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SIMONS' EQUATION AND MINIMAL HYPERSURFACES IN SPACE FORMS

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ABSTRACT. Let $n \geq 3$ be an integer, and let Σ^n be a non-totally geodesic complete minimal hypersurface immersed in the $(n+1)$-dimensional space form $\overline{M}^{n+1}(c)$, where the constant c denotes the sectional curvature of the space form. If Σ^n satisfies the Simons' equation (3.9), then either (1) Σ^n is a catenoid if $c \leq 0$, or (2) Σ^n is a Clifford minimal hypersurface or a compact Ostuki minimal hypersurface if $c > 0$. This paper is motivated by a 2009 work of Tam and Zhou.

1. Introduction

In 1968 Simons [14] (see also [2, §7 or §9] and [18, §1.6]) showed that the second fundamental form of an immersed minimal hypersurface in the sphere or in the Euclidean space satisfies a second order elliptic partial differential equation, which can imply the famous Simons' inequality. The Simons' inequality enabled him to prove a gap phenomenon for minimal submanifolds in the sphere and the Bernstein's problem in $\mathbb{R}^n$ for $n \leq 7$. Since then the Simons' inequality has been used by various authors to study minimal immersions.

In this paper, we shall classify all complete minimal hypersurfaces immersed in the space form $\overline{M}^{n+1}(c)$ which satisfy the Simons' equation (3.9), where $n \geq 3$.

Roughly speaking, a *catenoid* is a minimal rotation hypersurface immersed in $\overline{M}^{n+1}(c)$. In the case when $c = 0$, Tam and Zhou [17, Theorem 3.1] proved that if a non-flat complete minimal hypersurface Σ^n immersed in $\mathbb{R}^{n+1}$ satisfies the Simons' equation (3.9) on all non-vanishing points of $|A|$, then Σ^n must be a catenoid.

Motivated by the ideas of Tam and Zhou, we generalize the result to the cases when $c \neq 0$. More precisely we will prove the following theorem.

Theorem 1.1. *Let $\overline{M}^{n+1}(c)$ be the space form of dimension $n+1$, where $n \geq 3$. Suppose that Σ^n is a non-totally geodesic complete minimal hypersurface immersed in $\overline{M}^{n+1}(c)$. If the Simons' equation* (3.9) *holds as an equation at all non-vanishing points of $|A|$ in Σ^n, then $\Sigma^n \subset \overline{M}^{n+1}(c)$ is either*

(1) *a catenoid if $c \leq 0$, or*
(2) *a Clifford minimal hypersurface or a compact Ostuki minimal hypersurface if $c > 0$.*

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Remark 1.2. The compact minimal rotation hypersurfaces in Theorem 2.3 are called the *Otsuki minimal hypersurfaces* (see [11]). The Otsuki minimal hypersurfaces are *immersed* catenoids.

Remark 1.3. Combining Proposition 3.3 and Proposition 3.4 with Theorem 1.1, we can see that the Clifford minimal hypersurfaces and the catenoids are the *only* complete minimal hypersurfaces satisfying the Simons' equation (3.9).

Remark 1.4. When we say that Σ^n is a *complete* minimal hypersurface in the space form $\overline{M}^{n+1}(c)$, we actually mean one of the following cases:

(1) if $c \leq 0$, then Σ^n is a non-compact hypersurface without boundary, that is, an open hypersurface, or
(2) if $c > 0$, then Σ^n is a compact hypersurface without boundary, that is, a closed hypersurface.

Plan of the paper. This paper is organized as follows: In §2 we define the catenoids and their generating curves in the space forms. In §3 we derive the Simons' identity (3.1), and we show that the Clifford minimal hypersurfaces and the catenoids satisfy (3.9). In §4 we prove Theorem 1.1. In §5 we offer some figures of the generating curves of the catenoids in the space forms.

2. Preliminary

A simply connected $(n+1)$-dimensional complete Riemannian manifold whose sectional curvature is equal to a constant c, denoted by $\overline{M}^{n+1}(c)$, is called a *space form*. There are three types of space forms:

(i) If $c > 0$, let

$$\overline{M}^{n+1}(c) = \mathbb{S}^{n+1}(c) = \{x \in \mathbb{R}^{n+2} \mid x_1^2 + \cdots + x_{n+2}^2 = 1/c\}\ . \tag{2.1}$$

(ii) If $c < 0$, let

$$\overline{M}^{n+1}(c) = \mathbb{B}^{n+1}(c) = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 < -1/c\}\ , \tag{2.2}$$

with the metric

$$ds^2 = \frac{4|dx|^2}{(1 + c|x|^2)^2}\ , \tag{2.3}$$

where $x = (x_1, \ldots, x_{n+1})$ and $|x|^2 = x_1^2 + \cdots + x_{n+1}^2$.

(iii) If $c = 0$, let

$$\overline{M}^{n+1}(0) = \mathbb{R}^{n+1} \tag{2.4}$$

be the $(n+1)$-dimensional Euclidean space.

In Theorem 2.2 and Theorem 2.3, the results that we quote are about minimal hypersurfaces immersed in the unit sphere $\mathbb{S}^{n+1}$, but these results can be generalized to the space forms $\mathbb{S}^{n+1}(c)$ for $c > 0$.

In fact, consider both $\mathbb{S}^{n+1}$ and $\mathbb{S}^{n+1}(c)$ $(c > 0)$ as the subsets of $\mathbb{R}^{n+2}$. Define the map $f : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ by $f(x) = x/\sqrt{c}$ for any $x \in \mathbb{R}^{n+2}$, where $c > 0$. It's easy to verify $\mathbb{S}^{n+1}(c) = f(\mathbb{S}^{n+1})$. For any hypersurface Σ^n immersed in $\mathbb{S}^{n+1}$, let $\Sigma^n(c) = f(\Sigma^n)$; then $\Sigma^n(c)$ is a hypersurface immersed in $\mathbb{S}^{n+1}(c)$.

