PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 2, February 2018, Pages 589–600
http://dx.doi.org/10.1090/proc/13709
Article electronically published on October 23, 2017

CHARACTERIZATION OF POLYNOMIALS WHOSE LARGE POWERS HAVE ALL POSITIVE COEFFICIENTS

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(Communicated by Franc Forstnerič)

Abstract. We give a criterion which characterizes a homogeneous real multivariate polynomial to have the property that all sufficiently large powers of the polynomial (as well as their products with any given positive homogeneous polynomial) have all positive coefficients. Our result generalizes a result of De Angelis, which corresponds to the case of homogeneous bivariate polynomials, as well as a classical result of Pólya, which corresponds to the case of a specific linear polynomial. As an application, we also give a characterization of certain polynomial spectral radius functions of the defining matrix functions of Markov chains.

1. Introduction and main results

Positivity conditions for polynomials with real coefficients are relevant in several branches of pure and applied mathematics, including real algebraic geometry, convex geometry, probability theory and optimization, and have been extensively studied (see e.g. [3, 7, 9, 15, 16, 18, 20, 25] and the references therein). An important class of polynomials are those whose coefficients are positive.

De Angelis [9] characterized those univariate polynomials p such that p^m has all positive coefficients for all sufficiently large m in terms of certain positivity conditions on p itself. As an application, he obtained a characterization of certain univariate polynomials p for which there exists an irreducible (or aperiodic) Markov chain whose defining matrix has p as its spectral radius function [8, Theorem 6.7]. This spectral radius function is an important invariant in the study of Markov shifts (see e.g. [16]). As such, it is interesting and natural to ask whether similar results hold in the multivariate setting.

In this paper, we generalize both the aforementioned results of De Angelis to the case of homogeneous multivariate polynomials. Let $n \geq 1$. A homogeneous polynomial $f = \sum_{|I|=d} c_I x^I \in \mathbb{R}[x_1, \ldots, x_n]$ with real coefficients and of degree d is said to *have all positive coefficients* if $c_I > 0$ for all $|I| = d$. Here $I = (I_1, \ldots, I_n)$ is a multi-index of length $|I| := I_1 + \cdots + I_n$ and $x^I = x_1^{I_1} x_2^{I_2} \cdots x_n^{I_n}$. Next we let $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}$ denote the closed positive orthant in the real Euclidean space $\mathbb{R}^n$ (for simplicity, we also write $\mathbb{R}_+ := \mathbb{R}^1_+$). The circle group $U(1) := \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ acts via pointwise multiplication on the

Received by the editors January 8, 2017 and, in revised form, February 20, 2017.
2010 *Mathematics Subject Classification.* Primary 26C05, 12D99, 32T15.
Key words and phrases. Polynomials, positive coefficients, strongly pseudoconvex.
The second author was partially supported by the research grant R-146-000-142-112 from the National University of Singapore and the Ministry of Education.

complex Euclidean space $\mathbb{C}^n$, given by $e^{i\theta} \cdot z := (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$ for $e^{i\theta} \in U(1)$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The $U(1)$-invariant subset of $\mathbb{C}^n$ generated by $\mathbb{R}^n_+$ is given by $U(1) \cdot \mathbb{R}^n_+ := \{e^{i\theta} \cdot x \,|\, e^{i\theta} \in U(1),\ x \in \mathbb{R}^n_+\}$. For $k = 1, \ldots, n$, we denote the k-th facet of $\mathbb{R}^n_+$ by $F_k(\mathbb{R}^n_+) := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \,|\, x_k = 0\}$.

Our main result in this paper is the following:

Theorem 1.1. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonconstant homogeneous polynomial. The following two statements are equivalent:*

(a) p satisfies the following three conditions:

(Pos1): $p(1, 0, \ldots, 0), p(0, 1, 0, \ldots, 0), \ldots, p(0, \ldots, 0, 1) > 0$.

