# EXAMPLES OF NON- $F S Z$ p-GROUPS FOR PRIMES GREATER THAN THREE 

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#### Abstract

For any prime $p>3$ and $j \in \mathbb{N}$ we construct examples of non$F S Z_{p^{j}}$ groups of order $p^{p^{j}+2 j-1}$. In the special case of $j=1$ this yields groups of order $p^{p+1}$, which is the minimum possible order for a non-FSZ $p$-group.


## 1. Introduction

The study of the representation categories of semisimple Hopf algebras, and many other more general contexts, have brought forth an interesting invariant of monoidal categories known as (higher) Frobenius-Schur indicators [1,3, 5, 9, 12, 17, 19]. These form generalizations of the classical Frobenius-Schur indicators for a finite group $G$, which for a character $\chi$ of $G$ over $\mathbb{C}$ and any $m \in \mathbb{N}$ are defined by

$$
\begin{equation*}
\nu_{m}(\chi)=\frac{1}{|G|} \sum_{g \in G} \chi\left(g^{m}\right) . \tag{1.1}
\end{equation*}
$$

When applied to the Hopf algebra $\mathcal{D}(G)$, the Drinfel'd double of the finite group $G$ over $\mathbb{C}$, these indicators can be expressed entirely in group theoretical terms. Schauenburg [18 has obtained an intriguing description of the FS-indicators of $\mathcal{D}(G)$ in terms of the character tables of centralizers, in particular. Thus, while the FS-indicators are motivated by Hopf algebraic concerns, they also yield a new and interesting invariant for finite groups. Frobenius-Schur indicators are guaranteed to be algebraic integers in a certain cyclotomic field, and the Galois action on the field also acts on the indicators [9, Proposition 3.3]. In full generality these indicators need not even be real numbers [9, Example 7.5] [7], but in the case of $\mathcal{D}(G)$ they are guaranteed to be so [8, Remark 2.8]. All of the first examples computed in the case of group doubles [4, 10, 11] yielded indicator values in $\mathbb{Z}$. Since the higher indicators for $G$ itself are classically known to be integers, this raised the question of whether or not the indicators for $\mathcal{D}(G)$ were always integers for arbitrary $G$.

Iovanov, Mason, and Montgomery [8] investigated this question, ultimately finding that there were exactly 32 non-isomorphic groups of order $5^{6}$ with non-integer indicators. They dubbed the property of having all integer FS-indicators the FSZ property. They also defined the $F S Z_{m}$ property, which holds whenever all $m$-th indicators are integers. For our purposes, Theorem 2.8 below will be taken as our definition of the $F S Z_{m}$ properties, and therefore the $F S Z$ property. Iovanov et al. 8] also established that several large families of groups were $F S Z$, including but not limited to the symmetric groups $S_{n} ; P S L_{2}(q)$ for a prime power $q$; and
all regular $p$-groups. On the other hand, the regular wreath product $\mathbb{Z}_{p} \imath \mathbb{Z}_{p}$ is an irregular $p$-group for all primes $p$, and this was shown to be $F S Z$ [8, Example 4.4], thereby establishing that the class of $F S Z p$-groups properly contains the class of regular $p$-groups. It is interesting to ask what can be said about the properties of irregular non- $F S Z$ p-groups, or alternatively of irregular $F S Z$ p-groups.

It is the goal of this note to exhibit an infinite family of non-FSZ $p$-groups for arbitrary primes $p>3$. The construction, in particular, establishes that there are always non- $F S Z$-groups of order $p^{p+1}$ when $p>3$, which is well known to be the minimum order possible for an irregular $p$-group.

We will take $\mathbb{N}=\{1,2, \ldots\}$ to be the set of positive integers.

