# ALMOST PERIODIC SOLUTIONS OF SUBLINEAR HEAT EQUATIONS 

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#### Abstract

This paper is concerned with the existence and uniqueness of positive almost periodic solutions of a class of sublinear heat equations. The results of this paper are the analogues of the corresponding results in the periodic case.


## 1. Introduction

The purpose of this paper is to study the existence and uniqueness of positive almost periodic solutions of the sublinear heat equations subject to the Dirichlet boundary condition on the whole real line

$$
\begin{cases}\frac{\partial u}{\partial t}-\Delta u=\alpha(x, t) u^{q}(x, t), & (x, t) \in \Omega \times \mathbb{R}  \tag{1.1}\\ u(x, t)=0, & (x, t) \in \partial \Omega \times \mathbb{R}\end{cases}
$$

where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with smooth boundary $\partial \Omega, 0<q<1$, and $\alpha$ is a positive bounded measurable function on $\Omega \times \mathbb{R}$.

The time almost periodic dependence reflects the effects of certain "seasonal" variations which are roughly but not exactly periodic. In contrast to periodic solutions, almost periodic solutions have more application in physics, and the research is more difficult (see for example [3, 8, [15, 16, 21]), since the uniform topology in time in the whole space should be used.

Periodic solutions of the problem (1.1) have been widely investigated over the past 30 years and there have been a great number of results. Beltramo and Hess considered in 2 the linear case of (1.1) (i.e., $q=1$ ), and later the authors dealt with the superlinear case of (1.1) (i.e., $q>1$ ) in [5, 6, 17]. In 2010, Yin and Jin 23] gave a rather complete characterization for the evolutionary $p$-Laplacian, in terms of the parameter $p>1$ and the exponent $q>0$ of the source, of whether or not the positive periodic solutions exist. The existence of positive periodic solutions for both the superlinear case and the sublinear case of (1.1) can be obtained as a special case if one takes $p=2$ in [23. However, the problem on almost periodic solutions of (1.1) has received much less attention until recently. It is worth mentioning that the

[^0]authors studied in [10] the existence of almost automorphic solutions of sublinear evolution equations, which includes (1.1) as one of the most important examples. In general, the almost periodic function belongs to the almost automorphic function class, but not vice versa. Later the results in [10] have been improved by Diagana (4) to the almost periodic case. Unfortunately, those results in [4, 10 are based on an incorrect compact embedding theorem (see Proposition 3.3 in [10]), and the corresponding corrigendum can be found in [11]. This means that the existence of almost periodic solutions for sublinear parabolic equations remains unsolved. And that is exactly where our motivation for studying the existence of almost periodic solutions to the problem (1.1) comes from.

It is well known that almost periodic solutions are closely connected with global bounded solutions (cf. [1, 7, 8, 12, 13, 18, 19]). In the present paper, the upper and lower solutions method and the energy estimate method are used to study the existence of the global bounded solution. By constructing an appropriate auxiliary function $z(x, t)=u^{q}(x, t)-u^{q}(x, t+\tau)$, we get the relationship between the $\varepsilon$ translation sets of $\alpha$ and $u^{q}$, which guarantees the almost periodicity of the solution $u$.

By the way, the authors considered in [20, 22, 23] the periodic solution of some nonlinear degenerate parabolic equations. Therefore, we will continue to study the almost periodicity of the solution to the evolutionary $p$-Laplacian with nonlinear sources in future articles.

The paper is organized as follows. In Section 2, we give some notation and definitions, as well as our main results. In Section 3, we first establish several estimates of positive solutions defined on the half time axis and then prove the theorems.

## 2. Definitions and main results

2.1. Definitions. First, we introduce the definition of the almost periodic function as in [14.

Definition 2.1. Let $X$ be a Banach space. We say that a function $u(\cdot, t) \in C(\mathbb{R} ; X)$ is $X$ almost periodic, denoted by $u(\cdot, t) \in A P(X)$, if for any $\varepsilon>0$, the $\varepsilon$-translation set

$$
T(\varepsilon, u)=\left\{\tau \in \mathbb{R}: \sup _{t \in \mathbb{R}}\|u(\cdot, t+\tau)-u(\cdot, t)\|_{X}<\varepsilon\right\}
$$

is relatively dense; i.e., there is a number $l=l(\varepsilon)>0$ such that any interval of length $l$ contains at least one number from $T(\varepsilon, u)$.

$$
\text { For }-\infty \leq s<t \leq+\infty \text {, denote } Q_{s}^{t}=\Omega \times(s, t)
$$

Next we give several definitions of solutions to the problem (1.1) in the following weak sense, which will be referred to on different occasions. In order to ensure the nonlinear source term $u^{q}$ makes sense, the solution should be nonnegative. By the property of infinite propagation of disturbances of the heat conduct equation, the nontrivial and nonnegative solution must be positive on $\Omega \times(-\infty,+\infty)$, so we call it the positive solution.

Definition 2.2. Let $-\infty<t_{0}<T<+\infty$. A function $u$ is called a positive solution of the problem (1.1) on $Q_{t_{0}}^{T}$ if $u \in \stackrel{\circ}{V}_{2}^{1,0}\left(Q_{t_{0}}^{T}\right), u>0$ in $Q_{t_{0}}^{T}$, and for any function
$\varphi \in \stackrel{\circ}{W}_{2}^{1,1}\left(Q_{t_{0}}^{T}\right)$ with $\left.\varphi\left(\cdot, t_{0}\right)\right|_{\Omega}=\left.\varphi(\cdot, T)\right|_{\Omega}=0$ the following integral equality holds:

$$
\begin{equation*}
\iint_{Q_{t_{0}}^{T}}\left(-u \frac{\partial \varphi}{\partial t}+\nabla u \nabla \varphi-\alpha u^{q} \varphi\right) d x d t=0 . \tag{2.1}
\end{equation*}
$$

A function $u$ is called a positive solution of the problem (1.1) on $Q_{-\infty}^{+\infty}$, provided that for any $-\infty<s<t<+\infty, u$ is a positive solution of the problem (1.1) on $Q_{s}^{t}$.

