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A NOTE ON THE ERDŐS-HAJNAL PROPERTY FOR STABLE GRAPHS

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Abstract. In this note we give a proof of the Erdős–Hajnal conjecture for families of finite (hyper-)graphs without the m-order property. This theorem is in fact implicitly proved by M. Malliaris and S. Shelah (2014), however we use a new technique of independent interest combining local stability and pseudo-finite model theory.

1. Introduction

By a graph G we mean, as usual, a pair (V, E), where E is a symmetric subset of $V \times V$. If G is a graph, then a *clique* in G is a set of vertices all pairwise adjacent, and an *anti-clique* in G is a set of vertices such that any two different vertices from it are non-adjacent.

As usual, for a graph H we say that a graph G is H-free if G does not contain an induced subgraph isomorphic to H.

It is well known that every graph on n vertices contains either a clique or an anti-clique of size $\frac{1}{2}\log n$, and that this is optimal in general. However, the following famous conjecture of Erdős and Hajnal says that one can do much better in a family of graphs omitting a certain fixed graph H.

Conjecture 1.1 (Erdős-Hajnal conjecture [3])**.** *For every finite graph H there is a real number $\delta = \delta(H) > 0$ such that every finite H-free graph $G = (V, E)$ contains either a clique or an anti-clique of size at least $|V|^{\delta}$.*

It is known to hold for some choices of H, but is widely open in general (see [2, 4] for a survey). A variation of this conjecture starts with a finite set of finite graphs $\mathcal{H} = \{H_1, \dots, H_k\}$ and asks for the existence of a real constant $\delta = \delta(\mathcal{H}) > 0$ such that every finite graph G which is $\mathcal{H}$-free (that is, omits all of the $H_i \in \mathcal{H}$ *simultaneously*), contains either a clique or an anti-clique of size at least $|V|^{\delta}$. The aim of this note is to prove this conjecture for certain $\mathcal{H}$ connected to the model-theoretic notion of *stability*.

Definition 1.2. Given $m \in \mathbb{N}$, we say that a graph $G = (V, E)$ has the *m-order property* if there are some vertices $a_1, \dots, a_m, b_1, \dots, b_m$ from V such that $a_i E b_j$ holds if and only if $i < j$.

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Note that in this definition we make no requirement on the edges between a_i, a_j for $i \neq j$, and between b_i, b_j for $i \neq j$. The following theorem is implicitly proved in [6, Theorem 3.5].

Theorem 1.3. *For every $m \in \mathbb{N}$ there is a constant $\delta = \delta(m) > 0$ such that every finite graph $G = (V, E)$ without the m-order property contains either a clique or an anti-clique of size at least $|V|^{\delta}$.*

In this note we provide a new model-theoretic proof of the above theorem (and a version of it for hypergraphs) using a new technique combining classical local stability with Hrushovski's pseudo-finite dimension.

Theorem 1.3 implies an instance of Conjecture 1.1 for certain $\mathcal{H}$. We consider the following graphs, for each $m \in \mathbb{N}$:

(1) Let H_m be the *half-graph* on $2m$ vertices. Namely, the vertices of H_m are $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$, and the edges are $\{(a_i, b_j) : i < j\}$.
(2) Let H'_m be the complement graph of H_m. Namely, the vertices of H'_m are $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$, and the edges are $\{(a_i, b_j) : i \geq j\} \cup \{(a_i, a_j) : i \neq j\} \cup \{(b_i, b_j) : i \neq j\}$.
(3) Let H''_m have $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$ as its vertices, and $\{(a_i, b_j) : i < j\} \cup \{(a_i, a_j) : i \neq j\}$ as its edges.

Finally, let $\mathcal{H}_m = \{H_m, H'_m, H''_m\}$.

Corollary 1.4. *For every $m \in \mathbb{N}$, the Erdős–Hajnal conjecture holds for the family of all $\mathcal{H}_m$-free graphs.*

Proof. In view of Theorem 1.3, it is enough to show that for every $m \in \mathbb{N}$ there is some $m' \in \mathbb{N}$ such that if a finite graph G is $\mathcal{H}_m$-free, then it doesn't have the m'-order property.