Lemma 2.1. *If Σ^n is a minimal hypersurface immersed in $\mathbb{S}^{n+1}$, then $\Sigma^n(c)$ is a minimal hypersurface immersed in $\mathbb{S}^{n+1}(c)$.*

Proof. Let g_{ij} and $\tilde{g}_{ij}$ be the first fundamental forms of the hypersurfaces $\Sigma^n \subset \mathbb{S}^{n+1}$ and $\Sigma^n(c) \subset \mathbb{S}^{n+1}(c)$ respectively; then $\tilde{g}_{ij} = g_{ij}/c$ and $\tilde{g}^{ij} = cg^{ij}$ for $1 \leq i, j \leq n$. It's easy to verify that the Laplacians on Σ^n and $\Sigma^n(c)$ satisfy the equation $\Delta_{\Sigma^n(c)} = c\Delta_{\Sigma^n}$. Let x and $\tilde{x}$ be the position functions of Σ^n and $\Sigma^n(c)$ in $\mathbb{R}^{n+2}$ respectively; then $\tilde{x} = x/\sqrt{c}$.

If Σ is a minimal hypersurface immersed in $\mathbb{S}^{n+1}$, then by Theorem 3 in [16] (see also [5, p. 101] or [8, Theorem 3.10.2]) we have $\Delta_{\Sigma^n} x = -nx$. On the other hand, we have the following identities:

$$\Delta_{\Sigma^n(c)}\tilde{x} = c\Delta_{\Sigma^n}\left(\frac{x}{\sqrt{c}}\right) = \sqrt{c}\,\Delta_{\Sigma^n} x = \sqrt{c}\,(-nx) = -nc\tilde{x}\ ,$$

which can imply that $\Sigma^n(c)$ is a minimal hypersurface immersed in $\mathbb{S}^{n+1}(c)$ by applying [16, Theorem 3] again. ☐

As we will see that any complete minimal hypersurface Σ^n immersed in $\overline{M}^{n+1}(c)$ satisfying the equation (3.9) at all non-vanishing points of $|A|$ is a minimal rotation hypersurface unless Σ^n is a Clifford minimal hypersurface in the case when $c > 0$, so at first we shall study the Clifford minimal hypersurfaces and minimal rotation hypersurfaces in the space forms.

2.1. **Clifford minimal hypersurfaces in $\mathbb{S}^{n+1}(c)$.** In this subsection, we shall define the Clifford minimal hypersurfaces in the space forms $\mathbb{S}^{n+1}(c)$ for $c > 0$. Let $S^q(r)$ be a q-dimensional sphere in $\mathbb{R}^{q+1}$ with radius r. In particular $\mathbb{S}^q(c) = S^q(1/\sqrt{c})$ for $c > 0$.

For $c > 0$ and $m = 1, \ldots, n-1$, a *Clifford minimal hypersurface* embedded in $\mathbb{S}^{n+1}(c)$ is defined as follows:

$$\mathscr{M}_{m,n-m}(c) = S^m\left(\sqrt{\frac{m}{cn}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{cn}}\right)\ . \tag{2.5}$$

In particular, $\mathscr{M}_{m,n-m} = \mathscr{M}_{m,n-m}(1)$ is a Clifford minimal hypersurface embedded in $\mathbb{S}^{n+1}$ (see also [2] and [8, pp.229–230]). The following result is well known.

Theorem 2.2 ([3, 9])**.** *The Clifford minimal hypersurfaces $\mathscr{M}_{m,n-m}$ are the only compact minimal hypersurfaces in $\mathbb{S}^{n+1}$ with $|A|^2 = n$.*

Furthermore the second fundamental form A has two distinct constant eigenvalues with multiplicities m and n respectively.

2.2. **Catenoids in space forms.** In this subsection we shall follow Hsiang [6, 7] to derive the differential equations of the generating curves of catenoids in the space form $\overline{M}^{n+1}(c)$, and solve the differential equations in the case when $c \leq 0$.

Let $G = \mathsf{SO}(n)$ be a subgroup of the orientation preserving isometry group of $\overline{M}^{n+1}(c)$ which pointwise fixes a given geodesic $M^1 \subset \overline{M}^{n+1}(c)$. We call G the *spherical group* of $\overline{M}^{n+1}(c)$ and M^1 the *rotation axis* of G. A hypersurface in $\overline{M}^{n+1}(c)$ that is invariant under G is called a *rotation hypersurface*; if the rotation hypersurface in $\overline{M}^{n+1}(c)$ is a complete minimal hypersurface, then it is called a *spherical catenoid* or just *catenoid*, denoted by $\mathcal{C}$.

The following 2-dimensional half space is well defined

$$M^2_+(c) = \overline{M}^{n+1}(c)/G\ . \tag{2.6}$$

There are three types of half spaces:

(1) $\mathbb{S}^2_+(c) = \{x_1^2 + x_{n+1}^2 + x_{n+2}^2 = 1/c \mid x_1 \geq 0\} = M^2_+(c)$ if $c > 0$.
(2) $\mathbb{B}^2_+(c) = \{x_1^2 + x_{n+1}^2 < -1/c \mid x_1 \geq 0\} = M^2_+(c)$ if $c < 0$.
(3) $\mathbb{R}^2_+ = \{(x_1, x_{n+1}) \in \mathbb{R}^2 \mid x_1 \geq 0\} = M^2_+(0)$ if $c = 0$.

It's easy to see that the rotation axis M^1 is the boundary of $M^2_+(c)$.

The orbital distance metric on $\overline{M}^{n+1}(c)/G$ is the same as the restriction metric of $M^2_+(c)$. Let $d(\cdot,\cdot)$ be the distance function defined on $M^2_+(c)$. We shall parametrize $M^2_+(c)$ by the following coordinate system: Choose a base point $O \in M^1$ and let x be the arc length on M^1 travelling in the positive orientation of $M^1 = \partial M^2_+(c)$. To each point $p \in M^2_+(c)$, there is a (unique) point $q \in M^1$ such that the length of the geodesic arc connecting p and q, denoted by $\overline{pq}$, is equal to $d(p, M^1)$. We shall assign to the point p the coordinate (x, y), where $x = d(O, q)$ and $y = d(p, q) =$ the length of the geodesic arc $\overline{pq}$ (see Figure 1).