Definition 2.3. A function $u$ is called a positive upper solution of the problem (1.1) on $Q_{t_{0}}^{T}$ with the initial value $u\left(x, t_{0}\right)=u_{0}(x)$ if $u \in V_{2}^{1,0}\left(Q_{t_{0}}^{T}\right), u>0$ in $Q_{t_{0}}^{T}$, and for any nonnegative function $\varphi \in \stackrel{\circ}{W}{ }_{2}^{1,1}\left(Q_{t_{0}}^{T}\right)$ with $\left.\varphi\left(\cdot, t_{0}\right)\right|_{\Omega}=\left.\varphi(\cdot, T)\right|_{\Omega}=0$ the following inequalities hold:

$$
\begin{cases}\iint_{Q_{t_{0}}^{T}}\left(-u \frac{\partial \varphi}{\partial t}+\nabla u \nabla \varphi-\alpha u^{q} \varphi\right) d x d t \geq 0, & (x, t) \in Q_{t_{0}}^{T}  \tag{2.2}\\ u(x, t) \geq 0, & (x, t) \in \partial \Omega \times\left(t_{0}, T\right) \\ u\left(x, t_{0}\right) \geq u_{0}(x), & x \in \Omega\end{cases}
$$

Replacing " $\geq$ " by " $\leq$ " in (2.2), the definition of a positive lower solution follows. Furthermore, if $u$ is a positive upper solution as well as a positive lower solution with $u\left(x, t_{0}\right)=u_{0}(x)$, then we call it a positive solution of the problem (1.1) on $Q_{t_{0}}^{T}$ with $u\left(x, t_{0}\right)=u_{0}(x)$.

A function $u$ is called a positive solution of the problem (1.1) on $Q_{t_{0}}^{+\infty}$ with $u\left(x, t_{0}\right)=u_{0}(x)$ if, for any $T>t_{0}, u$ is a positive solution of the problem (1.1) on $Q_{t_{0}}^{T}$ with $u\left(x, t_{0}\right)=u_{0}(x)$.

It is easy to see that if $u$ is a positive solution of (1.1) on $Q_{t_{0}}^{T}$ with the initial value $u\left(x, t_{0}\right)=u_{0}(x)$, then it is also a positive solution of (1.1) on $Q_{t_{0}}^{T}$, but not vice versa.

Definition 2.4. A function $u$ is called a positive global $L^{\infty}(\Omega)$-bounded solution of the problem (1.1) if $u$ is a positive solution of (1.1) on $Q_{-\infty}^{+\infty}$ satisfying

$$
\begin{equation*}
\underset{t \in \mathbb{R}}{\operatorname{esssup}}\|u(\cdot, t)\|_{L^{\infty}(\Omega)}<+\infty \tag{2.3}
\end{equation*}
$$

2.2. Main results. Now we state the main results of this paper.

Theorem 2.1. Suppose that $\alpha \in L^{\infty}\left(Q_{-\infty}^{+\infty}\right)$, and $S=\inf _{\Omega \times \mathbb{R}} \alpha(x, t)>0$. Then the problem (1.1) admits uniquely a positive global $L^{\infty}(\Omega)$-bounded solution $u$. Moreover, for $r>\frac{1}{1-q}, u$ is asymptotically stable in the following sense:
$\left\|u^{1-q}(\cdot, t)-v^{1-q}(\cdot, t)\right\|_{L^{r}(\Omega)} \leq\left\|u^{1-q}\left(\cdot, t_{0}\right)-v_{0}^{1-q}(\cdot)\right\|_{L^{r}(\Omega)} e^{M\left(t_{0}-t\right)} \quad$ for all $t \geq t_{0}$, where $v$ is a positive solution of the problem (1.1) on $Q_{t_{0}}^{+\infty}$ with $v\left(x, t_{0}\right)=v_{0}(x)>0$ in $\Omega, v_{0} \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$, and $M$ is a positive constant depending only on $N, q$, $\Omega$ and $r$.

Theorem 2.2. Let $r>\frac{1}{1-q}$. In addition to the assumptions of Theorem 2.1, if $\alpha \in A P\left(L^{r}(\Omega)\right)$, then the positive global $L^{\infty}(\Omega)$-bounded solution $u$ of the problem (1.1) necessarily belongs to $A P\left(L^{r}(\Omega)\right)$.

Remark 2.1. Under the assumptions of Theorem 2.1, if $\alpha$ is $\omega$-periodic in $t$ additionally, i.e., $\alpha(x, t)=\alpha(x, t+\omega)$ a.e. $\Omega \times \mathbb{R}$, then the positive global $L^{\infty}(\Omega)$-bounded solution $u$ of the problem (1.1) is necessarily $\omega$-periodic in $t$. This is in accordance with Theorem 2.1 in [23]. However we also prove the uniqueness and stability of the positive periodic solution, and to our knowledge this has not been discovered in the previous results on the subject.

## 3. Proofs of theorems

It is well known that there exists a positive solution $u$ of the problem (1.1) on $Q_{t_{0}}^{+\infty}$ with the nontrivial and nonnegative initial value $u\left(\cdot, t_{0}\right)=u_{0}(\cdot) \in L^{2}(\Omega)$ (cf. [9, 24]).