Assume that G has the m'-order property. That is, there are some vertices $a_1, \ldots, a_{m'}, b_1, \ldots, b_{m'}$ in V such that $a_i E b_j$ holds if and only if $i < j$. If m' is large enough with respect to m, by the Ramsey theorem we can find some subsequences $A = \{a_{i_1}, \ldots, a_{i_{m+1}}\}$ and $B = \{b_{j_1}, \ldots, b_{j_{m+1}}\}$, $1 \leq i_1 < \ldots < i_{m+1} \leq m', 1 \leq j_1 < \ldots < j_{m+1} \leq m'$, such that each of A, B is either a clique or an anti-clique.

If both are anti-cliques, then the graph induced on $(A \cup B) \setminus \{a_{i_{m+1}}, b_{j_{m+1}}\}$ is isomorphic to H_m. If both are cliques, let $a'_k := b_{j_{k+1}}$ and $b'_l := a_{i_l}$ for $1 \leq k, l \leq m$. Then the graph induced on $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m\}$ is isomorphic to H'_m. If A is a clique and B is an anti-clique, then the graph induced on $(A \cup B) \setminus \{a_{i_{m+1}}, b_{j_{m+1}}\}$ is isomorphic to H''_m. Finally, if A is an anti-clique and B is a clique, let $a'_k := b_{j_{m+1-k}}$ and $b'_l := a_{i_{m+1-l}}$ for $1 \leq k, l \leq m$. Then the graph induced on $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m\}$ is again isomorphic to H''_m. In any of the cases, G is not $\mathcal{H}_m$-free. □

Remark 1.5. We remark that the (strong) Erdős-Hajnal property for semialgebraic graphs (and more generally, for graphs definable in arbitrary distal structures) can also be established using model-theoretic methods [1], and that the strong Erdős-Hajnal property need not hold under the assumptions of Theorem 1.3 (see [1, Section 6]).

2. Preliminaries

In this paper by a *pseudo-finite set* V we mean an **infinite** set that is an ultraproduct $V = \prod_{i\in I} V_i/\mathcal{F}$ of **finite** sets $V_i, i \in I$, with respect to a non-principal ultrafilter $\mathcal{F}$ on I.

Working in "set theory", for a pseudo-finite set $V = \prod_{i\in I} V_i/\mathcal{F}$ and a subset $A \subseteq V^k$ we say that A is *definable* (or "internal", in the terminology of non-standard analysis) if $A = \prod_{i\in I} A_i/\mathcal{F}$ for some $A_i \subseteq V_i^k$.

Let $V = \prod_{i\in I} V_i/\mathcal{F}$ be pseudo-finite and $A \subseteq V$ a definable non-empty subset. We define the "dimension" $\delta(A)$ ($\delta_{C_0}(A)$ in the notation of [5]) to be the number in $[0, 1]$ that is the standard part of $\log(|A|)/\log(|V|)$. As an alternative definition, write A as $A = \prod_{i\in I} A_i/\mathcal{F}$, where each A_i is a non-empty subset of V_i. For each $i \in I$ let $l_i = \log(|A_i|)/\log(|V_i|)$ (so $|A_i| = |V_i|^{l_i}$). Then $\delta(A)$ is the unique number $l \in [0, 1]$ such that for any $\varepsilon > 0$ in $\mathbb{R}$, the set $\{i \in I : l - \varepsilon < l_i < l + \varepsilon\}$ is in $\mathcal{F}$. We extend δ to the empty set by setting $\delta(\emptyset) := -\infty$.

In the following lemma we state some basic properties of δ that we need. Their proofs are not difficult and we refer to [5] for more details.