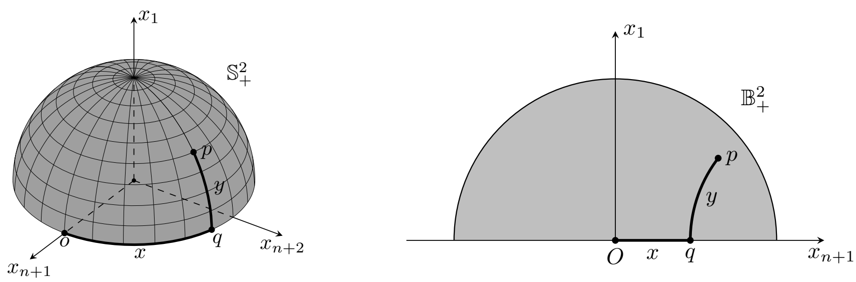

FIGURE 1. The warped product metric on the half space $M^2_+(c)$ for $c = \pm 1$. In each half space, $x = d(O, q)$ and $y = d(p, q) =$ the length of the geodesic arc $\overline{pq}$.

According to the above definition of x and y, we have

$$\begin{cases} -\infty < x < \infty \text{ and } 0 \leq y < \infty\ , & \text{if } c \leq 0\ , \\ -\dfrac{\pi}{\sqrt{c}} \leq x < \dfrac{\pi}{\sqrt{c}} \text{ and } 0 \leq y \leq \dfrac{\pi}{2\sqrt{c}}\ , & \text{if } c > 0\ . \end{cases}$$

In the case $c > 0$, the coordinate of the center is $\left(x, \dfrac{\pi}{2\sqrt{c}}\right)$, where x is arbitrary. The *warped product metric* on $M^2_+(c)$ is written in the form

$$ds^2 = (f'(y))^2 \cdot dx^2 + dy^2\ , \tag{2.7}$$

where $f' = df/dy$ and

$$f(y) = \begin{cases} \dfrac{1}{\sqrt{-c}} \sinh(\sqrt{-c}\, y)\ , & \text{if } c < 0 \text{ (hyperbolic case)}\ , \\ y\ , & \text{if } c = 0 \text{ (Euclidean case)}\ , \\ \dfrac{1}{\sqrt{c}} \sin(\sqrt{c}\, y)\ , & \text{if } c > 0 \text{ (spherical case)}\ . \end{cases} \tag{2.8}$$

Let Σ^n be a rotation hypersurface in $\overline{M}^{n+1}(c)$ with respect to the geodesic M^1; then the curve $\gamma = \Sigma^n \cap M^2_+(c)$ is called the *generating curve* of Σ^n. Suppose that γ is given by the parametric equations: $x = x(s)$ and $y = y(s)$, $a \leq s \leq b$, where s

is the arc length of γ and $y(s) > 0$. Let α be the angle between the unit tangent vector of $\gamma(s)$ and $\partial/\partial y$ (see Figure 2).

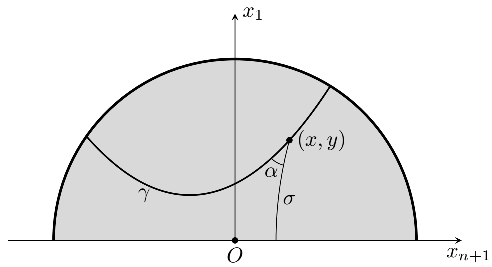

FIGURE 2. In the hyperbolic half space $\mathbb{B}^2_+$, α is the angle between the parametrized curve γ and the geodesic σ at the point (x, y), where σ is perpendicular to the x_{n+1}-axis.

Now suppose that the mean curvature of the rotation hypersurface $\Sigma^n \subset \overline{M}^{n+1}(c)$ generated by the curve $\gamma \subset M^2_+(c)$ is zero; then we get the following differential equations of γ (see [6, pp. 487–488] for the details):

$$\frac{f^{n-1} \cdot (f')^2}{\sqrt{(f')^2 + (y')^2}} = f^{n-1} \cdot f' \cdot \sin\alpha = k \text{ (constant)} , \tag{2.9}$$

where $y' = dy/dx$.

Without loss of generality we may consider the differential equations (2.9) with initial data $y(0) = a > 0$ and $y'(0) = 0$. Plugging the initial conditions into (2.9), we get $k = f^{n-1}(a)f'(a)$, which implies the following equation:

$$\sin\alpha = \frac{f^{n-1}(a)f'(a)}{f^{n-1}(y)f'(y)} . \tag{2.10}$$

Now we assume $c \leq 0$; then we can write dx/dy as follows:

$$\frac{dx}{dy} = \frac{1}{f'(y)} \cdot \frac{dy}{\sqrt{\left(\frac{f(y)}{f(a)}\right)^{2n-2} \cdot \left(\frac{f'(y)}{f'(a)}\right)^2 - 1}} .$$

Integrating both sides in terms of y, we have the following function:

$$x(y) = \int_a^y \frac{1}{f'(t)} \cdot \frac{dt}{\sqrt{\left(\frac{f(t)}{f(a)}\right)^{2n-2} \cdot \left(\frac{f'(t)}{f'(a)}\right)^2 - 1}} , \tag{2.11}$$

where $a \leq y < \infty$.

2.3. **Catenoids in $\mathbb{S}^{n+1}(c)$.** In this subsection we follow Otsuki [11] to study the generating curves of the compact immersed minimal rotation hypersurfaces in $\mathbb{S}^{n+1}$. See also [1] in *equivariant language*.

Suppose that $\gamma \subset \mathbb{S}^2_+$ is the generating curve of a minimal rotation hypersurface in $\mathbb{S}^{n+1}$. Project orthogonally the curve γ into the $x_{n+1}x_{n+2}$-plane, and denote it by σ. Let $h(\theta)$ be the support function of the curve σ. Otsuki [11] proved that h satisfies the differential equation

$$nh(1-h^2)\frac{d^2h}{d\theta^2}+\left(\frac{dh}{d\theta}\right)^2+(1-h^2)(nh^2-1)=0 \tag{2.12}$$

with the initial conditions $h(0)=a\le 1/\sqrt{n}$ and $h'(0)=0$ (see Figure 3).