Lemma 3.1. Suppose $\alpha^{*}=\sup _{t \in \mathbb{R}} \iint_{Q_{t}^{t+1}} \alpha^{\frac{2}{1-q}}(x, \tau) d x d \tau<+\infty, u_{0} \in L^{2}(\Omega)$, and $u_{0}$ is nontrivial and nonnegative. Let $u$ be a positive solution of the problem (1.1) on $Q_{t_{0}}^{+\infty}$ with the initial value $u\left(x, t_{0}\right)=u_{0}(x)$. Then the following estimates hold:

$$
\begin{equation*}
\sup _{t \geq t_{0}}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\sup _{t \geq t_{0}}\|\nabla u\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.1}
\end{equation*}
$$

$$
\begin{align*}
& \sup _{t_{0} \leq t \leq t_{0}+1}\left(t-t_{0}\right)\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\sup _{t \geq t_{0}+1}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right)  \tag{3.2}\\
& \int_{t_{0}}^{t_{0}+1}\left(s-t_{0}\right)\left\|\frac{\partial u(\cdot, s)}{\partial s}\right\|_{L^{2}(\Omega)}^{2} d s+\sup _{t \geq t_{0}+1}\left\|\frac{\partial u}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.3}
\end{align*}
$$

If $u_{0} \in H_{0}^{1}(\Omega)$ additionally, then

$$
\begin{equation*}
\sup _{t \geq t_{0}}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\sup _{t \geq t_{0}}\left\|\frac{\partial u}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.4}
\end{equation*}
$$

Here we denote by $C$ a positive constant which only depends on $N, q, \Omega$.
Proof. In order to obtain some necessary estimates, we might as well assume that the solution is appropriately smooth, since the same result can be obtained by considering a related approximate problem.
(i) Multiplying the first equation in (1.1) by $u$ and then integrating over $\Omega$ by parts, we derive

$$
\frac{1}{2} \frac{d}{d t}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2}=\int_{\Omega} \alpha(x, t) u^{1+q}(x, t) d x, \quad t \geq t_{0}
$$

Then it follows from Young's inequality and Poincaré's inequality that

$$
\begin{equation*}
\frac{d}{d t}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq C\|\alpha(\cdot, t)\|_{L^{2 /(1-q)(\Omega)}}^{2 /(1-q)}, \quad t \geq t_{0} \tag{3.5}
\end{equation*}
$$

Thus

$$
\frac{d}{d t}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\lambda_{1}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq C\|\alpha(\cdot, t)\|_{L^{2 /(1-q)}(\Omega)}^{2 /(1-q)}, \quad t \geq t_{0}
$$

where $\lambda_{1}>0$ is the first eigenvalue of the problem $-\Delta \psi=\lambda \psi$ in $\Omega$ with $\psi=0$ on $\partial \Omega$. Multiplying the above inequality by $e^{\lambda_{1} t}$ and then integrating over $[s, t]$, we get
$\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2} e^{\lambda_{1} t}-\|u(\cdot, s)\|_{L^{2}(\Omega)}^{2} e^{\lambda_{1} s} \leq C \int_{s}^{t}\|\alpha(\cdot, \tau)\|_{L^{2 /(1-q)(\Omega)}}^{2 /(1-q)} e^{\lambda_{1} \tau} d \tau, \quad t>s \geq t_{0}$.
Now let $t=s+1$, and consequently

$$
\begin{aligned}
\|u(\cdot, s+1)\|_{L^{2}(\Omega)}^{2} & \leq e^{-\lambda_{1}}\|u(\cdot, s)\|_{L^{2}(\Omega)}^{2}+C \int_{s}^{s+1}\|\alpha(\cdot, \tau)\|_{L^{2 /(1-q)}(\Omega)}^{2 /(1-q)} d \tau \\
& \leq e^{-\lambda_{1}}\|u(\cdot, s)\|_{L^{2}(\Omega)}^{2}+C \alpha^{*}, \quad s \geq t_{0}
\end{aligned}
$$

We therefore see that for any $n \in \mathbb{N}^{+}$,

$$
\|u(\cdot, s+n)\|_{L^{2}(\Omega)}^{2} \leq e^{-n \lambda_{1}}\|u(\cdot, s)\|_{L^{2}(\Omega)}^{2}+\frac{C \alpha^{*}}{1-e^{-\lambda_{1}}}, \quad s \geq t_{0}
$$

Next let $s=t_{0}$ in (3.6) to get

$$
\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+C \alpha^{*}, \quad t_{0} \leq t \leq t_{0}+1
$$

It follows from the above two inequalities that

$$
\begin{equation*}
\sup _{t \geq t_{0}}\|u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+C \alpha^{*} \tag{3.7}
\end{equation*}
$$

Integrating (3.5) over $[t, t+1]$ and using (3.7), we have

$$
\begin{equation*}
\sup _{t \geq t_{0}}\|\nabla u\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+C \alpha^{*} \tag{3.8}
\end{equation*}
$$

Combining (3.7)-(3.8) leads us to the estimate (3.1).
(ii) Multiplying the first equation in (1.1) by $\frac{\partial u}{\partial t}$ and then integrating over $\Omega$, we derive

$$
\begin{equation*}
\left\|\frac{\partial u(\cdot, t)}{\partial t}\right\|_{L^{2}(\Omega)}^{2}+\frac{1}{2} \frac{d}{d t}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2}=\int_{\Omega} \alpha(x, t) u^{q}(x, t) \frac{\partial u(x, t)}{\partial t} d x, \quad t \geq t_{0} \tag{3.9}
\end{equation*}
$$

By Young's inequality, we have
$\left\|\frac{\partial u(\cdot, t)}{\partial t}\right\|_{L^{2}(\Omega)}^{2}+\frac{d}{d t}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq q \int_{\Omega} u^{2}(x, t) d x+(1-q) \int_{\Omega} \alpha^{\frac{2}{1-q}}(x, t) d x, \quad t \geq t_{0}$.
We continue by multiplying (3.9) by $t-s$ and then integrating over $[s, \tau]\left(t_{0} \leq s<\tau\right)$ to find that

$$
\begin{align*}
& \int_{s}^{\tau}(t-s)\left\|\frac{\partial u(\cdot, t)}{\partial t}\right\|_{L^{2}(\Omega)}^{2} d t+(\tau-s)\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2}  \tag{3.10}\\
\leq & (\tau-s)\left(q\|u\|_{L^{2}\left(Q_{s}^{\tau}\right)}^{2}+(1-q)\|\alpha\|_{L^{2 /(1-q)}\left(Q_{s}^{\tau}\right)}^{2 /(1-q)}\right)+\|\nabla u\|_{L^{2}\left(Q_{s}^{\tau}\right)}^{2} .
\end{align*}
$$

