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TRUNCATED TOEPLITZ OPERATORS AND COMPLEX SYMMETRIES

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ABSTRACT. We show that truncated Toeplitz operators are characterized by a collection of complex symmetries. This was conjectured by Kliś-Garlicka, Lanucha, and Ptak, and proved by them in some special cases.

1. Introduction

The systematic study of truncated Toeplitz operators was initiated by Sarason [7]. Given an inner function u on the unit disc, this class, denoted by $\mathcal{T}_u$ consists of those bounded operators on $K_u = H^2 \ominus uH^2$ that are compressions of multiplication operators. A recent survey of results in this area is contained in [5].

Sarason observed that, while every operator in $\mathcal{T}_u$ is complex symmetric (relative to the natural conjugation on K_u; see [4]), not every complex symmetric operator on K_u belongs to $\mathcal{T}_u$. Operators in $\mathcal{T}_u$ satisfy additional complex symmetry conditions and the authors of [6] conjectured that every operator on K_u that satisfies these additional symmetries necessarily belongs to $\mathcal{T}_u$. This conjecture is proved in [6] in many cases in which u is a Blaschke product. The purpose of this note is to provide a proof of this conjecture for arbitrary inner functions u. In the case in which u has at least one zero, it turns out that the operators in $\mathcal{T}_u$ are characterized by the fact that they satisfy just two complex symmetries. In case u is singular, one needs to require a countable collection of complex symmetries.

2. Notation and preliminaries

We denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc, and by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. As usual, we view the Hardy space H^2 on $\mathbb{D}$ as a subspace of $L^2 = L^2(\mathbb{T})$ (relative to the normalized arclength measure on $\mathbb{T}$) by identifying functions analytic in $\mathbb{D}$ with their radial limits (which exist almost everywhere). Similarly, the algebra H^∞ of bounded analytic functions in $\mathbb{D}$ can be viewed as a closed subalgebra of $L^\infty = L^\infty(\mathbb{T})$. We denote by S the shift operator in H^2, defined by $(Sf)(z) = zf(z)$, $f \in H^2$, $z \in \mathbb{D}$.

A function $u \in H^\infty$ is said to be *inner* if $|u| = 1$ almost everywhere on $\mathbb{T}$. For instance, the function $\chi \in H^\infty$ defined by $\chi(z) = z$, $z \in \mathbb{D}$, is inner. If u is an inner function, the model space K_u (often denoted $\mathcal{H}(u)$ in the literature) is defined by $K_u = H^2 \ominus uH^2$ and $P_{K_u} : L^2 \to K_u$ denotes the orthogonal projection onto K_u.

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Given an arbitrary bounded operator A on a Hilbert space $\mathcal{H}$, we denote by Q_A the quadratic form on $\mathcal{H}$ defined by $Q_A(f) = \langle Af, f\rangle$, $f \in \mathcal{H}$. A *conjugation* on a Hilbert space $\mathcal{H}$ is an isometric, conjugate linear involution, that is, $C \circ C = I_{\mathcal{H}}$ and $\langle Ch, Ck\rangle = \langle k, h\rangle$ for $h, k \in \mathcal{H}$. An operator A is then said to be C-*symmetric* [4] or simply *complex symmetric* when C is understood, if $A^* = CAC$. This condition is easily seen to be equivalent to $Q_A(f) = Q_A(Cf)$, $f \in \mathcal{H}$.

Given an arbitrary inner function u, there is a conjugation C_u on L^2 defined by $C_u f = u\overline{\chi f}$. This conjugation maps K_u bijectively onto itself and therefore it also defines a conjugation on this space. We record for further use the following result whose proof is a simple calculation.

Lemma 2.1. *Suppose that u and v are inner functions in H^∞ and v divides u. Then for every $f \in L^2$ we have*

$$C_u(C_{u/v}(f)) = vf.$$

The space K_u is a reproducing kernel space of analytic functions on $\mathbb{D}$. The following well-known lemma is the H^2 version of a result that holds in arbitrary reproducing kernel Hilbert spaces. We sketch the proof for the reader's convenience.