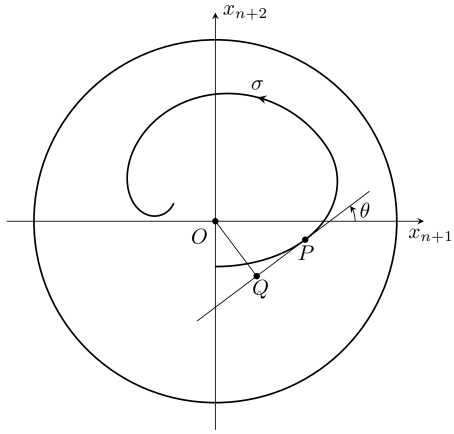

FIGURE 3. The generating curve for the spherical catenoid in $\mathbb{S}^{n+1}$, its support function is $h(\theta)=d(O,Q)$ and $h'(\theta)=d(P,Q)$. The coordinates (x_{n+1},x_{n+2}) of the point P (inside the unit disk) are given by $x_{n+1}=h\sin\theta+h'\cos\theta$ and $x_{n+2}=-h\cos\theta+h'\sin\theta$ (see equation (4.2) in [11]).

If $a=1/\sqrt{n}$, then the minimal rotation hypersurface in $\mathbb{S}^{n+1}$ generated by the curve σ satisfying (2.12) with this initial data is $\mathscr{M}_{n-1,1}$, one of the Clifford minimal hypersurfaces (see [11, p. 160]).

From now on we may assume that $0<a<1/\sqrt{n}$, and set

$$C(a)=(a^2)^{1/n}(1-a^2)^{1-1/n}=a^{2/n}(1-a^2)^{1-1/n} \tag{2.13}$$

for $0<a<1/\sqrt{n}$. Otsuki [11, 12] (see also [10]) proved that the support function $h(\theta)$, which is the solution to the differential equation (2.12) with the initial conditions $h(0)=a\le 1/\sqrt{n}$ and $h'(0)=0$, is a periodic function whose period is

$$T(a)=2\int_{a_0}^{a_1}\frac{dx}{\sqrt{1-x^2-C(a)\left(\dfrac{1}{x^2}-1\right)^{1/n}}}\ , \tag{2.14}$$

where $a_0=a\in(0,1/\sqrt{n})$, and $a_1\in(1/\sqrt{n},1)$ is a solution to the equation

$$1-x^2-C(a)\left(\frac{1}{x^2}-1\right)^{1/n}=0\ .$$

It was proved in [10–12] that the period T satisfies the following conditions:

(1) $T(a)\in(\pi,2\pi)$ is differentiable on $(0,1/\sqrt{n})$,
(2) $\lim_{a\to 0^+}T(a)=\pi$ and $\lim_{a\to(1/\sqrt{n})^-}T(a)=\sqrt{2}\,\pi$.

Moreover the generating curve σ is a simple closed curve if and only if $T(a) = 2\pi/k$ for $k = 1, 2, \ldots$, and σ is a closed curve (not necessarily simple) if and only if $T(a)$ is a (positive) rational multiple of π.

In conclusion we have the following results:

Theorem 2.3 ([1, 10–12])**.** *Let $n \geq 3$ be an integer.*

(1) *There is no closed minimally embedded rotation hypersurface of $\mathbb{S}^{n+1}$ other than the Clifford minimal hypersurface $\mathscr{M}_{n-1,1}$ and the round geodesic sphere $\mathbb{S}^n$.*
(2) *There are countably infinitely many closed minimal rotation hypersurfaces immersed in $\mathbb{S}^{n+1}$ (see also [7]).*

3. Simons' equation and catenoids in space forms

Suppose that Σ^n is a hypersurface immersed in the $(n+1)$-dimensional space form $\overline{M}^{n+1}(c)$. Let A be the second fundamental form of Σ^n and ∇A be the covariant derivative of A, and let h_{ij} and h_{ijk} be the components of A and ∇A in an orthonormal frame respectively.

The following lemma was proved by Tam and Zhou [17, Lemma 3.1] for the case when $c = 0$, but it's also true for the case when $c \neq 0$.

Lemma 3.1. *Let Σ^n be a minimal hypersurface immersed in the space form $\overline{M}^{n+1}(c)$. At a point where $|A| > 0$, we have*

$$|A|\Delta|A| + |A|^4 = \frac{2}{n}\,|\nabla|A||^2 + nc|A|^2 + E\ , \tag{3.1}$$

with $E \geq 0$. Moreover, in an orthonormal frame such that $h_{ij} = \lambda_i\delta_{ij}$, then $E = E_1 + E_2 + E_3$, where

$$\begin{aligned} E_1 &= \sum_{j\neq i, k\neq i, k\neq j} h_{ijk}^2\ ,\\ E_2 &= \frac{2}{n}\sum_{j\neq i, k\neq i, k\neq j}(h_{kki} - h_{jji})^2\ ,\\ E_3 &= \left(1 + \frac{2}{n}\right)|A|^{-2}\sum_k\sum_{i\neq j}(h_{ii}h_{jjk} - h_{jj}h_{iik})^2\ . \end{aligned} \tag{3.2}$$

Proof. For any point $p \in \Sigma^n$, we choose an orthonormal frame field $e_1, \ldots, e_{n+1}$ such that, restricted to Σ^n, the vectors $e_1, \ldots, e_n$ are tangent to Σ^n and the vector e_{n+1} is perpendicular to Σ^n, and the second fundamental form of Σ^n is diagonalized by $h_{ij} = \lambda_i\delta_{ij}$, where $1 \leq i, j \leq n$.