Removing the first term on the left hand side of (3.10) and utilizing (3.7) and (3.8), we obtain

$$
\begin{equation*}
\sup _{s \leq \tau \leq s+2}(\tau-s)\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right), \quad \forall s \geq t_{0} \tag{3.11}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
& \quad \sup _{s+1 \leq \tau \leq s+2}\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \\
& \leq \sup _{s+1 \leq \tau \leq s+2}(\tau-s)\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \\
& \leq \sup _{s \leq \tau \leq s+2}(\tau-s)\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \\
& \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right), \quad \forall s \geq t_{0}
\end{aligned}
$$

From the arbitrariness of $s$, we thereby see that

$$
\sup _{\tau \geq t_{0}+1}\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right)
$$

Now let $s=t_{0}$ in (3.10) to get that

$$
\sup _{t_{0} \leq \tau \leq t_{0}+1}\left(\tau-t_{0}\right)\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right)
$$

We thereupon conclude from the above two inequalities that

$$
\begin{equation*}
\sup _{t_{0} \leq t \leq t_{0}+1}\left(t-t_{0}\right)\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2}+\sup _{t \geq t_{0}+1}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.12}
\end{equation*}
$$

Similarly, remove the second term on the left hand side of (3.10) to deduce that

$$
\begin{align*}
& \int_{t_{0}}^{t_{0}+1}\left(s-t_{0}\right)\left\|\frac{\partial u(\cdot, s)}{\partial s}\right\|_{L^{2}(\Omega)}^{2} d s+\sup _{t \geq t_{0}+1} \int_{t}^{t+1}\left\|\frac{\partial u(\cdot, s)}{\partial s}\right\|_{L^{2}(\Omega)}^{2} d s  \tag{3.13}\\
& \quad \leq C\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+\alpha^{*}\right)
\end{align*}
$$

(iii) If $u_{0} \in H_{0}^{1}(\Omega)$ additionally, integrating (3.9) over $\left[t_{0}, \tau\right]$ and applying (3.7), we have

$$
\begin{align*}
&\left\|\frac{\partial u}{\partial s}\right\|_{L^{2}\left(Q_{t_{0}}^{\tau_{0}}\right)}^{2}+\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2}  \tag{3.14}\\
& \leq\left\|\nabla u_{0}\right\|_{L^{2}(\Omega)}^{2}+(1-q)\|\alpha\|_{L^{2 /(1-q)\left(Q_{t_{0}}^{\tau}\right)}}^{2 /(1-q)}+q\left(\tau-t_{0}\right)\left(\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}+C \alpha^{*}\right)
\end{align*}
$$

Hence remove the first term on the left hand side of (3.14) to find that

$$
\sup _{t_{0} \leq \tau \leq t_{0}+1}\|\nabla u(\cdot, \tau)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(\Omega)}^{2}+\alpha^{*}\right)
$$

which together with (3.12) implies that

$$
\begin{equation*}
\sup _{t \geq t_{0}}\|\nabla u(\cdot, t)\|_{L^{2}(\Omega)}^{2} \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.15}
\end{equation*}
$$

Similarly, removing the second term on the left hand side of (3.14), we get

$$
\int_{t_{0}}^{t_{0}+2}\left\|\frac{\partial u(\cdot, t)}{\partial t}\right\|_{L^{2}(\Omega)}^{2} d t \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(\Omega)}^{2}+\alpha^{*}\right)
$$

which together with (3.13) leads to

$$
\begin{equation*}
\sup _{t \geq t_{0}}\left\|\frac{\partial u}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\left\|u_{0}\right\|_{H_{0}^{1}(\Omega)}^{2}+\alpha^{*}\right) \tag{3.16}
\end{equation*}
$$

Consequently (3.4) follows from (3.15) and (3.16). The proof is complete.

Proof of Theorem 2.1. (i) We will construct a pair of bounded upper and lower solutions of the problem (1.1) as in [23]. Choose $r$ to be sufficiently large such that $\Omega \subset B_{r / 2}$. Then let $\lambda_{1}, \lambda_{1 r}$ be the first eigenvalue of the Laplacian equation with homogeneous Dirichlet boundary value conditions on $\Omega$ and $B_{r}$ respectively, $\psi_{1}(x)$, $\psi_{1 r}(x)$ with $\left\|\psi_{1}\right\|_{L^{\infty}(\Omega)}=\left\|\psi_{1 r}\right\|_{L^{\infty}\left(B_{r}\right)}=1$ being the corresponding eigenfunction corresponding to $\lambda_{1}$ and $\lambda_{1 r}$. Precisely speaking, $\psi_{1}, \psi_{1 r}$ satisfy that

$$
\begin{gathered}
\begin{cases}-\Delta \psi_{1}(x)=\lambda_{1} \psi_{1}(x) & \text { in } \Omega, \\
\psi_{1}(x)=0 & \text { on } \partial \Omega\end{cases} \\
\begin{cases}-\Delta \psi_{1 r}(x)=\lambda_{1 r} \psi_{1 r}(x) & \text { in } B_{r} \\
\psi_{1 r}(x)=0 & \text { on } \partial B_{r}\end{cases}
\end{gathered}
$$

It is well known that $\psi_{1}(x)>0$ for $x \in \Omega, \psi_{1 r}(x)>0$ for $x \in B_{r}$. Thus there exists a constant $\gamma>0$ such that $\psi_{1 r}(x)>\gamma$ for $x \in \Omega$. Let $\Phi(x)=\kappa_{1} \psi_{1}(x)$ with $\kappa_{1}=\left(\frac{S}{\lambda_{1}}\right)^{1 /(1-q)}$, and $\Psi(x)=\kappa_{2} \psi_{1 r}(x)$ with $\kappa_{2}=\max \left\{\frac{1}{\gamma}\left(\frac{L}{\lambda_{1 r}}\right)^{1 /(1-q)}, \frac{\kappa_{1}}{\gamma}\right\}$, where $L=\underset{\Omega \times \mathbb{R}}{\operatorname{esssup}} \alpha(x, t)$.