Lemma 2.1. *Let V be a pseudo-finite set.*

(1) $\delta(V) = 1$.
(2) $\delta(A_1 \cup A_2) = \max\{\delta(A_1), \delta(A_2)\}$ *for any definable* $A_1, A_2 \subseteq V$.
(3) Let $Y \subseteq V \times V^m$ and $Z \subseteq V$ be definable. Assume that $\delta(Z) = \alpha$ and for all pairwise distinct $a_1, \dots, a_m \in Z$ we have $\delta(\{x \in V : (x, a_1, \dots, a_m) \in Y\}) \leq \beta$. Then

$$\delta(\{x \in V : \exists z_1, \dots, z_m \in Z \bigwedge_{i\neq j} z_i \neq z_j \,\&\, (x, z_1, \dots z_m) \in Y\}) \leq m\alpha + \beta.$$

In the next section we will prove the following "non-standard" version of the main theorem (and in fact a more general version of it for hypergraphs).

Theorem 2.2. *Let V be a pseudo-finite set and $E \subseteq V \times V$ a definable symmetric subset. Assume that the graph (V, E) does not have the m-order property for some $m \in \mathbb{N}$. Then there is definable $A \subseteq V$ such that $\delta(A) > 0$ and either $(a, a') \in E$ for all $a \neq a' \in A$ or $(a, a') \notin E$ for all $a \neq a' \in A$.*

We explain how Theorem 1.3 follows from Theorem 2.2. Assume that Theorem 1.3 fails. This means that for a fixed m, for every $r \in \mathbb{N}$ there is some finite graph $G_r = (V_r, E_r)$ of size at least r which does not have the m-order property and does not have a homogeneous subset of size at least $|V_r|^{\frac{1}{r}}$. Let $G = (V, E)$ be an ultraproduct of the G_r's modulo some non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. It follows by Łos's theorem that G also does not have the m-order property. Thus, we can apply Theorem 2.2 and obtain a definable homogeneous set $A \subseteq V$, let's say a clique, with $\delta(A) > \alpha > 0$. By definability $A = \prod_{r\in\mathbb{N}} A_r/\mathcal{F}$ for some $A_r \subseteq V_r$, and by the definition of the δ-dimension we have that $|A_r| \geq |V_r|^{\alpha}$ for almost all r, contradicting the assumption.

3. Proof of Theorem 2.2

We fix a pseudo-finite set $V = \prod_{i\in I} V_i/\mathcal{F}$ and a definable symmetric subset $E = \prod_{i\in I} E_i/\mathcal{F}$ of V^n (where "symmetric" means that it is closed under permutation of the coordinates).

We follow standard model-theoretic notation. For $v_1, \dots, v_{n-1} \in V$ and a subset $X \subseteq V$ we let $E(v_1, \dots, v_{n-1}, X) := \{x \in X : V \models E(v_1, \dots, v_{n-1}, x)\}$. By a partitioned formula we mean a first-order formula $\phi(x_1, \dots, x_k; y_1, \dots, y_l)$ with two distinguished groups of variables $\bar{x}$ and $\bar{y}$, and it is *stable* if the bi-partite graph (R, V^k, V^l) with $R := \{(\bar{a}, \bar{b}) \in V^k \times V^l : V \models \phi(\bar{a}; \bar{b})\}$ does not have the m-order property for some m. We say that a definable set $X \subseteq V$ is *large* if $\delta(X) > 0$, and we say that X is *small* if $\delta(X) \leq 0$.

We prove the following proposition, in particular establishing Theorem 2.2.

Proposition 3.1. *Assume that $E(x_1; x_2, \dots, x_n)$ is stable. Then there is a large definable set $A \subseteq V$ such that either $(a_1, \dots, a_n) \in E$ for all pairwise distinct $a_1, \dots, a_n \in A$ or $(a_1, \dots, a_n) \notin E$ for all pairwise distinct $a_1, \dots, a_n \in A$.*

We will use some basic local stability such as definability of types and Shelah's 2-rank $R_\Delta(-) := R(-, \Delta, 2)$ (and refer to [7, Chapter II] for the details).