(Pos2): *For all $k = 1, \ldots, n$, $\dfrac{\partial p}{\partial x_k}(x) > 0$ for all $x \in F_k(\mathbb{R}^n_+) \setminus \{0\}$.*

(Pos3): $|p(z)| < p(|z_1|, \ldots, |z_n|)$ *for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}^n_+)$.*

(b) For each homogeneous polynomial $q \in \mathbb{R}[x_1, \ldots, x_n]$ such that $q(x) > 0$ whenever $x \in \mathbb{R}^n_+ \setminus \{0\}$, there exists $m_o > 0$ such that for each integer $m \geq m_o$, $p^m \cdot q$ has all positive coefficients.

The implication (a) $\Longrightarrow$ (b) may be regarded as a Positivstellensatz for those homogeneous polynomials q which are strictly positive on $\mathbb{R}^n_+ \cap \{p = 1\}$, as the latter condition is certified by the algebraic property that $p^m \cdot q$ has all positive coefficients for some $m \geq 1$.

The bulk of our proof of the implication (a) $\Longrightarrow$ (b) consists of showing that a certain Hermitian bihomogeneous polynomial P associated to p satisfies the sufficient conditions of a Hermitian Positivstellensatz of Catlin-D'Angelo [4] (see also Theorem 2.1 below), which enables us to apply the latter result. As mentioned above, Theorem 1.1 for the case of $n = 2$ dehomogenizes to De Angelis' Positivstellensatz [9, Theorem 6.6]. Thus our proof settles affirmatively a debate in MathOverflow [11] on whether De Angelis' Positivstellensatz is a consequence of Catlin-D'Angelo's Positivstellensatz.

Several examples of homogeneous real polynomials p which satisfy the three positivity conditions in (a) can be found in the literature. A classical example is the linear form $p = x_1 + \cdots + x_n$; in this case, Theorem 1.1 is the Positivstellensatz of Pólya on the simplex ([18]). A more general example is given by any homogeneous polynomial p which has all positive coefficients. A different kind of example is the polynomial

$$p_\lambda(x_1, x_2) := (x_1 + x_2)^{2k} - \lambda x_1^k x_2^k \quad \text{with } \binom{2k}{k} < \lambda < 2^{2k-1} \text{ and } k \geq 2 \tag{1.1}$$

given by D'Angelo-Varolin in [6, Theorem 3], for which the coefficient of $x_1^k x_2^k$ in p_λ is negative (we will skip the verification that p_λ satisfies the three positivity conditions in (a), which is similar to the calculations given in [6]).

The three positivity conditions in (a) are independent, in the sense that, any two of these conditions do not imply the third one. Consider the following polynomials (with $n \geq 2$):

$$(x_1 + \cdots + x_n)^3 - x_1^3, \tag{1.2}$$

$$x_1^2(x_1 + \cdots + x_n) + (x_2 + \cdots + x_n)^3, \tag{1.3}$$

$$(x_1 + \cdots + x_n)^4 - 8x_1^2 x_2^2. \tag{1.4}$$

As the reader can verify easily, (1.2) satisfies (Pos2) and (Pos3) but violates (Pos1) at the point $(1, 0, \dots, 0)$, (1.3) satisfies (Pos1) and (Pos3) but violates (Pos2) on the facet $F_1(\mathbb{R}^n_+)$, and (1.4) satisfies (Pos1) and (Pos2) but violates (Pos3) at the point $(-1, 1, 0, 0, \dots, 0)$.

As pointed out by the referee, in the special case when $p = x_1 + \cdots + x_n$ and q is as in Theorem 1.1(b), the result of Halfpap-Lebl in [14] yields a lower bound for m_o in terms of the signature of q.

In view of Theorem 1.1, it is natural and interesting to ask for a similar characterization of homogeneous polynomials whose large powers have all nonnegative coefficients. It appears to the authors that the method in this paper does not generalize readily to handle such a borderline case, and new ideas are needed to tackle the problem. To illustrate the subtlety of this problem, we mention that the polynomial in (1.4) (which, in the case when $n = 2$, corresponds to a limiting case of the family of polynomials in (1.1) with $k = 2$ and $\lambda = 8$) satisfies a weaker version of (Pos3) (with '$<$' there replaced by '$\leq$'), but it is easy to check that none of its powers has all nonnegative coefficients.