## 2. The construction

Fix an odd prime $p$ and an integer $j \in \mathbb{N}$.
Consider the abelian $p$-group

$$
P_{p, j}=\mathbb{Z}_{p^{j+1}} \times \mathbb{Z}_{p}^{p^{j}-2}
$$

with generators $a_{1}, \ldots, a_{p^{j}-1}$ where $a_{1}$ has order $p^{j+1}$ and the rest have order $p$. We define an endomorphism $b_{p, j}$ of $P_{p, j}$ by

$$
\begin{aligned}
a_{1} & \mapsto a_{1} a_{2}^{-1} \\
a_{k} & \mapsto a_{k} a_{k+1}, 1<k<p^{j}-1 \\
a_{p^{j}-1} & \mapsto a_{p^{j}-1} a_{1}^{-p^{j}} .
\end{aligned}
$$

It is convenient to write $b_{p, j}$ as a matrix $B_{p, j}$ which acts on the left in the obvious fashion, and whose first row of entries can be taken modulo $p^{j+1}$ and the remaining entries may be taken modulo $p$. We have

$$
B_{p, j}=\left(\begin{array}{ccccccc}
1 & 0 & 0 & \cdots & 0 & 0 & -p^{j} \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & & & & & & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{array}\right)
$$

The entries of $B_{p, j}^{k}$ for $1 \leq k \leq p^{j}-2$ are then naturally described by the values

$$
T_{i, k}=\binom{i}{k}
$$

of Pascal's Triangle. For example

$$
B_{p, j}^{2}=\left(\begin{array}{ccccccc}
1 & 0 & 0 & \cdots & 0 & -p^{j} & -2 p^{j} \\
-2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & & & & & & \vdots \\
0 & 0 & 0 & \cdots & 2 & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 2 & 1
\end{array}\right)
$$

and $B_{p, j}^{p^{j}-2}$ is given by

$$
\left(\begin{array}{ccccc}
1 & -p^{j} & \cdots & -T_{p^{j}-2, p^{j}-4} p^{j} & -T_{p^{j}-2, p^{j}-3} p^{j} \\
-T_{p^{j}-2,1} & 1 & \cdots & 0 & 0 \\
-T_{p^{j}-2,2} & T_{p^{j}-2,1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-T_{p^{j}-2, p^{j}-3} & T_{p^{j}-2, p^{j}-3} & \cdots & 1 & 0 \\
-1 & T_{p^{j}-2, p^{j}-3} & \cdots & T_{p^{j}-2,1} & 1
\end{array}\right)
$$

Indeed, the entries of $B_{p, j}^{k}$ are determined by Pascal's Triangle for arbitrary $k$, just that for $k>p^{j}-2$ we can no longer fit entire rows of the triangle in the rows or columns. Nevertheless the pattern is straightforward. We remind the reader that $T_{p^{t}, k}=\binom{p^{t}}{k}$ is divisible by $p$ for all $t \in \mathbb{N}$ and $0<k<p^{t}$, and is equal to 1 for $k=0$ and $k=p^{t}$. This elementary property is essential to several of the calculations we will do, as it regularly ensures that many entries are zero, and will be used without further mention.

Lemma 2.1. The endomorphism $b_{p, j} \in \operatorname{End}\left(P_{p, j}\right)$ is an automorphism of order $p^{j}$.
Proof. The formula for $B_{p, j}^{p^{j}-2}$ above shows that the upper left entry for $B^{p^{j}-1}=$ $B \cdot B^{p^{j}-2}$ is $p^{j}+1$, which is not congruent to 1 modulo $p^{j+1}$. Since the powers $B^{k}$ for $1 \leq k<p^{j}-1$ have a -1 entry in the first column, it follows that $B^{k}$ is not the identity matrix for $1 \leq k \leq p^{j}-1$. To finish showing that $B$ has order $p^{j}$, we write

$$
B=I+S
$$

where $I$ is the identity matrix and $S$ is the matrix

$$
S=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & -p^{j}  \tag{2.1}\\
-1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{array}\right)
$$