Now consider the following problem:

$$
\begin{cases}\frac{\partial u}{\partial t}-\Delta u=\alpha(x, t) u^{q}(x, t), & (x, t) \in \Omega \times(-l,+\infty)  \tag{3.17}\\ u(x, t)=0, & (x, t) \in \partial \Omega \times(-l,+\infty) \\ u(x,-l)=\Phi(x), & x \in \Omega\end{cases}
$$

By the choice of $\Phi(x), \Psi(x), \kappa_{1}$ and $\kappa_{2}$, it is easy to check that $\Phi(x)$ and $\Psi(x)$ are a pair of lower and upper solutions of the problem (3.17) with $0<\Phi(x) \leq \Psi(x) \leq \kappa_{2}$ for $x \in \Omega$. Thus for any $l \in \mathbb{N}^{+}$, the problem (3.17) admits a solution $u^{(l)}(x, t)$ satisfying

$$
\begin{equation*}
\Phi(x) \leq u^{(l)}(x, t) \leq \Psi(x), \quad(x, t) \in \Omega \times(-l,+\infty) \tag{3.18}
\end{equation*}
$$

Furthermore applying Lemma 3.1, we have

$$
\begin{equation*}
\sup _{t \geq-l}\left\|u^{(l)}(\cdot, t)\right\|_{H_{0}^{1}(\Omega)}^{2}+\sup _{t \geq-l}\left\|\frac{\partial u^{(l)}}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\alpha^{*}+\|\Phi\|_{H_{0}^{1}(\Omega)}^{2}\right) \tag{3.19}
\end{equation*}
$$

where $C$ is a constant independent of $l$.
(ii) Fix $t_{0}=-1$. Due to (3.19), there exist a subsequence $\left\{u^{\left(l_{1}(i)\right)}\right\}_{i=1}^{+\infty} \subset$ $\left\{u^{(l)}\right\}_{l=1}^{+\infty}$ and a limit function $u_{(1)} \in L^{\infty}\left((-1,+\infty) ; H_{0}^{1}(\Omega)\right)$ with $\frac{\partial u_{(1)}}{\partial t} \in L^{2}\left(Q_{s}^{T}\right)$ for any $-1 \leq s<T<+\infty$, such that

$$
\begin{cases}u^{\left(l_{1}(i)\right)} \rightarrow u_{(1)} & \text { strongly in } L^{2}\left(Q_{s}^{T}\right), \\ u^{\left(l_{1}(i)\right)} \rightarrow u_{(1)} & \text { a.e. in } Q_{s}^{T} \\ \nabla u^{\left(l_{1}(i)\right)} \rightarrow \nabla u_{(1)} & \text { weakly in } L^{2}\left(Q_{s}^{T}\right)\end{cases}
$$

as $i \rightarrow+\infty$. It follows from (3.18) and (3.19) that

$$
\left\{\begin{array}{l}
\sup _{t \geq-1}\left\|u_{(1)}(\cdot, t)\right\|_{H_{0}^{1}(\Omega)}^{2}+\sup _{t \geq-1}\left\|\frac{\partial u_{(1)}}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\alpha^{*}+\|\Phi\|_{H_{0}^{1}(\Omega)}^{2}\right) \\
\Phi(x) \leq u_{(1)}(x, t) \leq \Psi(x), \quad(x, t) \in \Omega \times(-1,+\infty)
\end{array}\right.
$$

By repeating the process above, we see that for $t_{0}=-j, j=2,3, \cdots$, there exist a family of subsequences $\left\{u^{\left(l_{j}(i)\right)}\right\}_{i=1}^{+\infty} \subset\left\{u^{\left(l_{j-1}(i)\right)}\right\}_{i=1}^{+\infty} \subset \cdots \subset\left\{u^{\left(l_{1}(i)\right)}\right\}_{i=1}^{+\infty}$ (where $l_{j}(i) \geq j$ for $i=1,2, \cdots$ ) and a function sequence $\left\{u_{(j)}\right\}_{j=1}^{+\infty}, u_{(j)} \in$ $L^{\infty}\left((-j,+\infty) ; H_{0}^{1}(\Omega)\right)$ with $\frac{\partial u_{(j)}}{\partial t} \in L^{2}\left(Q_{s}^{T}\right)$ for any $-j \leq s<T<+\infty$, such that

$$
\begin{cases}u^{\left(l_{j}(i)\right)} \rightarrow u_{(j)} & \text { strongly in } L^{2}\left(Q_{s}^{T}\right),  \tag{3.20}\\ u^{\left(l_{j}(i)\right)} \rightarrow u_{(j)} & \text { a.e. in } Q_{s}^{T}, \\ \nabla u^{\left(l_{j}(i)\right)} \rightarrow \nabla u_{(j)} & \text { weakly in } L^{2}\left(Q_{s}^{T}\right)\end{cases}
$$

as $i \rightarrow+\infty$, and

$$
\left\{\begin{array}{l}
\sup _{t \geq-j}\left\|u_{(j)}(\cdot, t)\right\|_{H_{0}^{1}(\Omega)}^{2}+\sup _{t \geq-j}\left\|\frac{\partial u_{(j)}}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\alpha^{*}+\|\Phi\|_{H_{0}^{1}(\Omega)}^{2}\right)  \tag{3.21}\\
\Phi(x) \leq u_{(j)}(x, t) \leq \Psi(x), \quad(x, t) \in \Omega \times(-j,+\infty)
\end{array}\right.
$$

Notice that $u_{(j)}(x, t)=u_{(j-1)}(x, t)$ in $\Omega \times(-(j-1),+\infty)(j=2,3, \cdots)$. That is to say, $u_{(j)}$ is an extension of $u_{(j-1)}$ to $\Omega \times(-j,+\infty)$.