Let $\mathbb{Z}_+ := \{k \in \mathbb{Z} \,|\, k \geq 0\}$, and denote by $\mathbb{Z}_+[x_1, \dots, x_n]$ the semiring of polynomials in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}_+$. Let A be an *irreducible* (resp. *aperiodic*) square matrix over $\mathbb{Z}_+[x_1, \dots, x_n]$ (see e.g. Section 5 for the definitions). Denote the spectral radius function of A by $\beta_A = \beta_A(x_1, \dots, x_n)$.

As an application of Theorem 1.1, we have

Corollary 1.2. *Let $p \in \mathbb{Z}[x_1, \dots, x_n]$ be a homogeneous polynomial which satisfies* (Pos1) *and* (Pos2). *The following statements are equivalent:*

(i) *p satisfies* (Pos3).
(ii) *$p = \beta_A$ for some irreducible square matrix A over $\mathbb{Z}_+[x_1, \dots, x_n]$.*
(iii) *$p = \beta_A$ for some aperiodic square matrix A over $\mathbb{Z}_+[x_1, \dots, x_n]$.*

As mentioned earlier, the case of $n = 2$ in Corollary 1.2 dehomogenizes to De Angelis' result [8, Theorem 6.7]. We refer the reader to Section 5 for the interpretation of Corollary 1.2 in terms of Markov chains.

De Angelis' Positivstellensatz [9, Theorem 6.6] has been applied by Bergweiler-Eremenko [1] to study the distribution of zeros of polynomials with positive coefficients (see also [12]). An effective version of Pólya's Positivstellensatz by Powers-Reznick [19] was applied by Schweighofer [22] to obtain complexity bounds on a Positivstellensatz of Schmüdgen [21], and by de Klerk-Pasechnik [10] to estimate the rate of convergence of a certain hierarchy of conic linear programs to the stability number of a graph. As a generalization of the Positivstellensatze of De Angelis and Pólya, Theorem 1.1 may also have similar applications, which will not be pursued here.

The organization of this paper is as follows. In Section 2, we recall some background material on bihomogeneous polynomials. In Section 3, we relate some positivity properties of a homogeneous real polynomial with those of its associated bihomogeneous polynomial. In Section 4, we give the proof of Theorem 1.1. In Section 5, we give the deduction of Corollary 1.2.

2. Bihomogeneous polynomials and Catlin-D'Angelo's Positivstellensatz

In this section, we recall some background material regarding bihomogeneous polynomials, which is mostly taken from [3–5, 26]. Throughout this section, we fix a positive integer $n \geq 2$. Denote by $\mathbb{C}[z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}]$ the complex polynomial algebra in the indeterminates $z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}$. For $d \geq 0$, a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}]$ is said to be *bihomogeneous of bidegree* (d, d) if

$$P(\zeta z, \overline{\mu w}) = \zeta^d \overline{\mu}^d P(z, \overline{w}) \tag{2.1}$$

for all $\zeta, \mu \in \mathbb{C}$ and $z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Such P is said to be *Hermitian* if $\overline{P(z, \overline{w})} = P(w, \overline{z})$ for all $z, w \in \mathbb{C}^n$. Furthermore, P is said to be *positive on* $\mathbb{C}^n \setminus \{0\}$ if $P(z, \overline{z}) > 0$ for all $z \in \mathbb{C}^n \setminus \{0\}$.

For $d \geq 0$, we denote by $\mathbb{C}[z_1, \ldots, z_n]_d$ the complex vector space of homogeneous holomorphic polynomials in $\mathbb{C}^n$ of degree d. A Hermitian bihomogeneous polynomial P is said to be a *maximal squared norm* if there exists a basis $\{g_1, \ldots, g_N\}$ of $\mathbb{C}[z_1, \ldots, z_n]_d$ (with $N = \dim_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n]_d = \binom{d+n-1}{n-1}$) such that

$$P(z, \overline{w}) = \sum_{k=1}^{N} g_k(z) \cdot \overline{g_k(w)} \quad \text{for all } z, w \in \mathbb{C}^n \tag{2.2}$$

(so that $P(z, \overline{z}) = \sum_{k=1}^{N} |g_k(z)|^2$ for all $z \in \mathbb{C}^n$).