By the binomial formula,

$$
B^{p^{j}}=I+S^{p^{j}}+\sum_{k=1}^{p^{j-1}}\binom{p^{j}}{k} S^{k}
$$

The binomial coefficients in the summation are all divisible by $p$, whence only the first row of the powers $S^{k}$ can possibly contribute a non-zero term to the summation. $S$ itself behaves very much like a circulant matrix, and in particular can have its arbitrary powers computed easily: for each successive power, shift the entries of the previous power to the left one column, set the last column to all zeros, and multiply all entries in the new first column by -1 . For example,

$$
S^{2}=\left(\begin{array}{cccccc}
0 & 0 & \cdots & 0 & -p^{j} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0
\end{array}\right)
$$

We can then see that, for $1 \leq k<p^{j}$, the first row of $S^{k}$ has a single non-zero entry which is equal to $-p^{j}$ for $k<p^{j}-1$, and is equal to $p^{j}$ for $k=p^{j}-1$. Since this is multiplied by the binomial coefficient and taken modulo $p^{j+1}$, we see that every term in the summation vanishes. Lastly we see that $S^{p^{j}}$ is the zero matrix. Thus, $B^{p^{j}}$ is the identity matrix, as desired.

We can now define the family of groups whose $F S Z$ properties we wish to study.
Definition 2.2. Let $p$ be an odd prime and $j \in \mathbb{N}$. Define

$$
S(p, j)=P_{p, j} \rtimes\left\langle b_{p, j}\right\rangle
$$

This is a group of order $p^{p^{j}+2 j-1}$.
By considering the eigenvectors of $B$ for the eigenvalue 1, we see that $S(p, j)$ has center $\left\langle a_{1}^{p}\right\rangle \cong \mathbb{Z}_{p^{j}}$. We identify $P_{p, j}$ and $\left\langle b_{p, j}\right\rangle$ as subgroups of $S(p, j)$ in the usual fashion, and for simplicity we denote $b_{p, j}$ by simply $b$ whenever convenient, and similarly for $B_{p, j}$.

The group $S(p, j)$ is defined in such a way as to make computing $p^{j}$-th powers relatively easy, after a bit of initial work. To help us investigate how to take $p^{j}$-th powers in $S(p, j)$, we introduce the following.
Definition 2.3. For $0 \leq k<j$ we define the matrices

$$
Y_{p, j}\left(p^{k}\right)=\sum_{m=0}^{p^{j-k}-1} B^{m p^{k}}
$$

These can be viewed as endomorphisms of $P_{p, j}$ by acting on the left, in the same fashion that $B$ acts.

Lemma 2.4. We have a block decomposition with a $1 \times 1$ entry in the upper left corner

$$
Y_{p, j}(1)=\left(\begin{array}{cc}
2 p^{j} & 0  \tag{2.2}\\
0 & 0
\end{array}\right)
$$

Proof. We have the identity $B Y_{p, j}(1)=Y_{p, j}(1)$. Thus the columns of $Y_{p, j}(1)$ are all eigenvectors of $B$ with eigenvalue 1 , and it is easily checked that all such eigenvectors have zeros in every entry except (possibly) the first one, which must be divisible by $p$. It follows that we have a block decomposition with a $1 \times 1$ entry in the upper left corner given by

$$
Y_{p, j}=\left(\begin{array}{cc}
c * p & v \\
0 & 0
\end{array}\right)
$$

for some integer $c$ and some (row) vector of integers $v$. Indeed, every entry of $v$ must be divisible by $p$. As noted in the proof of Lemma 2.1] it is easily seen that the $(1,1)$ entry of $B_{p, j}^{p^{j}-1}=B * B^{p^{j}-2}$ is exactly $p^{j}+1$, from which it follows that the $(1,1)$ entry of $Y_{p, j}(1)$ is $2 p^{j}$. It remains to show that $v$ is the zero vector (modulo $\left.p^{j+1}\right)$.