Recall that the curvature tensor $\overline{R}_{ABCD}$ of $\overline{M}^{n+1}(c)$ is given by

$$\overline{R}_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\ , \quad 1 \leq A, B, C, D \leq n+1\ , \tag{3.3}$$

where δ_{AB} is the Kronecker delta. According to [3, (3.1)] and [13, (1.21) and (1.27)], we have

$$\sum_{i,j} h_{ij}\Delta h_{ij} = -|A|^4 + nc|A|^2\ , \tag{3.4}$$

and

$$|A|\Delta|A| + |\nabla|A||^2 = \frac{1}{2}\Delta|A|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\ , \tag{3.5}$$

where h_{ijk} denotes the component of the covariant derivative of the second fundamental form A of Σ^n for $1 \le i, j, k \le n$. Therefore we have

$$|A|\Delta|A| + |\nabla|A||^2 = |\nabla A|^2 - |A|^4 + nc|A|^2 \ , \tag{3.6}$$

where $|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2$. We claim that

$$|\nabla A|^2 = \left(1 + \frac{2}{n}\right) |\nabla|A||^2 + E \ . \tag{3.7}$$

In fact, according to the computation in [17, pp. 3456–3457], we have

$$\begin{aligned} |\nabla A|^2 - |\nabla|A||^2 &= E_1 + 2\sum_{i\neq k} h_{iik}^2 + \frac{n}{n+2}\, E_3 \\ &= E_1 + \frac{2}{n}\left(|\nabla|A||^2 + \frac{n}{n+2}\, E_3 + \frac{n}{2}\, E_2\right) + \frac{n}{n+2}\, E_3 \\ &= E_1 + \frac{2}{n}|\nabla|A||^2 + E_2 + E_3 \ , \end{aligned}$$

where we use the fact $h_{ijk} = h_{ikj}$ since the sectional curvature of $\overline{M}^{n+1}(c)$ is constant (see [3, (2.12)] or [13, (1.10)]).

Combining (3.6) and (3.7) together, we have (3.1). □

Obviously, the term E in (3.1) is always non-negative, which implies the famous Simons' inequality

$$|A|\Delta|A| + |A|^4 \ge \frac{2}{n}\, |\nabla|A||^2 + nc|A|^2 \ . \tag{3.8}$$

If $E \equiv 0$ in (3.1), we get the *Simons' equation*

$$|A|\Delta|A| + |A|^4 = \frac{2}{n}\, |\nabla|A||^2 + nc|A|^2 \ . \tag{3.9}$$

If $n = 2$, then $E \equiv 0$ in (3.1), so we have the following corollary.

Corollary 3.2. *If Σ^2 is a minimal surface immersed in the 3-dimensional space form $\overline{M}^3(c)$, then Σ^2 satisfies the following Simons' equation:*

$$|A|\Delta|A| + |A|^4 = |\nabla|A||^2 + 2c|A|^2 \ . \tag{3.10}$$

We may ask what kinds of non-totally geodesic minimal hypersurfaces in the space form $\overline{M}^{n+1}(c)$ satisfy the Simons' equation (3.9). Proposition 3.3 and Proposition 3.4 show that the Clifford minimal hypersurfaces (2.5) and the minimal rotation hypersurfaces (i.e., the catenoids) satisfy (3.9).

On the other hand, Theorem 1.1 shows that the Clifford minimal hypersurfaces and catenoids are the only non-totally geodesic complete minimal hypersurfaces satisfying (3.9).

Proposition 3.3. *When $c > 0$, the second fundamental form $|A|$ of each Clifford minimal hypersurface* (2.5) *in $\mathbb{S}^{n+1}(c)$ satisfies* (3.9).

Proof. We can prove the statement by direct computation (see [3, pp. 68-70] or [8, pp. 229–230] for the details). Because of Lemma 2.1, we will just prove the statement for the case when $c = 1$.

For $m = 1, \dots, n-1$, we may embed the Clifford minimal hypersurface $\mathscr{M}_{m,n-m}$ into $\mathbb{S}^{n+1}$ as follows. Let (u, v) be a point of $\mathscr{M}_{m,n-m}$ where u is a vector in $\mathbb{R}^{m+1}$

of length $\sqrt{m/n}$, and v is a vector in $\mathbb{R}^{n-m+1}$ of length $\sqrt{(n-m)/n}$. We can consider (u,v) as a vector in $\mathbb{R}^{n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n-m+1}$ of length 1. Then we may choose an orthonormal basis on $\mathscr{M}_{m,n-m}$ such that the second fundamental form of $\mathscr{M}_{m,n-m}$ can be written as follows:

$$h_{ij} = \operatorname{diag}\left(\underbrace{\sqrt{\frac{n-m}{n}},\dots,\sqrt{\frac{n-m}{n}}}_{m}, \underbrace{-\sqrt{\frac{m}{n-m}},\dots,-\sqrt{\frac{m}{n-m}}}_{n-m}\right) .$$

Since each component of h_{ij} is a constant, we have $E=0$ in (3.1). On the other hand, it's easy to get

$$|A|^2 = m\cdot\frac{n-m}{m} + (n-m)\cdot\frac{m}{n-m} = n ,$$

so the Clifford minimal hypersurfaces satisfy (3.9). □

Next we shall verify that any catenoid in the space form $\overline{M}^{n+1}(c)$ satisfies the Simons' equation (3.9). In the case when $c=0$, Proposition 3.4 was proved by Tam and Zhou [17, Proposition 2.1 (iv)].

Proposition 3.4. *The second fundamental form $|A|$ of each catenoid $\mathcal{C}$ in the space form $\overline{M}^{n+1}(c)$ satisfies the Simons' equation* (3.9).

Proof. We shall prove the statement in the unified way by using the argument in [4, § 2 and §3]. Consider the space form $\overline{M}^{n+1}(c)$ as a subset of $\mathbb{R}^{n+2}$ as follows:

(i) If $c>0$, let

$$\mathbb{S}^{n+1}(c) = \{x\in\mathbb{R}^{n+2} \mid g_1(x,x) = 1/c\} = \overline{M}^{n+1}(c) ,$$

where $g_1(x,y) = x_1y_1 + \cdots + x_{n+1}y_{n+1} + x_{n+2}y_{n+2}$ for $x,y\in\mathbb{R}^{n+2}$.