We will use Δ to denote a finite set of (non-partitioned) formulas. By a Δ-formula $\psi(\bar{x})$ over a set of parameters $W \subseteq V$ we mean a Boolean combination of formulas of the form $\phi(\bar{x}, \bar{a})$ where $\phi(\bar{x}, \bar{y})$ is a formula from Δ and $\bar{a}$ is a tuple of elements from W. We let $\Delta(W)$ denote the set of all Δ-formulas over W. By a complete Δ-type $p(\bar{x})$ over W we mean a maximal consistent collection of Δ-formulas of the form $\psi(\bar{x})$ over W (p is axiomatized by specifying, for every $\phi(\bar{x}, \bar{y}) \in \Delta$ and $\bar{a} \in W^{|\bar{y}|}$, whether $\phi(\bar{x}, \bar{a}) \in p$ or $\neg\phi(\bar{x}, \bar{a}) \in p$).

For any permutation $\sigma \in \mathrm{Sym}(n)$, let $\phi_\sigma(x_1, \dots, x_n) = E(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. From now on we fix $\Delta = \{\phi_\sigma(x_1, \dots, x_n) : \sigma \in \mathrm{Sym}(n)\} \cup \{x_1 = x_2\}$.

Our assumption is that the *partitioned* formula $\phi(x; \bar{y}) = E(x, y_1, \dots, y_{n-1})$ is stable. By the basic properties of stable formulas we then have the following:

(1) Every partitioned $\Delta(V)$-formula $\phi(x; \bar{y})$ is stable, where x is a single variable. This follows from the assumption since E is symmetric and the set of stable formulas is closed under Boolean combinations and under replacing some of the variables by a fixed parameter.
(2) Every complete Δ-type $p(x)$ over V, with x a single variable, is definable using Δ-formulas over V. Indeed, for a partitioned $\Delta(V)$-formula $\phi(x; \bar{y})$, which is stable by (1), the type $p \restriction \phi$ is definable by a Boolean combination of instances of the formula $\phi^*(\bar{y}; x) = \phi(x; \bar{y})$, with parameters in V, which is also a $\Delta(V)$-formula.
(3) For any complete Δ-type $p(x)$ over V and $k \in \mathbb{N}$ we have a complete Δ-type $p^{(k)}(x_1, \dots, x_k)$ over V — the type of a *Morley sequence* in p. Namely, as p is definable by (2), say using $\Delta(V_0)$-formulas for some countable $V_0 \subseteq V$, we take $p^{(k)} = \bigcup\{\mathrm{tp}_\Delta(a_k, \dots, a_1/V') : V_0 \subset V' \subset V \text{ countable}, a_{i+1} \models p \restriction_{V' a_0 \dots a_i} \text{ for } i < k\}$. By a standard argument $p^{(k)}$ is well defined.

Consider $\Delta' = \{\phi(x; \bar{y}) : \phi(x, \bar{y}) \in \Delta, |x| = 1\}$, a finite set of partitioned formulas. Slightly abusing the notation, we will write $R_\Delta(-)$ to refer to $R_{\Delta'}(-)$. As every partitioned formula in Δ' is stable by (1), $R_\Delta(x = x)$ is finite. Let $S \subseteq V$ be a large definable subset of the smallest R_Δ-rank. By Lemma 2.1(2), S cannot be covered by finitely many definable sets of smaller R_Δ-rank, hence by compactness there is a complete Δ-type $p(x)$ over V such that $R_\Delta(S(x) \cap p(x)) = R_\Delta(S)$ (and in fact p is the unique type with this property).

Claim 3.2. For any formula $r(x_1, \dots, x_k) \in \Delta(V)$, if $p^{(k)} \vdash r(x_1, \dots, x_k)$, then there is a large definable $A \subseteq S$ such that $\models r(a_1, \dots, a_k)$ holds for any pairwise distinct $a_1, \dots, a_k$ from A.

Proof. We prove the claim by induction on k.