From (2.1), one easily sees that a Hermitian bihomogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}]$ of bidegree (d, d) may be regarded as a Hermitian form on the dual vector space of $\mathbb{C}[z_1, \ldots, z_n]_d$. In particular, with respect to any basis $\{h_1, \ldots, h_N\}$ of $\mathbb{C}[z_1, \ldots, z_n]_d$, there exists a unique $N \times N$ Hermitian matrix $C = (c_{k\bar{l}})_{1 \leq k, l \leq N}$ such that

$$P(z, \overline{w}) = \sum_{1 \leq k, l \leq N} c_{k\bar{l}} h_k(z) \overline{h_l(w)} \tag{2.3}$$

for all $z, w \in \mathbb{C}^n$. It is easy to see that P is a maximal squared norm if and only if its associated matrix $C = (c_{k\bar{l}})$ with respect to some (and hence any) basis of $\mathbb{C}[z_1, \ldots, z_n]_d$ is positive definite. Note that a Hermitian bihomogeneous polyomial positive on $\mathbb{C}^n \setminus \{0\}$ need not be a maximal squared norm.

Following [4], a Hermitian bihomogeneous polynomial P is said to satisfy *the strong global Cauchy-Schwarz (in short, SGCS) inequality* if

$$|P(z, \overline{w})|^2 < P(z, \overline{z}) P(w, \overline{w}) \quad \text{for all linearly independent } z, w \in \mathbb{C}^n, \tag{2.4}$$

i.e., the above inequality holds whenever z and w are not scalar multiples of each other. (Note that the Hermitian bihomogeneity of P implies that $|P(z, \overline{w})|^2 = P(z, \overline{z}) P(w, \overline{w})$ whenever z and w are linearly dependent.) We recall the following result of Catlin-D'Angelo:

Theorem 2.1 ([4, Theorem 1, Corollary and its proof]). *Let $P \in \mathbb{C}[z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}]$ be a nonconstant Hermitian bihomogeneous polynomial such that (i) P is positive on $\mathbb{C}^n \setminus \{0\}$, (ii) the domain $\{z \in \mathbb{C}^n : P(z, \overline{z}) < 1\}$ is strongly pseudoconvex, and (iii) P satisfies the SGCS inequality. Then for each Hermitian bihomogeneous polynomial $Q \in \mathbb{C}[z_1, \ldots, z_n, \overline{w_1}, \ldots, \overline{w_n}]$ positive on $\mathbb{C}^n \setminus \{0\}$, there exists $m_o > 0$ such that for each integer $m \geq m_o$, $P^m \cdot Q$ is a maximal squared norm.*

3. Homogeneous polynomials and associated bihomogeneous polynomials

Throughout this section, we fix a positive integer $n \geq 2$. For each homogeneous real polynomial $p = \sum_{|I|=d} c_I x^I \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, we have an associated Hermitian bihomogeneous polynomial P of bidegree (d, d) given by

$$P(z, \overline{w}) := p(z_1\overline{w_1}, \ldots, z_n\overline{w_n}) = \sum_{|I|=d} c_I z^I \overline{w^I} \tag{3.1}$$

for all $z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. We remark that the bihomogeneous polynomial P is indeed Hermitian, since the c_I's are real. First we make a simple observation as follows:

Proposition 3.1. *p has all positive coefficients if and only if P is a maximal squared norm.*

Proof. With notation as in Section 2, the monomials $\{z^I\}_{|I|=d}$ form a basis of the complex vector space $\mathbb{C}[z_1, \ldots, z_n]_d$. With respect to this basis, it follows readily from (3.1) that the square matrix associated to P (as in (2.3)) is given by the real diagonal matrix $C := \operatorname{diag}(c_I)_{|I|=d}$. Then, as remarked in Section 2, P is a maximal squared norm if and only if the matrix C is positive definite. In turn, the latter condition holds if and only if $c_I > 0$ for all $|I| = d$. $\square$

Our main result in this section is the following proposition:

Proposition 3.2. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonconstant homogeneous real polynomial, and let P be its associated Hermitian bihomogeneous polynomial as in* (3.1)*. If p satisfies* (Pos1), (Pos2)*, and* (Pos3)*, then the following statements hold:*

(i) P is positive on $\mathbb{C}^n \setminus \{0\}$.
(ii) The domain $\Omega_{P<1} := \{z \in \mathbb{C}^n \mid P(z, \overline{z}) < 1\}$ is strongly pseudoconvex.
(iii) P satisfies the SGCS inequality.