Lemma 2.2. *Suppose that $\{f_n\}_{n\in\mathbb{N}} \subset H^2$. Then:*

(i) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to a function $f \in H^2$ if and only if $\sup_{n\in\mathbb{N}} \|f_n\| < +\infty$ and $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in \mathbb{D}$.*

(ii) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in norm to a function $f \in H^2$ if and only if $\lim_{n\to\infty} \|f_n\| = \|f\|$ and $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in \mathbb{D}$.*

Proof. To prove (i), suppose first that $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to f. Then the sequence must be bounded by the uniform boundedness principle and $\lim_{n\to\infty} f_n(z) = f(z)$ follows because $f(z) = \langle f, k_z\rangle$, $z \in \mathbb{D}$, where k_z denotes the Szegö kernel. Conversely, if the sequence is bounded, then it has weak limit points, and the relation $\lim_{n\to\infty} f_n(z) = f(z)$ shows that f is the unique limit point. Part (ii) follows from (i) and standard Hilbert space arguments. ☐

We recall [7] that a bounded linear operator A on K_u is called a *truncated Toeplitz operator* if there exists a function $\varphi \in L^2$ (called a *symbol* of A) such that

$$Af = P_{K_u}(\varphi f)$$

for every bounded function $f \in K_u$. The truncated Toeplitz operators on K_u form a weakly closed subspace $\mathcal{T}_u$ of $\mathcal{L}(K_u)$. There is a simple characterization of the operators in $\mathcal{T}_u$ that does not require a symbol. The space

$$K_u^0 = \{g \in K_u : Sg \in K_u\}.$$

is closed in K_u (since $K_u^0 = K_u \cap S^{-1}(K_u)$) and $K_u \ominus K_u^0$ is generated by the vector $S^*u = \overline{\chi}(u - u(0))$ (see, for instance, [7]). The following result is [7, Theorem 8.1].

Lemma 2.3. *A bounded linear operator A on K_u belongs to $\mathcal{T}_u$ if and only if*

$$Q_A(f) = Q_A(Sf) \tag{2.1}$$

for every $f \in K_u^0$.

Fix $a \in \mathbb{D}$ and denote by $b_a(z) = (z - a)/(1 - \bar{a}z)$, $z \in \mathbb{D}$, the corresponding Blaschke factor. The following result is used in [2, Section 4] as well as [6].

Lemma 2.4. *There is a unitary operator* $\omega_a : K_u \to K_{u\circ b_a}$ *defined by*

$$\omega_a(f) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}\chi} f \circ b_a, \quad f \in K_u. \tag{2.2}$$

We have $\omega_a C_u = C_{u\circ b_a}\omega_a$, *and* $\omega_a \mathcal{T}_u \omega_a^* = \mathcal{T}_{u\circ b_a}$.

3. Truncated Toeplitz operators and conjugations

Suppose that u is an inner function and $A \in \mathcal{T}_u$. Then A is C_u symmetric, that is, $Q_A(f) = Q_A(C_u f)$ for every $f \in K_u$. Let v be an inner divisor of the function u. Then $K_v \subset K_u$ and it was observed in [6] that $P_v A|K_v$ is also C_v-symmetric. The authors of [6] formulated the following conjecture.

Conjecture 3.1. *A bounded linear operator* A *on* K_u *belongs to* $\mathcal{T}_u$ *if and only if, for every inner divisor* v *of* u, *the compression* $P_v A|K_v$ *is* C_v*-symmetric.*

This conjecture is proved in [6] for certain Blaschke products u, namely, Blaschke products with a single zero, finite Blaschke products with simple zeros, and interpolating Blaschke products. The arguments rely on a characterization [3] of the class $\mathcal{T}_u$ in terms of its matrix entries in a particular orthonormal basis for K_u. In this section, we prove the conjecture for those inner functions u that have at least one zero. The case of singular inner functions is treated in the following section.

Theorem 3.2. *Suppose that* $u \in H^\infty$ *is an inner function and* $u(a) = 0$ *for some* $a \in \mathbb{D}$. *Then an operator* $A \in \mathcal{L}(K_u)$ *belongs to* $\mathcal{T}_u$ *if and only if it is* C_u*-symmetric and* $Q_A(C_{u/b_a} f) = Q_A(f)$ *for every* $f \in K_{u/b_a}$.