(ii) If $c<0$, let

$$\mathbb{H}^{n+1}(c) = \{x\in\mathbb{R}^{n+2} \mid g_{-1}(x,x) = 1/c, x_{n+2}>0\} = \overline{M}^{n+1}(c) ,$$

where $g_{-1}(x,y) = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2}$ for $x,y\in\mathbb{R}^{n+2}$.

(iii) If $c=0$, let

$$\mathbb{R}^{n+1} = \{x\in\mathbb{R}^{n+2} \mid x_{n+2}=0\} = \overline{M}^{n+1}(0) .$$

Let $e_i = (0,\cdots,0,\underset{i\text{-th}}{1},0,\cdots,0)$ be the i-th vector in the space $\mathbb{R}^{n+2}$ for $i = 1,\dots,n+2$. Let P^2 be a subspace of $\mathbb{R}^{n+2}$ spanned by either e_{n+1} and e_{n+2} if $c\neq 0$ or e_{n+1} if $c=0$, and let $\mathsf{O}(P^2)$ be the set of metric-preserving transformations of $(\mathbb{R}^{n+2}, g_1)$ if $c>0$, $(\mathbb{R}^{n+2}, g_{-1})$ if $c<0$ or $\mathbb{R}^{n+1}$ if $c=0$, which leave P^2 pointwise fixed. Let P^3 be a subspace of $\mathbb{R}^{n+2}$ spanned by either e_1, e_{n+1} and e_{n+2} if $c\neq 0$ or e_1 and e_{n+1} if $c=0$.

Let $M^2(c) = \overline{M}^{n+1}(c)\cap P^3$, and let γ be a smooth curve in $M^2(c)$ that does not meet P^2. The orbit of γ under the action of $\mathsf{O}(P^2)$ is a rotation hypersurface generated by γ, and the curve γ is the generating curve of the rotation hypersurface.

Suppose that the generating curve γ is parametrized by either $x_1 = x_1(s)$, $x_{n+1} = x_{n+1}(s)$ and $x_{n+2} = x_{n+2}(s)$ if $c\neq 0$ or $x_1 = x_1(s)$ and $x_{n+1} = x_{n+1}(s)$ if $c=0$, where s is the arc length parameter of the curve γ. Let $\mathbf{I}$ be either a straight line if $c \le 0$ or a closed curve immersed in a plane if $c > 0$. Let $f : \mathbb{S}^{n-1} \times \mathbf{I} \to \overline{M}^{n+1}(c) \subset \mathbb{R}^{n+2}$ be the minimal spherical rotation hypersurface generated by γ, which is parametrized as follows:

$$f(t_1, \ldots, t_{n-1}, s) = (x_1(s)\varphi_1, \ldots, x_1(s)\varphi_n, x_{n+1}(s), x_{n+2}(s)) \tag{3.11}$$

if $c \neq 0$, or

$$f(t_1, \ldots, t_{n-1}, s) = (x_1(s)\varphi_1, \ldots, x_1(s)\varphi_n, x_{n+1}(s)) \tag{3.12}$$

if $c = 0$, where $\varphi(t_1, \ldots, t_{n-1}) = (\varphi_1, \ldots, \varphi_n)$ is the orthogonal parametrization of the $(n-1)$-unit sphere of the subspace of $\mathbb{R}^{n+2}$ spanned by $e_1, \ldots, e_n$.

Let $\mathcal{C} = f(\mathbb{S}^{n-1} \times \mathbf{I})$ be the minimal rotation hypersurface immersed in $\overline{M}^{n+1}(c) \subset \mathbb{R}^{n+2}$. According to the computation in [4, §3], we have the first fundamental form of $\mathcal{C}$:

$$g_{ij} = \begin{cases} \alpha_{ij} x_1^2(s), & 1 \le i, j \le n-1 \ , \\ 0, & i = n, j \neq n \text{ or } i \neq n, j = n \ , \\ 1, & i = j = n \ , \end{cases} \tag{3.13}$$

where $\alpha_{ij} = \sum\limits_{k=1}^{n} \dfrac{\partial \varphi_k}{\partial t_i} \dfrac{\partial \varphi_k}{\partial t_j}$ for $1 \le i, j \le n-1$. According to Proposition 3.2 in [4], the principal curvatures of $\mathcal{C}$ are

$$\lambda_1 = \cdots = \lambda_{n-1} = -\frac{\sqrt{1 - cx_1^2 - \dot{x}_1^2}}{x_1} \quad \text{and} \quad \mu = \frac{\ddot{x}_1 + cx_1}{\sqrt{1 - cx_1^2 - \dot{x}_1^2}} \ , \tag{3.14}$$

where $\dot{x}_1$ and $\ddot{x}_1$ are the first and second derivatives of x_1 on s respectively. Since $\mathcal{C}$ is a minimal rotation hypersurface, i.e., a catenoid, in $\overline{M}^{n+1}(c)$, then by [4, (3.13) and (3.16)] we have

$$\dot{x}_1^2 = 1 - cx_1^2 - a^2 x_1^{2-2n} \ , \tag{3.15}$$

and

$$\ddot{x}_1 = -cx_1 + a^2(n-1)x_1^{1-2n} \ , \tag{3.16}$$

where $a > 0$ is a constant. Therefore we have the following identities:

$$|A|^2 = (n-1) \cdot \frac{1 - cx_1^2 - \dot{x}_1^2}{x_1^2} + \frac{(\ddot{x}_1 + cx_1)^2}{1 - cx_1^2 - \dot{x}_1^2} = a^2 n(n-1) x_1^{-2n} \ , \tag{3.17}$$

where we use the equations (3.15) and (3.16) for the last equality.

