PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 1, January 2018, Pages 145–152
http://dx.doi.org/10.1090/proc/13478
Article electronically published on September 27, 2017

SUPPORT PROPERTIES OF THE INTERTWINING AND THE MEAN VALUE OPERATORS IN DUNKL THEORY

LÉONARD GALLARDO AND CHAABANE REJEB

(Communicated by Mourad Ismail)

Abstract. In this paper we show that the representing measures of the Dunkl intertwining operator associated to a Coxeter-Weyl group W in $\mathbb{R}^d$ and to a multiplicity function $k \geq 0$, have W-invariant supports under the condition $k > 0$. This property enables us to determine explicitly the supports of the measures representing the volume mean operator, a fundamental tool for the study of harmonic functions relative to the Dunkl-Laplacian operator.

1. Introduction and statement of the results

Let R be a (finite) root system in $\mathbb{R}^d$ with associated Coxeter-Weyl group W (see [7] or [9] for details on root systems) and for $\xi \in \mathbb{R}^d$, let D_ξ be the Dunkl operator defined by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\, \langle \alpha, \xi \rangle \,\frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^d),$$

where R_+ is a subsystem of positive roots, σ_α is the reflection directed by the root $\alpha \in R_+$, k is a nonnegative multiplicity function (defined on R) and $\partial_\xi f$ is the usual ξ-directional derivative of f.

These operators, introduced by C. F. Dunkl (see [1]), are related to partial derivatives by means of an intertwining operator V_k (see [3] or [4]) as follows:

$$\forall\ \xi \in \mathbb{R}^d, \quad D_\xi V_k = V_k \partial_\xi. \tag{1.1}$$

We know that V_k is a topological isomorphism from the space $\mathcal{C}^\infty(\mathbb{R}^d)$ (carrying its usual Fréchet topology) onto itself satisfying (1.1) and $V_k(1) = 1$ (see [15]) and V_k commutes with the W-action (see [14]) i.e.

$$\forall\ \ f \in \mathcal{C}^\infty(\mathbb{R}^d), \ \ \forall\ \ g \in W, \quad g^{-1}.V_k(g.f) = V_k(f), \tag{1.2}$$

where $g.f(x) = f(g^{-1}x)$.

A fundamental fact due to M. Rösler (see [11] or [14]) is that for every $x \in \mathbb{R}^d$, there exists a unique compactly supported probability measure μ_x^k on $\mathbb{R}^d$ with

$$\text{supp}\ \mu_x^k \subset C(x) := co\{gx,\ g \in W\} \tag{1.3}$$

Received by the editors June 1, 2016 and, in revised form, September 13, 2016.
2010 *Mathematics Subject Classification.* Primary 31B05, 33C52, 47B39; Secondary 43A32, 51F15.
Key words and phrases. Dunkl-Laplacian operator, Dunkl's intertwining operator, generalized volume mean operator and harmonic kernel, Rösler's measure, Dunkl harmonic functions.

(the convex hull of the orbit of x under the group W) such that

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y)d\mu_x^k(y). \tag{1.4}$$

Note that the property (1.3) follows from the results in [8].

Throughout this paper, the notation $k > 0$ means that $k(\alpha) > 0$ for all $\alpha \in R$.

Concerning the measure μ_x^k (which we call Rösler's measure at point x), the first result of our paper is the following

Theorem A. *For every $x \in \mathbb{R}^d$, we have*

1) $x \in \operatorname{supp} \mu_x^k$.
2) *If $k > 0$, the support of μ_x^k is a W-invariant set.*
3) *If $k > 0$, then $W.x$ (the W-orbit of x) is contained in* $\operatorname{supp} \mu_x^k$.