Proof. Suppose first that $a = 0$ and thus $b_a = \chi$. If $f \in K_u$, then $Sf \in K_u$ if and only if $f \in K_{u/\chi}$. For such a function f we have $C_u C_{u/\chi} f = \chi f = Sf$ by Lemma 2.1. The two symmetry hypotheses in the statement imply that

$$Q_A(Sf) = Q_A(C_u C_{u/z} f) = Q_A(C_{u/\chi} f)) = Q_A(f).$$

It follows then from Lemma 2.3 that $A \in \mathcal{T}_u$.

For the general case $a \neq 0$ we use Lemma 2.4. The inner function $v = u \circ b_{-a}$ satisfies $v(0) = 0$, and the unitary map ω_a defined in (2.2) yields by restriction unitary maps from K_v onto K_u and from $K_{v/\chi}$ to K_{u/b_a} that intertwine the standard conjugations on these spaces. Therefore $\omega_a A \omega_a^*$ is C_v-symmetric, and its compression to $K_{v/\chi}$ is $C_{v/\chi}$-symmetric. By the first part of the proof, $\omega_a A \omega_a^* \in \mathcal{T}_v$. It follows from Lemma 2.4 that $A \in \mathcal{T}_u$. □

We have thus proved a stronger version of the conjecture in case u has a zero in $\mathbb{D}$: an operator A on K_u is a truncated Toeplitz operator if and only if the complex symmetry condition is satisfied by A as well as by a single one of its compressions to model spaces.

4. Singular inner functions

Given a positive, singular Borel measure ν on $\mathbb{T}$, we denote by e_ν the corresponding singular inner function, that is,

$$e_\nu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\nu(\zeta)\right), \quad z \in \mathbb{D}. \tag{4.1}$$

Lemma 4.1. *Let ν be a nonzero, positive, singular Borel measure on $\mathbb{T}$. Then there exist $\eta \in \mathbb{T}$ and a sequence of nonzero, positive Borel measures $\mu_n \le \nu$, $n \in \mathbb{N}$, such that:*

(i) $\lim_{n\to\infty} e_{\mu_n}(z) = 1$ *for every* $z \in \mathbb{D}$, *and*
(ii) *for every* $g \in H^2$, *the functions*

$$\frac{e_{\mu_n} - 1}{\mu_n(\mathbb{T})}(\chi - \eta)g, \quad n \in \mathbb{N},$$

converge weakly in H^2 *to* $(\chi + \eta)g$ *as* $n \to \infty$.

Proof. Choose $\eta \in \mathbb{T}$ and a sequence $\{I_n\}_{n\in\mathbb{N}}$ of arcs in $\mathbb{T}$, symmetric about η, with length $|I_n| = 1/n$, such that $\lim_{n\to\infty}(\nu(I_n)/|I_n|) = +\infty$. This is possible since ν is singular. Define the measures μ_n by

$$d\mu_n = \sqrt{\frac{|I_n|}{\nu(I_n)}}\,\chi_{I_n}\,d\nu, \quad n \in \mathbb{N}.$$

By the maximum modulus principle, condition (i) only needs to be verified at $z = 0$, and this is immediate because $e_{\mu_n}(0) = e^{-\mu_n(\mathbb{T})}$. In fact, we have

$$\lim_{n\to\infty} \frac{e_{\mu_n}(z) - 1}{\mu_n(\mathbb{T})} = \frac{z + \eta}{z - \eta}, \quad z \in \mathbb{D}.$$

Lemma 2.2 shows that (ii) is true as well once we verify that

$$\sup_{z\in\mathbb{D}, n\in\mathbb{N}} |z - \eta| \frac{|e_{\mu_n}(z) - 1|}{\mu_n(\mathbb{T})} < \infty. \tag{4.2}$$