Define $u: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$
u(x, t)=u_{(j)}(x, t) \quad \text { if }(x, t) \in \Omega \times(-j,+\infty)
$$

where $j=1,2, \cdots$.
(iii) The next goal is to show that $u$ is a positive solution of the problem (1.1) on $Q_{-\infty}^{+\infty}$. By (3.21), we have

$$
\left\{\begin{array}{l}
\sup _{t \in \mathbb{R}}\|u(\cdot, t)\|_{H_{0}^{1}(\Omega)}^{2}+\sup _{t \in \mathbb{R}}\left\|\frac{\partial u}{\partial s}\right\|_{L^{2}\left(Q_{t}^{t+1}\right)}^{2} \leq C\left(\alpha^{*}+\|\Phi\|_{H_{0}^{1}(\Omega)}^{2}\right)  \tag{3.22}\\
\Phi(x) \leq u(x, t) \leq \Psi(x), \quad \text { a.e. }(x, t) \in \Omega \times \mathbb{R} .
\end{array}\right.
$$

This implies that $u \in L^{\infty}\left(\mathbb{R} ; H_{0}^{1}(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^{2}\left(Q_{s}^{T}\right)$ for any $-j<s<T<+\infty$. For any $-\infty<s<T<+\infty$, there exists a $j_{0} \in \mathbb{N}^{+}$such that $(s, T) \subset\left(-j_{0},+\infty\right)$. From Definition 2.2, we have that for any function $\varphi \in \stackrel{\circ}{W}_{2}^{1,1}\left(Q_{s}^{T}\right)$ with $\left.\varphi(\cdot, s)\right|_{\Omega}=$ $\left.\varphi(\cdot, T)\right|_{\Omega}=0$, the following integral equality holds:

$$
\begin{equation*}
\iint_{Q_{s}^{T}}\left[-u^{\left(l_{j_{0}}(i)\right)} \frac{\partial \varphi}{\partial t}+\nabla u^{\left(l_{j_{0}}(i)\right)} \nabla \varphi-\alpha\left(u^{\left(l_{j_{0}}(i)\right)}\right)^{q} \varphi\right] d x d t=0 . \tag{3.23}
\end{equation*}
$$

Letting $i \rightarrow+\infty$ in (3.23) and using (3.20), one gets

$$
\iint_{Q_{s}^{T}}\left[-u_{\left(j_{0}\right)} \frac{\partial \varphi}{\partial t}+\nabla u_{\left(j_{0}\right)} \nabla \varphi-\alpha\left(u_{\left(j_{0}\right)}\right)^{q} \varphi\right] d x d t=0
$$

Note that $u=u_{\left(j_{0}\right)}$ on $\Omega \times\left(-j_{0},+\infty\right)$ and $Q_{s}^{T} \subset \Omega \times\left(-j_{0},+\infty\right)$. Therefore

$$
\iint_{Q_{s}^{T}}\left(-u \frac{\partial \varphi}{\partial t}+\nabla u \nabla \varphi-\alpha u^{q} \varphi\right) d x d t=0
$$

From the arbitrariness of $s$ and $T$, we see that $u$ is a positive solution of the problem (1.1) on $Q_{-\infty}^{+\infty}$.
(iv) Now we prove the uniqueness of positive global $L^{\infty}(\Omega)$-bounded solutions of the problem (1.1). Let $u_{1}$ and $u_{2}$ be two positive global $L^{\infty}(\Omega)$-bounded solutions
of the problem (1.1) and denote $w(x, t)=u_{1}^{1-q}(x, t)-u_{2}^{1-q}(x, t)$. We compute by integration by parts

$$
\begin{align*}
\frac{d}{d t}\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r}= & r(1-q) \int_{\Omega}|w|^{r-2} w\left(u_{1}^{-q} u_{1 t}-u_{2}^{-q} u_{2 t}\right) d x \\
= & r(1-q) \int_{\Omega}|w|^{r-2} w\left(u_{1}^{-q} \Delta u_{1}-u_{2}^{-q} \Delta u_{2}\right) d x \\
= & -r(1-q) \int_{\Omega} \nabla\left(|w|^{r-2} w\right) \cdot\left(u_{1}^{-q} \nabla u_{1}-u_{2}^{-q} \nabla u_{2}\right) d x \\
& -r(1-q) \int_{\Omega}|w|^{r-2} w\left(\nabla u_{1}^{-q} \cdot \nabla u_{1}-\nabla u_{2}^{-q} \cdot \nabla u_{2}\right) d x, \tag{3.24}
\end{align*}
$$

where $r>\frac{1}{1-q}$. Note that

$$
\begin{equation*}
\nabla\left(|w|^{r-2} w\right) \cdot\left(u_{1}^{-q} \nabla u_{1}-u_{2}^{-q} \nabla u_{2}\right)=\frac{4(r-1)}{r^{2}(1-q)}\left|\nabla\left(|w|^{\frac{r-2}{2}} w\right)\right|^{2} . \tag{3.25}
\end{equation*}
$$