A question strongly related to the support of Rösler's measures concerns the volume mean operator introduced by the authors in [6] in the study of harmonic functions for the Dunkl-Laplacian operator $\Delta_k = \sum_{i=1}^d D_i^2$ where $D_i = D_{e_i}$ with $(e_i)_{1\le i\le d}$ an orthonormal basis of $\mathbb{R}^d$. Precisely for $x \in \mathbb{R}^d$ and $r > 0$, the mean value of a continuous function f at (x, r) is defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0,r))} \int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy,$$

where $y \mapsto h_k(r, x, y)$ is the compactly supported measurable function (a generalized translate of $\mathbf{1}_{B(0,r)}$) called the harmonic kernel (see Section 2) given by

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle}\right)d\mu_y^k(z), \tag{1.5}$$

m_k is the measure $dm_k(x) := \omega_k(x)dx$ and ω_k is the weight function

$$\omega_k(x) := \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}. \tag{1.6}$$

In particular we have shown that a $\mathcal{C}^2(\mathbb{R}^d)$-function u is Δ_k-harmonic in $\mathbb{R}^d$ if and only if for all $(x, r) \in \mathbb{R}^d \times \mathbb{R}_+$, $u(x) = M_B^r(u)(x)$. For a further thorough study of Δ_k-harmonicity on a general W-invariant open set, it would be crucial to get information on the supports of the representing measures of the volume mean operators. We already know that the measures

$$d\eta_{x,r}^k = \frac{1}{m_k(B(0,r))} h_k(r,x,y)\omega_k(y)dy \quad (x \in \mathbb{R}^d, r > 0), \tag{1.7}$$

are probability measures with compact support equal to $\operatorname{supp} h_k(r, x, .)$ and satisfying the following inclusion ([6]):

$$\operatorname{supp} h_k(r,x,.\,) \subset B^W(x,r) := \bigcup_{g\in W} B(gx,r), \tag{1.8}$$

where $B(x, r)$ denotes the usual closed ball of radius r centered at x.

In fact, the second main result of this paper, intimately related to Theorem A, is a precise description of the support of $h_k(r,x,.)$. It states that

Theorem B. *Let $x \in \mathbb{R}^d$ and $r > 0$.*

1) *We have $B(x,r) \subset \operatorname{supp}\ h_k(r,x,.)$.*

2) *If $k > 0$, then we have*

$$\operatorname{supp}\ h_k(r,x,.) = B^W(x,r) := \bigcup_{g \in W} B(g.x, r).$$

We will call $B^W(x,r)$ the closed W-ball centered at x and with radius $r > 0$ associated to the Coxeter-Weyl group W.

2. The harmonic kernel and the mean value operator

In this section we recall some results of [6].

Let $(r,x,y) \mapsto h_k(r,x,y)$ be the harmonic kernel defined by (1.5). We note that in the classical case (i.e. $k = 0$), we have $\mu_y^k = \delta_y$ and $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$.

The harmonic kernel satisfies the following properties (see [6]):

(1) For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r,x,y) \leq 1$.

(2) For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ is right-continuous and nondecreasing on $]0,+\infty[$.

(3) Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon > 0$,

$$0 \leq \varphi_\varepsilon \leq 1,\ \varphi_\varepsilon = 1 \text{ on } B(0,r) \text{ and } \forall\, y \in \mathbb{R}^d, \quad \lim_{\varepsilon \to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y),$$

we have

$$\forall\, y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \widetilde{\varphi}_\varepsilon\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle}\right) d\mu_y^k(z),$$

where $\widetilde{\varphi}_\varepsilon$ is the profile function of φ_ε i.e. $\varphi_\varepsilon(x) = \widetilde{\varphi}_\varepsilon(\|x\|)$.

(4) For all $r > 0$, $x, y \in \mathbb{R}^d$ and $g \in W$, we have

$$h_k(r,x,y) = h_k(r,y,x) \quad \text{and} \quad h_k(r,gx,y) = h_k(r,x,g^{-1}y). \tag{2.1}$$

(5) For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r,x,.)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r,x,y)\omega_k(y)dy = m_k(B(0,r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \tag{2.2}$$

where d_k is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi) d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}.$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of $\mathbb{R}^d$ and c_k is the Macdonald-Mehta constant (see [10], [5]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x) dx.$$

(6) Let $r > 0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r,x,.)$ is upper semi-continuous on $\mathbb{R}^d$.