Observe first that, if $z \in \mathbb{D}$ and $|z - \eta| < 10/n = 10|I_n|$,

$$|z - \eta| \frac{|e_{\mu_n}(z) - 1|}{\mu_n(\mathbb{T})} \le \frac{20|I_n|}{\sqrt{|I_n|\nu(I_n)}} = 20\sqrt{\frac{|I_n|}{\nu(I_n)}},$$

and the last quantity tends to 0 by the choice of I_n. If $|z - \eta| \ge 10/n$, we use the inequalities

$$|e^\lambda - 1| \le |\lambda| e^{|\lambda|}, \quad \lambda \in \mathbb{C},$$

and

$$\left|\frac{\zeta - z}{\zeta + z}\right| < 3, \quad \zeta \in I_n, z \in \mathbb{D}, |z - \zeta| > \frac{10}{n},$$

to deduce that

$$|e_{\mu_n}(z) - 1| \le 3\mu_n(\mathbb{T}) e^{3\mu_n(\mathbb{T})}.$$

For such values of z we see that

$$|z - \eta| \frac{|e_{\mu_n}(z) - 1|}{\mu_n(\mathbb{T})} \le 6e^{3\mu_n(\mathbb{T})} < 6e^{3\nu(\mathbb{T})}.$$

This concludes the proof of the lemma. □

We need one more technical result before establishing Conjecture 3.1 for $u = e_\nu$. Recall that K_u^0 consists of those vectors $f \in K_u$ with the property that Sf also belongs to K_u. Clearly, $K_v^0 \subset K_u^0$ if v is an inner divisor of u.

Lemma 4.2. *Suppose that $u \in H^\infty$ is an inner function and $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of inner divisors of u such that u_{n+1} divides u_n, $n \in \mathbb{N}$, and $\lim_{n\to\infty} |u_n(0)| = 1$. Then $\bigcup_{n\in\mathbb{N}} K_{u/u_n}^0$ is dense in K_u^0.*

Proof. We can, and do, assume without loss of generality that $u_n(0) \geq 0$, $n \in \mathbb{N}$. The least common inner multiple of the functions $\{u/u_n\}_{n\in\mathbb{N}}$ is equal to u, and thus $\bigcap_{n\in\mathbb{N}}(u/u_n)H^2 = uH^2$ (see, for instance [1, Section 2.2]). It follows that $\bigcup_{n\in\mathbb{N}} K_{u/u_n}$ is dense in K_u and therefore the sequence $\{P_{u/u_n}\}_{n\in\mathbb{N}}$ converges to P_u in the strong operator topology.

It was noted earlier that the space $K_{u/u_n} \ominus K^0_{u/u_n}$ is generated by the vector $S^*(u/u_n)$. It follows from Lemma 2.2 that the sequence $\{u/u_n\}_{n\in\mathbb{N}}$ converges in the H^2 norm to u, and thus $\lim_{n\to\infty} S^*(u/u_n) = S^*u$. If we denote by P_n and P the orthogonal projections onto the spaces $K_{u/u_n} \ominus K^0_{u/u_n}$ and $K_u \ominus K^0_u$, respectively, it follows that the sequence $\{P_n\}_{n\in\mathbb{N}}$ converges to P in the strong operator topology. Given an arbitrary vector $f \in K^0_u$, we have $f_n = P_{u/u_n}f - P_nP_{u/u_n}f \in K^0_{u/u_n}$ and $\lim_{n\to\infty} f_n = f - Pf = f$. The lemma follows. □

We may now give the solution of the conjecture for singular inner functions.

Theorem 4.3. *Suppose that ν is a positive, singular Borel measure on $\mathbb{T}$. Let A be an operator on K_{e_ν} that is, C_{e_ν}-symmetric, and such that for every positive Borel measure $\mu \leq \nu$, the compression of A to K_{e_μ}, is C_{e_μ}-symmetric. Then $A \in \mathcal{T}_{e_\nu}$.*