To this end we note that we also have $Y_{p, j}(1) B=Y_{p, j}(1)$, or equivalently that $Y_{p, j}(1)(B-I)=0$, where $I$ is the identity matrix. As in Lemma 2.1] we write $S=B-I$, which takes the form given in (2.1). Thus the row vectors of $Y_{p, j}$ are annihilated by $S$ when acted on from the right. Writing $v=\left(v_{1}, \ldots, v_{p^{j}-2}\right)$, we see that $\left(2 p^{j}, v\right) S=\left(-v_{1}, v_{2}, \ldots, v_{p^{j}-2}, 0\right)=0 \bmod p^{j+1}$. This shows that $v$ is the zero vector, as desired, and so completes the proof.

We also have the following relations for the remaining $Y_{p, j}\left(p^{k}\right)$.
Lemma 2.5. For all $0<k<j$ we have a block decomposition with a $1 \times 1$ entry in the upper left corner

$$
p^{k} Y_{p, j}\left(p^{k}\right)=\left(\begin{array}{rr}
p^{j} & 0  \tag{2.3}\\
0 & 0
\end{array}\right), 1 \leq k<j
$$

Proof. Since all rows but the first are taken modulo $p$, the scalar multiplication by $p^{k}$ automatically forces all entries of $Y_{p, j}\left(p^{k}\right)$ other than possibly those on the first row to be zero. In particular, we have established the bottom row of the desired decomposition. Indeed, the scalar multiplication by $p^{k}$ means we need only consider the elements in the first row to calculate $p^{k} Y_{p, j}\left(p^{k}\right)$.

We now consider the entries in the first row. We easily see that, by assumptions on $k$, the $(1,1)$ entry of every matrix in the sum defining $Y_{p, j}\left(p^{k}\right)$ is 1 , and there are $p^{j-k}$ matrices in the summation, whence the $(1,1)$ entry of $p^{k} Y_{p, j}\left(p^{k}\right)$ is $p^{j}$. Recall that the first row is always taken modulo $p^{j+1}$. In the first row of $B^{p^{k}}$ the only non-zero entries are the first and last ones, which are 1 and $-p^{j}$ respectively. Indeed, all entries on the diagonal of $B^{p^{k}}$ are 1 , and the only non-zero entries below the diagonal are $\pm 1$ modulo $p$. If follows that every entry in the first row of any power of $B^{p^{k}}$, other than the first entry, is divisible by $p^{j}$. Subsequently, every entry in the first row of $p^{k} Y_{p, j}\left(p^{k}\right)$, except the first one, is divisible by $p^{j+k} \equiv 0$ $\bmod p^{j+1}$ since $k \geq 1$. This completes the proof.

We can now state how these matrices are used to describe arbitrary $p^{j}$-th powers in $S(p, j)$. Namely, fixing $q \in P_{p, j}$ and $b^{k} \in\langle b\rangle$ with $\left|b^{k}\right|=p^{j-t}$, then

$$
\begin{equation*}
\left(q b^{k}\right)^{p^{j}}=p^{t} Y_{p, j}\left(p^{t}\right) q \tag{2.4}
\end{equation*}
$$

The fact that $P_{p, j}$ is abelian is essential to this formula. In particular, it is needed to be sure that the value depends only on the order of $b^{k}$. The reader may find it worthwhile to see how this holds in the particular case of $k \equiv-1 \bmod p^{j+1}$, as we have previously noted that this is the only power of $B$ for which the $(1,1)$ entry is not congruent to 1 . As a result of this identity, all $p^{j}$-th powers in $S(p, j)$ yield the subgroup $\left\langle a^{p^{j}}\right\rangle \subseteq Z(S(p, j))$.