(7) The harmonic kernel satisfies the following geometric inequality: if $\|a-b\| \leq 2r$ with $r > 0$, then

$$\forall\ \xi \in \mathbb{R}^d, \quad h_k(r, a, \xi) \leq h_k(4r, b, \xi)$$

(see [6], Lemma 4.1). Note that in the classical case (i.e. $k = 0$), this inequality says that if $\|a - b\| \leq 2r$, then $B(a, r) \subset B(b, 4r)$.

(8) Let $x \in \mathbb{R}^d$. Then the family of probability measures $d\eta^k_{x,r}(y)$ defined by (1.7) is an approximation of the Dirac measure δ_x as $r \longrightarrow 0$. That is,

$$\forall\ \alpha > 0, \quad \lim_{r\to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0$$

and if f is a locally bounded measurable function on a W-invariant open neighborhood of x and if f is continuous at x, then (see [6], Proposition 3.2):

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y) d\eta^k_{x,r}(y) = \lim_{r\to 0} M^r_B(f)(x) = f(x). \tag{2.3}$$

3. Proof of the results

For convenience we group together the first items of Theorem A and Theorem B in the following proposition.

Proposition 3.1. *Let $x \in \mathbb{R}^d$. Then*

i) *for every $r > 0$, $x \in \operatorname{supp} h_k(r, x, .)$,*
ii) $x \in \operatorname{supp} \mu^k_x$,
iii) *for every $r > 0$, $B(x, r) \subset \operatorname{supp} h_k(r, x, .)$.*

Proof. **i)** Suppose that there exists $r > 0$ such that $x \notin \operatorname{supp} h_k(r, x, .)$. Then we can find $\varepsilon > 0$ such that $h_k(r, x, y) = 0$, for all $y \in B(x, \varepsilon)$. Let f be a nonnegative continuous function on $\mathbb{R}^d$ such that $\operatorname{supp} f \subset B(x, \varepsilon)$ and $f = 1$ on $B(x, \varepsilon/2)$.

Since $t \mapsto h_k(t, x, y)$ is increasing on $]0, +\infty[$, we deduce that

$$\forall\ t \in]0, r], \quad 0 \leq M^t_B(f)(x) \leq \frac{1}{m_k[B(0, t)]} \int_{\mathbb{R}^d} f(y) h_k(r, x, y) \omega_k(y) dy = 0.$$

Hence, we obtain $M^t_B(f)(x) = 0$, for all $t \in]0, r]$. Letting $t \to 0$ and using the relation (2.3), we get a contradiction.

ii) Let $x \in \mathbb{R}^d$ be fixed. At first, we claim that

$$\forall\ r > 0, \quad \forall\ y \in \mathbb{R}^d, \quad h_k(r, x, y) \leq \mu^k_x[B(y, r)]. \tag{3.1}$$

Indeed, from the inclusion $\operatorname{supp} \mu^k_x \subset B(0, \|x\|)$, we see that

$$\forall\ y \in \mathbb{R}^d, \quad \forall\ z \in \operatorname{supp} \mu^k_x, \quad \|y - z\|^2 \leq \|y\|^2 + \|x\|^2 - 2\langle y, z\rangle.$$

This implies for any $y \in \mathbb{R}^d$ and $r > 0$ that

$$\forall\ z \in \operatorname{supp} \mu^k_x, \quad \mathbf{1}_{[0,r]}\big(\sqrt{\|y\|^2 + \|x\|^2 - 2\langle y, z\rangle}\big) \leq \mathbf{1}_{[0,r]}(\|y - z\|) = \mathbf{1}_{B(y,r)}(z).$$

If we integrate the two terms of the previous inequality with respect to the measure μ^k_x, we obtain $h_k(r, y, x) \leq \mu^k_x\big(B(y, r)\big)$ and then (3.1) follows from (2.1).