Proof. Let $\eta \in \mathbb{T}$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be as in Lemma 4.1. By Lemmas 2.1 and 4.2, it suffices to show that $Q_A(Sf) = Q_A(f)$ for every $f \in \bigcup_{n\in\mathbb{N}} K^0_{e_\mu/e_{\mu_n}}$. Fix $n \in \mathbb{N}$, $f \in K^0_{e_\nu/e_{\mu_n}}$, and observe that then $(\chi - \eta)f \in K_{e_\nu/e_{\mu_n}}$. If $m \geq n$, we also have $e_{\mu_m}|e_{\mu_n}$ and $(\chi - \eta)f \in K_{e_\nu/e_{\mu_m}}$. Lemma 2.1 yields

$$C_{e_\nu}(C_{e_\nu/e_{\mu_m}}((\chi - \eta)f)) = e_{\mu_m}(\chi - \eta)f.$$

The complex symmetry of A and of its compression to $K_{e_\nu/e_{\mu_m}}$ shows that

$$Q_A(e_{\mu_m}(\chi - \eta)f) = Q_A((\chi - \eta)f), \quad m \geq n,$$

and therefore

$$\begin{aligned} 0 &= \frac{1}{\mu_m(\mathbb{T})}\big(\langle A(e_{\mu_m}(\chi - \eta)f), e_{\mu_m}(\chi - \eta)f\rangle - \langle A((\chi - \eta)f), (\chi - \eta)f\rangle\big) \\ &= \left\langle \frac{e_{\mu_m} - 1}{\mu_m(\mathbb{T})}(\chi - \eta)f, A^*(e_{\mu_m}(\chi - \eta)f) \right\rangle. \\ &\qquad + \left\langle A((\chi - \eta)f), \frac{e_{\mu_m} - 1}{\mu_m(\mathbb{T})}(\chi - \eta)f \right\rangle. \end{aligned}$$

By Lemma 4.1(ii) $(1/\mu_m(\mathbb{T}))(e_{\mu_m} - 1)(\chi - \eta)f$ tends weakly in H^2 to $(\chi + \eta)f$. On the other hand, $e_{\mu_m}(\chi - \eta)f$ tends pointwise on $\mathbb{D}$ to $(\chi - \eta)f$ and $\|e_{\mu_m}(\chi - \eta)f\| = \|(\chi - \eta)f\|$. By Lemma 2.2 $e_{\mu_m}(\chi - \eta)f$ tends to $(\chi - \eta)f$ in norm. We obtain then, by letting $m \to \infty$ in the last equality,

$$\langle A((\chi + \eta)f), (\chi - \eta)f\rangle + \langle A((\chi - \eta)f), (\chi + \eta)f\rangle = 0.$$

A simple calculation yields then $Q_A(Sf) = Q_A(f)$, thereby concluding the proof. □

We observe that the argument above only requires that A and its compressions to $K_{e_\nu/e_{\mu_n}}$, $n \in \mathbb{N}$, be complex symmetric.

References

[1] Hari Bercovici, *Operator theory and arithmetic in H^∞*, Mathematical Surveys and Monographs, vol. 26, American Mathematical Society, Providence, RI, 1988. MR954383

[2] Joseph A. Cima, Stephan Ramon Garcia, William T. Ross, and Warren R. Wogen, *Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity*, Indiana Univ. Math. J. **59** (2010), no. 2, 595–620, DOI 10.1512/iumj.2010.59.4097. MR2648079

[3] Joseph A. Cima, William T. Ross, and Warren R. Wogen, *Truncated Toeplitz operators on finite dimensional spaces*, Oper. Matrices **2** (2008), no. 3, 357–369, DOI 10.7153/oam-02-21. MR2440673

[4] Stephan Ramon Garcia and Mihai Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315, DOI 10.1090/S0002-9947-05-03742-6. MR2187654

[5] Stephan Ramon Garcia and William T. Ross, *Recent progress on truncated Toeplitz operators*, Blaschke products and their applications, Fields Inst. Commun., vol. 65, Springer, New York, 2013, pp. 275–319, DOI 10.1007/978-1-4614-5341-3_15. MR3052299

[6] Kamila Kliś-Garlicka, Bartosz Łanucha, and Marek Ptak, *Characterization of truncated Toeplitz operators by conjugations*, Oper. Matrices **11** (2017), no. 3, 807–822, DOI 10.7153/oam-11-57. MR3655687

[7] Donald Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526, DOI 10.7153/oam-01-29. MR2363975

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