Remark 2.6. The matrices $p^{k} Y_{p, j}\left(p^{k}\right)$, in particular $Y_{p, j}(1)$, are higher dimensional analogues of the integer parameter $d$ appearing in [10]. The parameter $d$ controlled the existence of negative indicators in the double of the groups under consideration in [10], in much the same way that $Y_{p, j}$ will dictate the existence of non-integer indicators here. More generally, such objects naturally arise when considering the $F S Z$ property for groups of the form $A \rtimes C$ where $A$ is abelian and $C$ is cyclic, such as in [8, Example 4.4].

Now that we understand how to take $p^{j}$-th powers in $S(p, j)$, we can begin investigating what Iovanov et al. [8] called the $F S Z_{p^{j}}$ property of $S(p, j)$. We recall the following.

Definition 2.7. For any group $G, n \in \mathbb{N}$, and $g, u \in G$, define

$$
G_{n}(u, g)=\left\{a \in G: a^{n}=\left(a u^{-1}\right)^{n}=g\right\}
$$

Of necessity, $G_{n}(u, g) \neq \emptyset$ implies $[u, g]=1$, and indeed for fixed $g$ they are subsets of $C_{G}(g)$. These sets characterize the $F S Z_{n}$ properties, as shown by the following.

Theorem 2.8 ([8, Corollary 3.2]). Let $n \in \mathbb{N}$ and $G$ be a finite group. Then $G$ is an $F S Z_{n}$-group if and only if for all commuting pairs of elements $u, g$ and all integers $m$ coprime to $|G|$ we have

$$
\left|G_{n}(u, g)\right|=\left|G_{n}\left(u, g^{m}\right)\right|
$$

We can now state and prove the main result of the paper.
Theorem 2.9. Let notation be as above and set $G=S(p, j)$ for any odd prime $p>3$ and $j \in \mathbb{N}$. Then $G_{p^{j}}\left(b a_{1}, a_{1}^{p^{j}}\right)=\emptyset$ and $G_{p^{j}}\left(b a_{1}, a_{1}^{2 p^{j}}\right) \neq \emptyset$.

In particular, $S(p, j)$ is non- $F S Z_{p^{j}}$.
Proof. We first note that the assumption $p>3$ is necessary, since when $p=3$ we have $a_{1}^{2 p^{j}}=a_{1}^{-p^{j}}$ and by [8, Lemma 2.7] we always have a bijection $G_{n}(u, g) \rightarrow$ $G_{n}\left(u, g^{-1}\right)$ for any $G, n \in \mathbb{N}$, and $u, g \in G$.

Fix $u=b a_{1}$ and set $Y=\left\langle a_{2}, \ldots, a_{p^{j}-1}\right\rangle$ for the remainder of the proof. Every element $a \in G$ can be uniquely written in the form $a=a_{1}^{j_{1}} y b^{-k} \in G$ for some $y \in Y$. By equations (2.2) to (2.4) the value of $a^{p^{j}}$ does not depend on $y$, so we may suppress elements of $Y$ for the rest of the proof. It follows that when determining the membership of $G_{p^{j}}\left(b a_{1}, g\right)$ we will naturally break things down into cases, depending on the orders of $b^{k}$ and $b^{k+1}$.

First consider the case that $\left|b^{k}\right|=1$. Then

$$
a^{p^{j}}=a_{1}^{j_{1} p^{j}}=g
$$

while

$$
\left(a u^{-1}\right)^{p^{j}}=Y_{p, j}(1)\left(a_{1}^{j_{1}-1}\right)=a_{1}^{2\left(j_{1}-1\right) p^{j}}=g .
$$

These equalities are consistent if and only if $j_{1} \equiv 2 \bmod p$. In particular, we have no contribution from elements of this form when $g=a_{1}^{p^{j}}$, but do have contributions from such elements when $g=a_{1}^{2 p^{j}}$.