Now, if $x \notin \operatorname{supp} \mu^k_x$, there exists $\epsilon > 0$ such that $\mu^k_x\big(B(x, \epsilon)\big) = 0$. Thus, we have $\mu^k_x\big(B(y, \epsilon/2)\big) = 0$ whenever $y \in B(x, \epsilon/2)$. Using (3.1), we deduce that $h_k(\epsilon/2, x, .) = 0$ on $B(x, \epsilon/2)$, a contradiction to the result of i).

iii) Let $y \in \mathbb{R}^d$ such that $\|x-y\| < r$. As $\lim_{z\to y}(\|x\|^2+\|y\|^2-2\langle x,z\rangle) = \|x-y\|^2$, there exists $\eta > 0$ such that

$$\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle} \le r \quad \text{for every } z \in B(y,\eta).$$

Therefore, by using the fact that $y \in \operatorname{supp} \mu_y^k$ we obtain

$$h_k(r,x,y) \ge \mu_y^k[B(y,\eta)] > 0.$$

□

Remark 3.1. For $\alpha \in R$, let

$$H_\alpha := \{x \in \mathbb{R}^d,\ \langle x, \alpha\rangle = 0\}$$

be the hyperplane directed by α. We note that in [12, Corollary 3.6], and under the condition $x \notin \bigcup_{\alpha\in R} H_\alpha$, Rösler has proved that $x \in \operatorname{supp} \mu_x^k$ by using the asymptotic behavior of the Dunkl kernel E_k which is defined by

$$E_k(x,y) := V_k\big(e^{\langle \cdot, y\rangle}\big)(x) = \int_{\mathbb{R}^d} e^{\langle z,y\rangle} d\mu_x^k(z).$$

We turn now to the second statement of Theorem A that we recall below:

Theorem 3.1. *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then the set* $\operatorname{supp} \mu_x^k$ *is W-invariant.*

Proof. In order to simplify the formulas, we will assume here that the root system R is normalized i.e. $\|\alpha\|^2 = 2$ for all $\alpha \in R$. In particular, for reflections we have $\sigma_\alpha x = x - \langle \alpha, x\rangle \alpha$.

We will prove that if $y \in \operatorname{supp} \mu_x^k$, then $\sigma_\alpha y \in \operatorname{supp} \mu_x^k$ for every $\alpha \in R$. Let then $y \in \operatorname{supp} \mu_x^k$ and suppose that there is a root $\alpha \in R$ such that $\sigma_\alpha y \notin \operatorname{supp} \mu_x^k$. Write $y' := \sigma_\alpha y$ to simplify notation. There is a ball $B(y',\epsilon)$ ($\epsilon > 0$) such that for all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ with compact support included in $B(y',\epsilon)$, we have

$$\textstyle\int_{\mathbb{R}^d} f(z)\mu_x(dz) = V_k f(x) = 0.$$

Let us denote by $C^\infty_{y',\epsilon}$ (resp. $C_{y',\epsilon}$) the set of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ (resp. $f \in \mathcal{C}(\mathbb{R}^d)$) with compact support in $B(y',\epsilon)$. For all $\xi \in \mathbb{R}^d$ and all $f \in C^\infty_{y',\epsilon}$, we also have $\partial_\xi f \in C^\infty_{y',\epsilon}$. By the intertwining relation (1.1) we obtain

$$\forall\ \xi \in \mathbb{R}^d, \quad \forall\ f \in C^\infty_{y',\epsilon}, \quad D_\xi V_k f(x) = 0.$$

Suppose $f \in C^\infty_{y',\epsilon}$ and $f \ge 0$ and let $g := V_k f$. We have $g \ge 0$ on $\mathbb{R}^d$ (because V_k preserves positivity) and

$$\forall\ \xi \in \mathbb{R}^d, \quad D_\xi g(x) = \partial_\xi g(x) + \sum_{\alpha\in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha\rangle} = 0. \tag{3.2}$$

But as $g(x) = 0$, x is a minimum of g so $\partial_\xi g(x) = 0$ and relation (3.2) implies

$$\forall\ \xi \in \mathbb{R}^d, \quad \sum_{\alpha\in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{g(x) - g(\sigma_\alpha x)}{\langle x, \alpha\rangle} = 0. \tag{3.3}$$

Now, consider the set

$$R_x := \{\alpha \in R_+;\ x \in H_\alpha\}.$$

There are two possible locations for x:

• **First case**: Suppose that $R_x = \emptyset$ i.e $x \notin \bigcup_{\alpha\in R} H_\alpha$ (i.e. for all roots $\alpha \in R$, $\langle x, \alpha\rangle \neq 0$). Applying (3.3) with $\xi = x$ and using the fact that $g(x) = 0$, we get

$$\textstyle\sum_{\alpha\in R_+} k(\alpha)g(\sigma_\alpha x) = 0.$$

As $g \geq 0$ and by the assumption $k > 0$, we obtain that $g(\sigma_\alpha x) = V_k f(\sigma_\alpha x) = 0$ for all $\alpha \in R_+$ and all $f \in C^\infty_{y',\epsilon}$ and $f \geq 0$. By uniform approximation, we deduce that for all $f \in C_{y',\epsilon}$ and $f \geq 0$, we also have $V_k f(\sigma_\alpha x) = 0$. Finally for every $f \in C_{y',\epsilon}$, by decomposing $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ and using the linearity and W-equivariance of V_k (relation (1.2)), we obtain that

$$\forall\ f \in C_{y',\epsilon}, \quad \forall\ \alpha \in R_+, \quad V_k f(\sigma_\alpha x) = V_k(\sigma_\alpha.f)(x) = 0,$$

where $\sigma_\alpha.f$ is the function $z \mapsto f(\sigma_\alpha z)$. As it is easy to see that $\sigma_\alpha.C_{y',\epsilon} = C_{\sigma_\alpha y',\epsilon}$, we deduce that

$$\forall\ \alpha \in R_+, \quad \forall f \in C_{\sigma_\alpha y',\epsilon}, \quad V_k f(x) = 0.$$

But this implies in particular that $V_k f(x) = 0$ for all $f \in C_{y,\epsilon}$ in contradiction to the hypothesis $y \in \mathrm{supp}\ \mu^k_x$. The result of the theorem follows in the first case.

• **Second case**: Suppose that $R_x \neq \emptyset$. For every $\beta \in R_x$, clearly we have $x = \sigma_\beta x$. Therefore, since $g(x) = 0$, we get $g(\sigma_\beta x) = 0$, for all $\beta \in R_x$. But, as x is a minimum of g, we have

$$\forall\ \beta \in R_x, \quad \frac{g(x) - g(\sigma_\beta x)}{\langle x, \beta\rangle} = \int_0^1 \partial_\beta g(x - t\,\langle x, \beta\rangle\,\beta)dt = \partial_\beta g(x) = 0.$$

Hence, the relation (3.3) with $\xi = x$ implies

$$\textstyle\sum_{\alpha\in R_+\setminus R_x} k(\alpha)g(\sigma_\alpha x) = 0.$$

Consequently, we obtain $g(\sigma_\alpha x) = 0$ for all $\alpha \in R$. The end of the proof of the first case applies and gives also the result in this case. This completes the proof of the theorem. □

From the W-invariance property of the support of μ^k_x and the fact that $x \in \mathrm{supp}\ \mu^k_x$, we obtain immediately the last assertion of Theorem A:

Corollary 3.1. *Let $x \in \mathbb{R}^d$ and assume that $k > 0$. Then, for all $g \in W$, $gx \in \mathrm{supp}\ \mu^k_x$.*

Now, we can turn to the proof of the second statement of Theorem B.

Corollary 3.2. *Let $x \in \mathbb{R}^d$ and $r > 0$. If $k > 0$, then*

$$\mathrm{supp}\ h_k(r, x, .) = B^W(x, r) := \bigcup_{g\in W} B(gx, r). \tag{3.4}$$

Proof. Let $g \in W$ and $y \in \mathbb{R}^d$ such that $\|gx - y\| < r$. We will proceed as in the proof of Proposition 3.1, iii). We have

$$\lim_{z\to g^{-1}y} \sqrt{\|x\|^2 + \|y\|^2 - 2\,\langle x, z\rangle} = \|x - g^{-1}y\|.$$

Hence, there exists $\eta > 0$ such that for all $z \in B(g^{-1}y, \eta)$, $\sqrt{\|x\|^2+\|y\|^2-2\,\langle x, z\rangle} \leq r$ and thus $h_k(r, x, y) \geq \mu^k_y[B(g^{-1}y, \eta)]$.