It follows from Cauchy's inequality that

$$
\begin{aligned}
& -w\left(\nabla u_{1}^{-q} \cdot \nabla u_{1}-\nabla u_{2}^{-q} \cdot \nabla u_{2}\right) \\
= & q\left(u_{1}^{-2 q}\left|\nabla u_{1}\right|^{2}+u_{2}^{-2 q}\left|\nabla u_{2}\right|^{2}\right)-q\left(u_{2}^{1-q} u_{1}^{-q-1}\left|\nabla u_{1}\right|^{2}+u_{1}^{1-q} u_{2}^{-q-1}\left|\nabla u_{2}\right|^{2}\right) \\
\leq & q\left(u_{1}^{-2 q}\left|\nabla u_{1}\right|^{2}+u_{2}^{-2 q}\left|\nabla u_{2}\right|^{2}-2 u_{1}^{-q} u_{2}^{-q} \nabla u_{1} \cdot \nabla u_{2}\right) \\
= & \frac{q}{(1-q)^{2}}|\nabla w|^{2},
\end{aligned}
$$

and hence that

$$
\begin{equation*}
-|w|^{r-2} w\left(\nabla u_{1}^{-q} \cdot \nabla u_{1}-\nabla u_{2}^{-q} \cdot \nabla u_{2}\right) \leq \frac{4 q}{r^{2}(1-q)^{2}}\left|\nabla\left(|w|^{\frac{r-2}{2}} w\right)\right|^{2} . \tag{3.26}
\end{equation*}
$$

We thereupon conclude from (3.24)-(3.26) that

$$
\begin{equation*}
\frac{d}{d t}\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq \frac{-4(-q r+r-1)}{r(1-q)} \int_{\Omega}\left|\nabla\left(|w|^{\frac{r-2}{2}} w\right)\right|^{2} d x . \tag{3.27}
\end{equation*}
$$

From Poincarés inequality, we observe that

$$
\begin{equation*}
\int_{\Omega}|w(\cdot, t)|^{r} d x \leq C_{0} \int_{\Omega}\left|\nabla\left(|w|^{\frac{r-2}{2}} w\right)\right|^{2} d x \tag{3.28}
\end{equation*}
$$

where $C_{0}$ is a positive constant depending only on $N$ and $\Omega$. Applying (3.28) and recalling $r>\frac{1}{1-q}$, we discover that

$$
\begin{equation*}
\frac{d}{d t}\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq-C_{1}\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r}, \quad t \in \mathbb{R} \tag{3.29}
\end{equation*}
$$

where $C_{1}$ is a positive constant depending only on $N, q, \Omega$ and $r$.
We claim that $u_{1}=u_{2}$. We argue by contradiction. Were the above claim false, there would exist $t_{0} \in \mathbb{R}$ such that

$$
\left\|w\left(\cdot, t_{0}\right)\right\|_{L^{r}(\Omega)}^{r}>0 .
$$

Now consider the following ODE corresponding to (3.29):

$$
\left\{\begin{array}{l}
\frac{d y}{d t}=-C_{1} y, \quad t \leq t_{0} \\
y\left(t_{0}\right)=\left\|w\left(\cdot, t_{0}\right)\right\|_{L^{r}(\Omega)}^{r}
\end{array}\right.
$$

Clearly

$$
y(t)=\left\|w\left(\cdot, t_{0}\right)\right\|_{L^{r}(\Omega)}^{r} e^{C_{1}\left(t_{0}-t\right)}, \quad t \leq t_{0} .
$$

Consequently

$$
\lim _{t \rightarrow-\infty} y(t)=+\infty
$$

Then applying the comparison principle of ODE we have

$$
\lim _{t \rightarrow-\infty}\|w(\cdot, t)\|_{L^{r}(\Omega)}=+\infty
$$

However this conclusion is at variance with the boundedness of $\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r}$. This contradiction confirms our claim, and so the uniqueness is proved.
(v) At last we intend to prove the stability of positive solutions. Assume that $v$ is a positive solution of the problem (1.1) on $Q_{t_{0}}^{+\infty}$ with the initial value $v\left(x, t_{0}\right)=$ $v_{0}(x) \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$. Let $w(x, t)=u^{1-q}(x, t)-v^{1-q}(x, t), r>\frac{1}{1-q}$. Then similarly as in the proof of (3.29), we have

$$
\frac{d}{d t}\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq-M\|w(\cdot, t)\|_{L^{r}(\Omega)}^{r}, \quad t \geq t_{0}
$$

where $M$ is a positive constant depending only on $N, q, \Omega$ and $r$. Hence

$$
\|w(\cdot, t)\|_{L^{r}(\Omega)} \leq\left\|u^{1-q}\left(\cdot, t_{0}\right)-v_{0}^{1-q}(\cdot)\right\|_{L^{r}(\Omega)} e^{M\left(t_{0}-t\right)}, \quad t \geq t_{0}
$$

That is to say, $\left\|u^{1-q}(\cdot, t)-v^{1-q}(\cdot, t)\right\|_{L^{r}(\Omega)} \leq\left\|u^{1-q}\left(\cdot, t_{0}\right)-v_{0}^{1-q}(\cdot)\right\|_{L^{r}(\Omega)} e^{M\left(t_{0}-t\right)} \rightarrow 0$ as $t \rightarrow+\infty$. The proof is complete.
Proof of Theorem 2.2. Let $u(x, t)$ be a positive global $L^{\infty}(\Omega)$-bounded solution of the problem (1.1). Then $v(x, t)=u(x, t+\tau)$ is a positive global $L^{\infty}(\Omega)$-bounded solution of the following problem:

$$
\begin{cases}\frac{\partial v}{\partial t}-\Delta v=\alpha(x, t+\tau) v^{q}(x, t), & (x, t) \in \Omega \times \mathbb{R}  \tag{3.30}\\ v(x, t)=0, & (x, t) \in \partial \Omega \times \mathbb{R}\end{cases}
$$