Now suppose $\left|b^{k}\right|=\left|b^{k+1}\right|=p^{j}$. Then

$$
a^{p^{j}}=Y_{p, j}(1) a_{1}^{j_{1}}=a_{1}^{2 j_{1} p^{j}}
$$

and

$$
\left(a u^{-1}\right)^{p^{j}}=Y_{p, j}(1) a_{1}^{j_{1}-1}=a_{1}^{2\left(j_{1}-1\right) p^{j}} .
$$

We point out to the reader that this identity holds even when $k=1$, as in this case putting $a u^{-1}$ into the desired form requires we apply $B^{-1}$ to $a_{1}$, and we have previously noted that this is the unique power of $B$ whose $(1,1)$ is $p^{j}+1$ instead of 1. Regardless, these values can never be equal, so we have no contributions from elements of this form to the sets $G_{p^{j}}(u, g)$ for any choice of $g$.

Next suppose $k \equiv-1 \bmod p^{j}$. Then

$$
a^{p^{j}}=Y_{p, j}(1) a_{1}^{j_{1}}=a_{1}^{2 j_{1} p^{j}}
$$

while

$$
\left(a u^{-1}\right)^{p^{j}}=a_{1}^{\left(j_{1}-1\right) p^{j}}
$$

These are equal if and only if $j_{1} \equiv-1 \bmod p$. Since $p>3$, we conclude that for $g \in\left\{a_{1}^{p^{j}}, a_{1}^{2 p^{j}}\right\}$ there are no contributions from elements of this form.

For the next case, suppose $\left|b^{k}\right|=p^{j-t}$ for some $0<t<j$, which implies $\left|b^{k+1}\right|=p^{j}$. It follows that

$$
a^{p^{j}}=a_{1}^{j_{1} p^{j}}
$$

and

$$
\left(a u^{-1}\right)^{p^{j}}=a_{1}^{2\left(j_{1}-1\right) p^{j}} .
$$

These are equal if and only if $j_{1} \equiv 2 \bmod p$. Therefore for $g=a_{1}^{p^{j}}$ there are no contributions from elements of this form. But for $g=a_{1}^{2 p_{j}}$ contributions from elements of this form do exist.

Finally, suppose $\left|b^{k+1}\right|=p^{j-t}$ for some $0<t<j$, which implies $\left|b^{k}\right|=p^{j}$. Then we have

$$
a^{p^{j}}=a_{1}^{2 p^{j} j_{1}}
$$

while

$$
\left(a u^{-1}\right)^{p^{j}}=a_{1}^{p^{j}\left(j_{1}-1\right)} .
$$

These values are equal if and only if $j_{1} \equiv-1 \bmod p$, so again since $p>3$ we conclude that for $g \in\left\{a_{1}^{p^{j}}, a_{1}^{2 p^{j}}\right\}$ there are no contributions from elements of this form.

This completes the proof except for the final claim, which follows immediately from Theorem 2.8.

Example 2.10. For $p>3$ we have that $S(p, 1)$ is a group of order $p^{p+1}$ that is not $F S Z_{p}$, and this is the minimum possible order for any non- $F S Z p$-group. Indeed, $S(5,1)$ is $\operatorname{SmallGroup}\left(5^{6}, 632\right)$ in GAP [6], which is the smallest id number amongst the 32 non- $F S Z$ groups of order $5^{6}$ found by Iovanov et al. [8.

For $p>3$ and $j>1$ we do not know if $S(p, j)$ has minimal order amongst the non- $F S Z_{p^{j}} p$-groups.
Example 2.11. Iovanov et al. [8 used GAP 6] to verify that there are no non$F S Z 2$-groups of order at most $2^{9}$. The author has verified, with the help of the GAP functions in [8, 18], that there are no non-FSZ 3 -groups of order at most $3^{7}$. It remains an open question if non- $F S Z 2$-groups or 3 -groups exist, and if they do what their minimum orders are. The constructions here, and several attempts at modifications thereof, run into the usual issues for the primes 2,3 .

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