But, from the fact that $g^{-1}y \in \mathrm{supp}\ \mu^k_y$ we deduce that $y \in \mathrm{supp}\ h_k(r, x, .)$.

This completes the proof. □

Remark 3.2. When $k \geq 0$, we will say that a root $\alpha \in R$ is active if $k(\alpha) > 0$. Let us denote by $R_A = \{\alpha \in R;\ k(\alpha) > 0\}$ the set of active roots and F the vector subspace of $\mathbb{R}^d$ generated by $\{\alpha,\ \alpha \in R_A\}$. Then we can generalize the results of Theorems A and B in the following form:
a) The set R_A is a root system. Indeed, using the fact that k is W-invariant, we can see that for every $\alpha, \beta \in R_A$, $k\big(\sigma_\alpha \beta\big) = k(\beta) > 0$. Thus

$$\forall\ \alpha \in R_A, \quad R_A \cap \mathbb{R}\alpha = \{\pm\alpha\} \quad \text{and} \quad \sigma_\alpha(R_A) = R_A.$$

b) Let W_A be the Coxeter-Weyl group associated to the root system R_A. Then the restriction k_A of k to R_A is clearly invariant under the W_A-action. In other words, it is a multiplicity function.
c) For any $\xi \in \mathbb{R}^d$, we will use the notation $\xi = \xi' + \xi'' \in F + F^\perp = \mathbb{R}^d$ (where $F^\perp$ is the orthogonal complement of F in $\mathbb{R}^d$).

• Let $x \in \mathbb{R}^d$. Rösler's measure μ_x^k is of the form (see [13])

$$\mu_x^k = \mu_{x'}^{k_A} \otimes \delta_{x''}, \tag{3.5}$$

where $\mu_{x'}^{k_A}$ is Rösler's measure associated to (R_A, k_A) and $\delta_{x''}$ is the Dirac measure at x''. We have

$$\text{supp } \mu_x^k = x'' + \text{supp } \mu_{x'}^{k_A}.$$

From (1.3), the support of $\mu_{x'}^{k_A}$ is contained in the convex hull of $W_A.x'$ (the W_A-orbit of x'). Furthermore, by Theorem A, it is invariant under the action of the group W_A and contains the whole orbit $W_A.x'$.
• Let $x \in \mathbb{R}^d$ and $r > 0$. According to (1.5) and (3.5) the harmonic kernel is given by

$$h_k(r,x,y) = \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\big(\sqrt{\|x''-y''\|^2 + \|x'\|^2 + \|y'\|^2 - 2\,\langle x', z'\rangle}\big)\, d\mu_{y'}^{k_A}(z'), \quad y \in \mathbb{R}^d.$$

The support of $h_k(r,x,.)$ takes the following form:

$$\text{supp } h_k(r,x,.) = x'' + B^{W_A}(x',r) = x'' + \bigcup_{g \in W_A} B(gx',r) = \bigcup_{g \in W_A} B(gx,r).$$

Example 3.1. Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$. Then, the set $R := \{\pm e_1, \pm e_2\}$ is a root system in $\mathbb{R}^2$, its Coxeter-Weyl group is $\mathbb{Z}_2^2$ and the multiplicity function can be identified to a pair $k = (k_1, k_2)$, with $k_i = k(e_i) \geq 0$, $i = 1, 2$. Take $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1, x_2 > 0$. In this case, according to [16], Rösler's measure is given by $\mu_x^k = \mu_{x_1}^{k_1} \otimes \mu_{x_2}^{k_2}$, where $\mu_{x_i}^{k_i} = \delta_{x_i}$ if $k_i = 0$ and

$$\langle \mu_{x_i}^{k_i}, f\rangle = \frac{\Gamma(k_i + 1/2)}{\sqrt{\pi}\Gamma(k_i)} \int_{-1}^{1} f(tx_i)(1-t)^{k_i - 1}(1+t)^{k_i}\, dt$$

if $k_i > 0$ (see [2]).