Denote

$$
z(x, t)=u^{1-q}(x, t)-v^{1-q}(x, t)
$$

For any $r>1$, we compute by integration by parts

$$
\begin{aligned}
\frac{d}{d t}\|z(\cdot, t)\|_{L^{r}(\Omega)}^{r}= & r(1-q) \int_{\Omega}|z|^{r-2} z\left(u^{-q} u_{t}-v^{-q} v_{t}\right) d x \\
= & r(1-q) \int_{\Omega}|z|^{r-2} z\left(u^{-q} \Delta u-v^{-q} \Delta v+\alpha(x, t)-\alpha(x, t+\tau)\right) d x \\
= & -r(1-q) \int_{\Omega} \nabla\left(|z|^{r-2} z\right) \cdot\left(u^{-q} \nabla u-v^{-q} \nabla v\right) d x \\
& -r(1-q) \int_{\Omega}|z|^{r-2} z\left(\nabla u^{-q} \cdot \nabla u-\nabla v^{-q} \cdot \nabla v\right) d x \\
& +r(1-q) \int_{\Omega}|z|^{r-2} z(\alpha(x, t)-\alpha(x, t+\tau)) d x .
\end{aligned}
$$

Set

$$
\begin{equation*}
F_{\tau, r}=\sup _{t \in \mathbb{R}}\|\alpha(\cdot, t)-\alpha(\cdot, t+\tau)\|_{L^{r}(\Omega)}, \quad \tau \in \mathbb{R}, 1 /(1-q)<r<+\infty \tag{3.32}
\end{equation*}
$$

Utilizing (3.25), (3.26) and Hölder's inequality, we deduce that (3.33)

$$
\frac{d}{d t}\|z(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq \frac{-4(-q r+r-1)}{r(1-q)} \int_{\Omega}\left|\nabla\left(|z|^{\frac{r-2}{2}} z\right)\right|^{2} d x+r(1-q) F_{\tau, r}\|z\|_{L^{r}(\Omega)}^{r-1}
$$

But inequality (3.28) allows us to estimate

$$
\begin{aligned}
\frac{d}{d t}\|z(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq\|z\|_{L^{r}(\Omega)}^{r-1}\left[\frac{-4(-q r+r-1)}{C_{0} r(1-q)}\|z\|_{L^{r}(\Omega)}+r(1-q) F_{\tau, r}\right] \\
r>1 /(1-q) .
\end{aligned}
$$

For the sake of simplicity of notation, we write $C_{1}=\frac{4(-q r+r-1)}{C_{0} r(1-q)}, C_{2}=r(1-q)$, whereupon the above inequality becomes

$$
\begin{equation*}
\frac{d}{d t}\|z(\cdot, t)\|_{L^{r}(\Omega)}^{r} \leq\|z\|_{L^{r}(\Omega)}^{r-1}\left(-C_{1}\|z\|_{L^{r}(\Omega)}+C_{2} F_{\tau, r}\right), \quad r>1 /(1-q) . \tag{3.34}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
\sup _{t \in \mathbb{R}}\|z(\cdot, t)\|_{L^{r}(\Omega)} \leq \frac{C_{2}}{C_{1}} F_{\tau, r} . \tag{3.35}
\end{equation*}
$$

We argue by contradiction. Were the above inequality false, there would exist $t_{0} \in \mathbb{R}$ such that

$$
\begin{equation*}
\left\|z\left(\cdot, t_{0}\right)\right\|_{L^{r}(\Omega)}>\frac{C_{2}}{C_{1}} F_{\tau, r} \tag{3.36}
\end{equation*}
$$

Now consider the following ODE:

$$
\left\{\begin{array}{l}
\frac{d y}{d t}=\left(-C_{1} y^{\frac{1}{r}}+C_{2} F_{\tau, r}\right) y^{\frac{r-1}{r}}, \quad t \leq t_{0} \\
y\left(t_{0}\right)=\left\|z\left(\cdot, t_{0}\right)\right\|_{L^{r}(\Omega)}^{r}
\end{array}\right.
$$

From (3.36), we see that $\gamma \triangleq-C_{1} y^{\frac{1}{r}}\left(t_{0}\right)+C_{2} F_{\tau, r}<0$. It follows from the decreasing property of $y(t)$ that

$$
\frac{d y}{d t} \leq \gamma y^{\frac{r-1}{r}}, \quad t \leq t_{0}
$$

This implies that

$$
r\left(y^{\frac{1}{r}}\left(t_{0}\right)-y^{\frac{1}{r}}(t)\right) \leq \gamma\left(t_{0}-t\right), \quad t \leq t_{0} .
$$

That is,

$$
r y^{\frac{1}{r}}(t) \geq r y^{\frac{1}{r}}\left(t_{0}\right)-\gamma\left(t_{0}-t\right), \quad t \leq t_{0}
$$

Consequently

$$
\lim _{t \rightarrow-\infty} y(t)=+\infty
$$

We get from the comparison principle of ODE that

$$
\left.\lim _{t \rightarrow-\infty} \| z \cdot t\right) \|_{L^{r}(\Omega)}=+\infty
$$

However this conclusion is at variance with the boundedness of $\|z(\cdot, t)\|_{L^{r}(\Omega)}$. This contradiction confirms (3.35).

According to the differential mean value theorem we have

$$
|z|=(1-q)[\theta u+(1-\theta) v]^{-q}|u-v| \geq(1-q)\|u\|_{L^{\infty}(\Omega \times \mathbb{R})}^{-q}|u-v|,
$$

where $\theta \in(0,1)$. Thus we obtain from the above inequality and (3.35) that

$$
\sup _{t \in \mathbb{R}}\|u(\cdot, t)-u(\cdot, t+\tau)\|_{L^{r}(\Omega)} \leq C \sup _{t \in \mathbb{R}}\|\alpha(\cdot, t)-\alpha(\cdot, t+\tau)\|_{L^{r}(\Omega)},
$$

where $C$ is independent of $\tau$. Hence $u \in A P\left(L^{r}(\Omega)\right)$. The proof is complete.

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