If $\phi = \phi(s)$ is a function on $\mathcal{C} \subset \overline{M}^{n+1}(c)$ depending only on the variable s, then the Laplacian and the square norm of the covariant derivative of ϕ with respect to the metric (3.13) on $\mathcal{C}$ respectively are

$$\Delta\phi = \ddot{\phi} + (n-1)\,\frac{\dot{x}_1}{x_1}\,\dot{\phi} \tag{3.18}$$

and

$$|\nabla\phi|^2 = \dot{\phi}^2 \ . \tag{3.19}$$

Since $|A| = a\sqrt{n(n-1)}\,x_1^{-n}$ is a function on $\mathcal{C}$ that depends only on s, we have the following equalities:

$$\begin{aligned} |A|\Delta|A| + |A|^4 &= a^2 n^2 (n-1) x_1^{-4n} (-2a^2 + 2x_1^{2n-2} - cx_1^{2n}) \\ &= \frac{2}{n}\,|\nabla|A||^2 + nc|A|^2 \ , \end{aligned}$$

where we use the equations (3.15) and (3.16) again. □

4. Proof of Theorem 1.1

Theorem 1.1. *Let $\overline{M}^{n+1}(c)$ be the space form of dimension $n+1$, where $n \geq 3$. Suppose that Σ^n is a non-totally geodesic complete minimal hypersurface immersed in $\overline{M}^{n+1}(c)$. If the Simons' equation* (3.9) *holds as an equation at all non-vanishing points of $|A|$ in Σ^n, then $\Sigma^n \subset \overline{M}^{n+1}(c)$ is either*

(1) *a catenoid if $c \leq 0$, or*
(2) *a Clifford minimal hypersurface or a compact Ostuki minimal hypersurface if $c > 0$.*

Proof. By the assumption Σ^n is not totally geodesic, so $|A|$ is a non-negative continuous function which does not vanish identically. Let p be a point such that $|A|(p) > 0$, there exists an open neighborhood U of p such that $|A| > 0$ in U.

We claim that $|\nabla|A|| \not\equiv 0$ in U unless Σ^n is a Clifford minimal hypersurface in the case when $c > 0$. We need to deal with two cases:

Case 1 $(c \leq 0)$. Assume $|\nabla|A|| \equiv 0$ in U. Since Σ^n satisfies the Simons' equation (3.9), and $|A|$ is a positive constant in U, we have $0 < |A|^2 = nc \leq 0$ in the open set U, which is a contradiction.

Case 2 $(c > 0)$. In this case, if $|\nabla|A|| \equiv 0$ in U, then $|A|$ is a non-zero constant in U, so $|A|^2 = nc$ in U according to (3.9) and the assumption $|A| \neq 0$ in U; then Σ^n is a Clifford minimal hypersurface according to [3,9]. In this case, we have $|\nabla|A|| \not\equiv 0$ in U if Σ^n is not a Clifford minimal hypersurface.

Therefore in each case, there is a point in U such that $|\nabla|A|| \neq 0$ if Σ^n is not a Clifford minimal hypersurface in the case when $c > 0$.

By shrinking U, we may assume that $|A| > 0$ and $|\nabla|A|| > 0$ in U if Σ^n is not a Clifford minimal hypersurface in the case when $c > 0$. By (3.1) and the fact that Σ^n satisfies (3.9) in U, we conclude that $E \equiv 0$ in U. According to the argument in [17, pp. 3457–3458], the eigenvalues of $A = (h_{ij})_{n\times n}$ are λ with multiplicity $n-1$ and $\mu = -(n-1)\lambda$ with $\lambda > 0$ since $|A| > 0$. According to [11, Theorem 5] and [4, Corollary 4.4], U is part of a catenoid $\mathcal{C}$ in $\overline{M}^{n+1}(c)$.

According to the maximal principle of minimal submanifolds, Σ^n is part of a catenoid $\mathcal{C}$ in $\overline{M}^{n+1}(c)$. We claim that Σ^n must coincide with the catenoid $\mathcal{C}$. Let $f : \Sigma^n \to \mathcal{C}$ be the inclusion. There are two cases:

Case 1 $(c \leq 0)$. In this case, each catenoid is a simply connected (since $n \geq 3$) complete minimal rotation hypersurface embedded in $\overline{M}^{n+1}(c)$. Since f is a local isometry, the inclusion f is a covering map by Lemma 8.14 in [15, p. 224]. Therefore f must be an identity map, i.e., $\Sigma^n = \mathcal{C}$.

Case 2 $(c > 0)$. In this case, since Σ^n is a closed minimal hypersurface immersed in $\mathbb{S}^{n+1}(c)$, we then have $\Sigma^n = \mathcal{C}$ according to Theorem 4 and Theorem 5 in [11]. □

5. Appendix

In the appendix, with the help of Mathematica and Ti*k*Z/PGF, we shall draw some figures of the generating curves of 3-dimensional catenoids in the space form $\overline{M}^4(c)$.

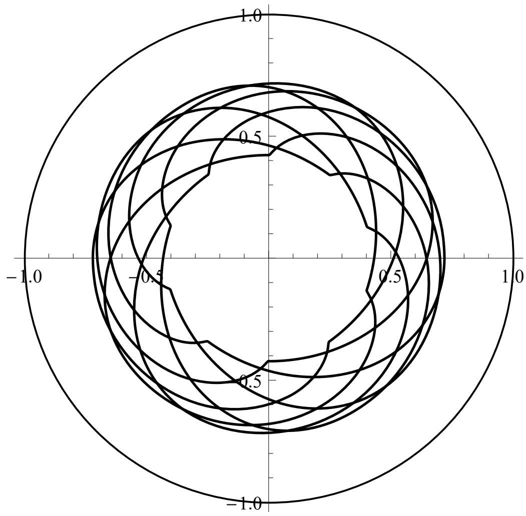

FIGURE 4. Generating curve of a compact Otsuki minimal surface in the unit sphere $\mathbb{S}^4$. In this figure, the support function h has initial conditions $h(0) = a_0 = 0.42231$ and $h'(0) = 0$, and the period of h is $T = 1.4\pi$.