• If $k = (0,0)$, $\mu_x^k = \delta_x$ and $h_k(r,x,y) = \mathbf{1}_{B(x,r)}(y)$.

• If $k = (k_1, 0)$ with $k_1 > 0$, then supp μ_x^k is the line segment between x and $\sigma_{e_1} x = (-x_1, x_2)$ and

$$\text{supp } h_k(r,x,.) = B(x,r) \cup B(\sigma_{e_1} x, r).$$

• If $k_1, k_2 > 0$, the support of μ_x^k is the convex hull of $\mathbb{Z}_2^2.x$ and the closed W-ball is given by

$$\begin{aligned} B^{\mathbb{Z}_2^2}(x,r) &= \text{supp } h_k(r,x,.) \\ &= B\big((x_1,x_2),r\big) \cup B\big((-x_1,x_2),r\big) \cup B\big((x_1,-x_2),r\big) \cup B\big((-x_1,-x_2),r\big). \end{aligned}$$

References

[1] Charles F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 167–183, DOI 10.2307/2001022. MR951883

[2] Charles F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math. **43** (1991), no. 6, 1213–1227, DOI 10.4153/CJM-1991-069-8. MR1145585

[3] Charles F. Dunkl, *Hankel transforms associated to finite reflection groups*, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991), Contemp. Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 123–138, DOI 10.1090/conm/138/1199124. MR1199124

[4] Charles F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001. MR1827871

[5] Pavel Etingof, *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups*, Math. Res. Lett. **17** (2010), no. 2, 275–282, DOI 10.4310/MRL.2010.v17.n2.a7. MR2644375

[6] Léonard Gallardo and Chaabane Rejeb, *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications*, Trans. Amer. Math. Soc. **368** (2016), no. 5, 3727–3753, DOI 10.1090/tran/6671. MR3451892

[7] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[8] M. F. E. de Jeu, *The Dunkl transform*, Invent. Math. **113** (1993), no. 1, 147–162, DOI 10.1007/BF01244305. MR1223227

[9] Richard Kane, *Reflection groups and invariant theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 5, Springer-Verlag, New York, 2001. MR1838580

[10] E. M. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Compositio Math. **85** (1993), no. 3, 333–373. MR1214452

[11] Margit Rösler, *Positivity of Dunkl's intertwining operator*, Duke Math. J. **98** (1999), no. 3, 445–463, DOI 10.1215/S0012-7094-99-09813-7. MR1695797

[12] Margit Rösler, *Short-time estimates for heat kernels associated with root systems*, Special functions (Hong Kong, 1999), World Sci. Publ., River Edge, NJ, 2000, pp. 309–323. MR1805992

[13] Margit Rösler and Marcel de Jeu, *Asymptotic analysis for the Dunkl kernel*, J. Approx. Theory **119** (2002), no. 1, 110–126, DOI 10.1006/jath.2002.3722. MR1934628

[14] Margit Rösler, *Dunkl operators: theory and applications*, Orthogonal polynomials and special functions (Leuven, 2002), Lecture Notes in Math., vol. 1817, Springer, Berlin, 2003, pp. 93–135, DOI 10.1007/3-540-44945-0_3. MR2022853

[15] Khalifa Trimèche, *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual*, Integral Transform. Spec. Funct. **12** (2001), no. 4, 349–374, DOI 10.1080/10652460108819358. MR1872375

[16] Yuan Xu, *Orthogonal polynomials for a family of product weight functions on the spheres*, Canad. J. Math. **49** (1997), no. 1, 175–192, DOI 10.4153/CJM-1997-009-4. MR1437206

Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Université de Tours, Campus de Grandmont, 37200 Tours, France

E-mail address: `Leonard.Gallardo@lmpt.univ-tours.fr`

Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Université de Tours, Campus de Grandmont, 37200 Tours, France – and – Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématiques et Applications LR11ES11, 2092 El Manar I, Tunis, Tunisia

E-mail address: `chaabane.rejeb@gmail.com`