Case $k = 1$. If $p(x_1) \vdash r(x_1)$ and $r(x_1) \in \Delta(V)$, then by the choice of p we have $R_\Delta(r(x_1) \cap S(x_1)) = R_\Delta(S(x_1))$. Thus $R_\Delta(\neg r(x_1) \cap S(x_1)) < R_\Delta(S(x_1))$ by the definition of rank, so $\delta(\neg r(x_1) \cap S(x_1)) = 0$ by the choice of S, so $\delta(r(x_1) \cap S(x_1)) > 0$. Thus we can take $A = r(S)$.

Assume $k > 1$.

By the definition of $p^{(k)}$ in (3) above, there is some $\psi(x_1, \dots, x_{k-1}) \in \Delta(V)$ such that $p \restriction_{r(x_1,\dots,x_{k-1};x_k)}$ is defined by $\psi(x_1, \dots, x_{k-1})$, i.e.,

$$r(v_1, \dots, v_{k-1}; x_k) \in p(x_k) \iff V \models \psi(v_1, \dots, v_{k-1})$$

for any $v_1, \dots, v_{k-1} \in V$.

Also $p^{(k-1)} \vdash \psi(x_1, \dots, x_{k-1})$ as $p^{(k)} \vdash r(x_1, \dots, x_k)$. By the inductive assumption, there is some large definable $B \subseteq S$ such that $V \models \psi(b_1, \dots, b_{k-1})$ holds for all pairwise distinct $b_1, \dots, b_{k-1} \in B$. As B is definable, there are some $B_i \subseteq S_i$ such that $B = \prod_{i \in I} B_i/\mathcal{F}$. For each i, let $A_i \subseteq B_i$ be maximal (under inclusion) such that $r_i(a_1, \dots, a_k)$ holds for all pairwise distinct $a_1, \dots, a_k \in A_i$, and let $A := \prod_{i \in I} A_i/\mathcal{F}$. We have:

(i) $A \subseteq B$.
(ii) $V \models r(a_1, \dots, a_k)$ for any pairwise distinct $a_1, \dots, a_k \in A$.
(iii) For any $b \in B \setminus A$ there are some pairwise distinct $a_1, \dots, a_{k-1}$ in A such that $V \not\models r(a_1, \dots, a_k, b)$.

We claim that A is large, so satisfies the conclusion of the claim. In fact, we show that $\delta(A) \geq \frac{1}{k-1}\delta(B)$. Assume not, say $\delta(A) = \alpha_1 < \frac{1}{k-1}\delta(B)$. For all pairwise distinct $a_1, \dots, a_{k-1} \in A$ we have $V \models \psi(a_1, \dots, a_{k-1})$, so $r(a_1, \dots, a_{k-1}, x_k) \in p$. By the choice of p, the R_Δ-rank of $r(a_1, \dots, a_{k-1}, S)$ is equal to the R_Δ-rank of S, so the R_Δ-rank of $\neg r(a_1, \dots, a_{k-1}, S)$ has to be smaller than the R_Δ-rank of S, which implies that $\delta(B \setminus r(a_1, \dots, a_{k-1}, B)) = 0$ by the choice of S. By the property (iii) above, the set $B \setminus A$ is covered by the family $\{B \setminus r(a_1, \dots, a_{k-1}, B) : a_1, \dots, a_{k-1} \in A, \bigwedge_{i \neq j} a_i \neq a_j\}$.

Then by Lemma 2.1(3),

$$\delta(B \setminus A) \leq (k-1)\delta(A) + 0 \leq (k-1)\alpha_1$$

which implies by Lemma 2.1(2) that $\delta(B) \leq (k-1)\alpha_1 < \alpha$ — a contradiction. □

Finally, as both $E(x_1, \dots, x_n)$ and $\neg E(x_1, \dots, x_n)$ are in Δ and either $p^{(n)} \vdash E(x_1, \dots, x_n)$ or $p^{(n)} \vdash \neg E(x_1, \dots, x_n)$, the proposition follows.

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