5.1. **Catenoids in $\mathbb{S}^{n+1}$.** In this case we draw the generating curve for a compact Otsuki minimal hypersurface in $\mathbb{S}^4$. The function $C(a)$ is given by

$$C(a) = a^{2/3}(1-a^2)^{2/3} \ , \tag{5.1}$$

where $0 < a < 1/\sqrt{3} \approx 0.57735$. Then the following equation:

$$1 - x^2 - C(a)\left(\frac{1}{x^2} - 1\right)^{1/3} = 0 \tag{5.2}$$

has two solutions $a_0 = a$ and $a_1 = (-a + \sqrt{4 - 3a^2})/2 \in (1/\sqrt{3}, 1)$. Let $a_0 = 0.42231$; then $a_1 = 0.71957$, and the period of $h(\theta)$ is

$$T = 2\int_{a_0}^{a_1} \frac{dx}{\sqrt{1 - x^2 - C(a_0)\left(\dfrac{1}{x^2} - 1\right)^{1/3}}} = 4.39823 \ ,$$

i.e. $T = 1.4\pi$. Therefore we have a closed immersed generating curve as shown in Figure 4, the rotation hypersurface in $\mathbb{S}^4$ is a compact immersed minimal hypersurface.

5.2. **Catenoids in $\mathbb{B}^{n+1}$.** In this case, $c = -1$ and $f(y) = \sinh y$, so $f'(y) = \cosh y$. Equation (2.11) becomes

$$x(y) = \int_a^y \frac{1}{\cosh t} \cdot \frac{dt}{\sqrt{\left(\dfrac{\sinh t}{\sinh a}\right)^{2n-2} \cdot \left(\dfrac{\cosh t}{\cosh a}\right)^2 - 1}} \ , \tag{5.3}$$

where $a \le y < \infty$.

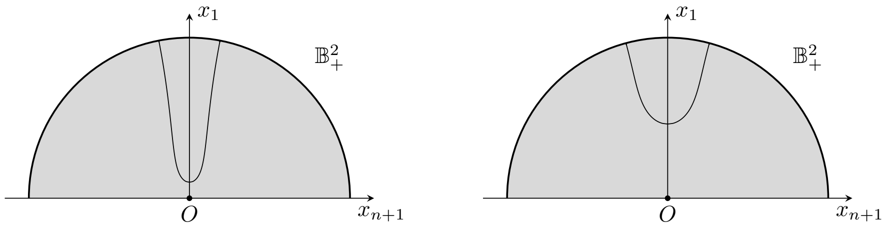

FIGURE 5. Two generating curves for the catenoids in the hyperbolic space $\mathbb{B}^4$. In these figures, $a = 0.2$ and $a = 1$ respectively. The rotation axis is the x_{n+1}-axis.

Now let $n = 3$, and let $a = 0.2$ and $a = 1$ respectively in (5.3); then we have two generating curves as shown in Figure 5.

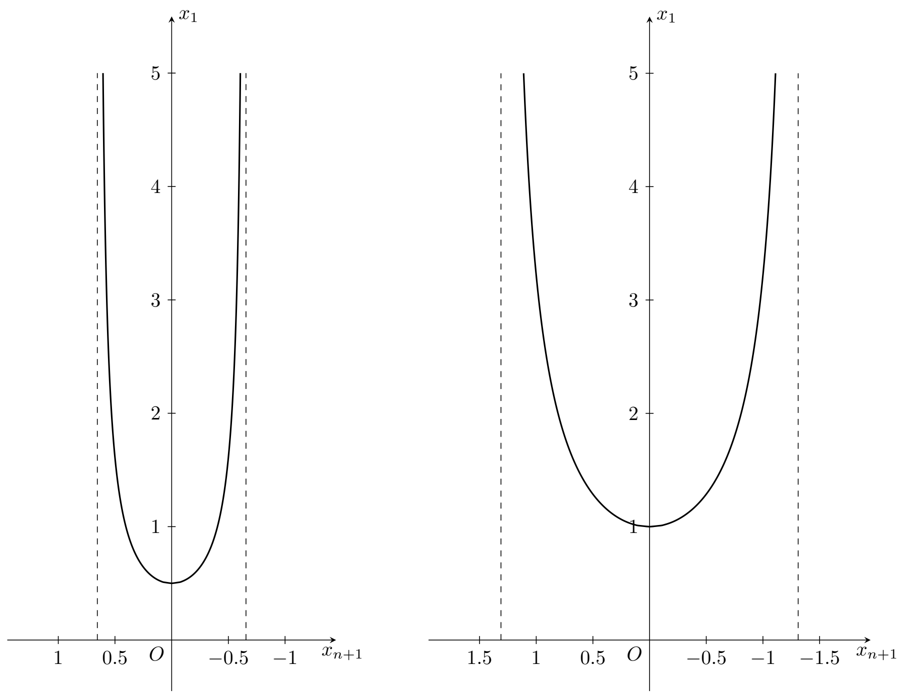

FIGURE 6. Two generating curves for the catenoids in the Euclidean space $\mathbb{R}^4$. In these figures, $a = 0.5$ and $a = 1$ respectively. The rotation axis is the x_{n+1}-axis.

5.3. **Catenoids in $\mathbb{R}^{n+1}$.** In this case, $c = 0$ and $f(y) = y$, so $f'(y) = 1$. Equation (2.11) becomes

$$x(y) = \int_a^y \frac{dt}{\sqrt{(t/a)^{2n-2} - 1}} \ , \tag{5.4}$$

where $a \leq y < \infty$. It's easy to see that the integral

$$x(a,\infty) = \int_a^\infty \frac{dt}{\sqrt{(t/a)^{2n-2}-1}} = a\int_1^\infty \frac{dt}{\sqrt{t^{2n-2}-1}} \tag{5.5}$$

is always finite if $n \geq 3$, where $a > 0$ is a constant. Actually if $n \geq 3$, then

$$\int_1^\infty \frac{dt}{\sqrt{t^{2n-2}-1}} \leq \int_1^\infty \frac{dt}{\sqrt{t^4-1}} < \int_1^\infty \frac{dt}{t\sqrt{t^2-1}} = \int_0^\infty \frac{dx}{\cosh x} = \frac{\pi}{2}\,,$$

where we use the substitution $t = \cosh x$.

Now let $n = 3$, and let $a = 0.5$ and $a = 1$ respectively in (5.4); then we have two generating curves as shown in Figure 6.

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