcal{A}_\sigma$ by $\mathcal{N}$, as constructed in Subsection 4.4, define the component ν_n of $\mu_{n,D}$ supported on the union of small disks with uniform size. In the same manner as (4.7), $\nu_n(\bigcup_{\ell=m_0}^{n-1} f^\ell(\mathcal{N} \cap D)) \geq \beta_0$. Consequently, there exist $\delta > 0$ and a subset $\mathcal{K}$ of these uniform disks in the support of ν_n such that

- the union of disks in $\mathcal{K}$ has ν_n mass larger than $\beta_0/2$;
- for any disk $B \subset f^n(D)$ coming from $\mathcal{K}$ for some n, $\mathrm{Leb}_B(B \cap f^n(\mathcal{N})) > \delta$.

Consider the accumulation K_∞ of disks in $\mathcal{K}$ formed by disjoint unstable disks (whose collection is denoted by $\mathcal{K}_\infty$) accumulated by disks from $\mathcal{K}$, it has ν measure larger than $\beta_0/2$. Here we assume μ, ν are the limit measures of $\mu_{n,D}$ and ν_n for some subsequence, respectively. Now, we take a point in the support of $\nu|K_\infty$, and then choose two cylinders U_1, U_2 around this point as in Theorem 4.1 such that

- U_1 is strictly contained in U_2 with the same height (the E^{cs}-direction, see Figure 1);
- if we let $\mathcal{J}$ be the family of disks in $\mathcal{K}_\infty$ crossing U_2, whose union is denoted by J, then $\nu(U_1 \cap J) > 0$.

Moreover, the conditional measures of $\nu|K_\infty$ restricted on U_2 along disks of $\mathcal{J}$ are absolutely continuous w.r.t. Lebesgue measures along these disks; see [1, Lemma 4.4] for details. Using Proposition 5.4, there is an ergodic Gibbs *cu*-state μ_* as a component of μ possessing positive measure on $U_1 \cap J$. Up to considering a subfamily of $\mathcal{J}$, we may assume that for every $\gamma \in \mathcal{J}$, $\mathrm{Leb}_{\gamma \cap U_2}$-a.e. point is contained in $B(\mu_*, f) \cap \mathcal{R}(f)$, where $\mathcal{R}(f)$[4] is the Lyapunov regular set of f and, moreover, $\mathrm{Leb}_{\gamma \cap U_1}$-a.e. point returns to $U_1 \cap J$ infinitely often.

[4]It is a set for which Lyapunov exponents are well defined, it has full measure for every invariant measure.

We claim that for every $\gamma \in \mathcal{J}$, Leb_γ-a.e. $x \in \gamma$ is in $B(\mu_*, f)$ and admits a Pesin stable manifold through x. This is because, if there exist $\gamma \in \mathcal{J}$ and a subset $S \subset \gamma$ with positive Leb_γ measure such that $S \subset M \setminus (B(\mu_*, f) \cap \mathcal{R}(f))$, then by the backward contraction of γ and the recurrent property, we have that $f^{-N}\gamma \subset \gamma_0 \cap U_2$ for some N sufficiently large and $\gamma_0 \in \mathcal{J}$. Thus, $f^{-N}(S) \subset \gamma_0 \cap U_2$ has positive $\mathrm{Leb}_{\gamma_0 \cap U_2}$ measure, and it is not contained in $B(\mu_*, f) \cap \mathcal{R}(f)$ by the invariance of $B(\mu_*, f)$ and $\mathcal{R}(f)$. This gives a contradiction.

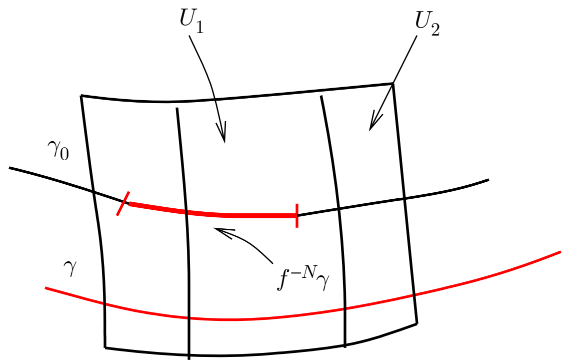

FIGURE 1. Two cylinders

Take any $\gamma \in \mathcal{J}$, there exists a sequence of disks $\gamma_{n_i} \subset f^{n_i}(D)$ of $\mathcal{K}$ converges to it by assumption. Using the absolute continuity of Pesin stable foliation and the claim above, we have $\mathrm{Leb}_{\gamma_{n_i}}(f^{n_i}(\mathcal{N} \cap D) \cap B(\mu_*, f)) > 0$, for i large enough. Hence, $\mathrm{Leb}_D(\mathcal{N} \cap B(\mu_*, f)) > 0$. □

Proof of Theorem B. By Theorem 5.1, let $\mu_1, \ldots, \mu_\ell$ be the finitely many ergodic Gibbs *cu*-states for f. Since (f, E^{cs}) has the $\mathcal{G}^-$ property, they are hyperbolic measures, by using the absolute continuity of Pesin stable foliation, we know all these measures are physical measures. Now we show that Leb-a.e. point of M is in the union of basins of these physical measures. By contradiction, the set $\mathcal{N} = \mathcal{A}_\sigma \setminus B(\mu_1, f) \cup \cdots \cup B(\mu_\ell, f)$ admits positive Lebesgue measure. By Proposition 5.5, one can take a disk D and ergodic Gibb *cu*-state μ_* such that $\mathrm{Leb}_D(\mathcal{N} \cap B(\mu_*, f)) > 0$, a contradiction. □

Now we can complete the proof of Theorem A.

Proof of Theorem A. By Proposition 3.3, there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $\mathcal{L}_{E^{cu}}(f^N, \mu) > \alpha$ for every $\mu \in G^u(f^N)$. Since (f, E^{cs}) satisfies the $\mathcal{G}^-$ property, by Lemma 2.4 we have that (f^N, E^{cs}) satisfies the $\mathcal{G}^-$ property. Thus, we can apply Theorem B by considering f^N instead. Therefore, there are finitely many ergodic physical measures $\mu_1, \ldots, \mu_\ell$ for f^N whose basins cover a full Lebesgue measure subset of M. Consider the measures $\widetilde{\mu}_i = \frac{1}{N}\sum_{j=0}^{N-1} f_*^j \mu_i$ for each $1 \le i \le \ell$, they are ergodic measures of f. In what follows, we prove that $\widetilde{\mu}_i, 1 \le i \le \ell$ are physical measures for f whose basins cover Leb-a.e. point of M. Observe that $B(\mu_i, f^N) \subset B(\widetilde{\mu}_i, f)$ for each $1 \le i \le \ell$, this together with the fact that Leb-a.e. point belongs to some basin $B(\mu_i, f^N)$ ensures that $\widetilde{\mu}_i$, $1 \le i \le \ell$ are physical measures for f whose basins cover a full Lebesgue measure subset of M. Moreover, since they are Gibbs cu-states, and have negative Lyapunov exponents along E^{cs}, they are SRB measures also. $\square$

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