Througout the rest of this section, which is devoted to the proof of the above proposition, we let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonconstant homogeneous real polynomial, and let P be its associated Hermitian bihomogeneous polynomial.

Proposition 3.3. *If p satisfies* (Pos1) *and* (Pos3)*, then*

(i) $p(x) > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$, and
(ii) P is positive on $\mathbb{C}^n \setminus \{0\}$.

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \setminus \{0\}$ be given. If $x_i > 0$ for only one i (with $1 \leq i \leq n$), then it follows readily from (Pos1) and the homogeneity of p that $p(x) > 0$. If $x_i, x_j > 0$ for some $1 \leq i < j \leq n$, then by permuting the coordinate functions, we may assume without loss of generality that $x_1, x_2 > 0$. Then one easily checks that $x' := (-x_1, x_2, x_3, \ldots, x_n) \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}^n_+)$ (since the equalities $e^{i\theta} \cdot x_1 = -x_1$ and $e^{i\theta} \cdot x_2 = x_2$ imply $e^{i\theta} = -1$ and $e^{i\theta} = 1$ respectively, which is a contradiction). Then by (Pos3), one has $|p(x')| < p(x)$, which implies that $p(x) > 0$ again. This finishes the proof of (i). For (ii), we let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}$ be given. Then one sees from (3.1) and (i) that $P(z, \overline{z}) = p(x) > 0$, where $x = (|z_1|^2, \ldots, |z_n|^2) \in \mathbb{R}^n_+ \setminus \{0\}$. Hence P is positive on $\mathbb{C}^n \setminus \{0\}$. $\square$

Next we recall a result of De Angelis [7]. For $\ell \geq 1$, we denote the interior of $\mathbb{R}^\ell_+$ by $(\mathbb{R}^\ell_+)^\circ := \{(s_1,\ldots,s_\ell) \in \mathbb{R}^\ell \,\big|\, s_i > 0,\ i = 1,\ldots,\ell\}$. Let $f(s) = \sum_I c_I s^I \in \mathbb{R}[s_1,\ldots,s_\ell]$ be a (possibly nonhomogeneous) polynomial such that $f(s) > 0$ for all $s = (s_1,\ldots,z_\ell) \in (\mathbb{R}^\ell_+)^\circ$. Consider the set $\mathrm{Log}(f) := \{I \in \mathbb{Z}^\ell \,\big|\, c_I \neq 0\}$, and recall that the *Newton polytope* $N(f)$ of f is defined as the convex hull of $\mathrm{Log}(f)$ in $\mathbb{R}^\ell$. We associate to f the $\ell \times \ell$ matrix-valued function $J_f : (\mathbb{R}^\ell_+)^\circ \to \mathbb{R}^{\ell^2}$ whose components are given by

$$
\begin{aligned}
(3.2)\qquad J_f(s)_{ij} &:= s_j \cdot \frac{\partial}{\partial s_j}\Big(s_i \cdot \frac{\partial}{\partial s_i}\big(\log f\big)\Big)(s) \\
&= s_i s_j \frac{\partial^2}{\partial s_i \partial s_j}\big(\log f\big)(s) + \delta_{ij} \cdot s_j \cdot \frac{\partial}{\partial s_i}\big(\log f\big)(s)
\end{aligned}
$$

for $s \in (\mathbb{R}^\ell_+)^\circ$, $1 \leq i,j \leq \ell$. Here δ_{ij} denotes the Kronecker delta. i.e., $\delta_{ij} = 1$ (resp. 0) if $i = j$ (resp. $i \neq j$). Next we introduce a change of variables, and consider the function $\widetilde{f} : \mathbb{R}^\ell \to \mathbb{R}$ associated to f given by

$$
(3.3)\qquad \widetilde{f}(t) = f(e^{t_1},\ldots,e^{t_\ell}) \quad \text{for } t = (t_1,\ldots,t_\ell) \in \mathbb{R}^\ell.
$$

Using (3.2), one easily checks that the Hessian matrix of $\log \widetilde{f}$ coincides with J_f, i.e., one has

$$
(3.4)\qquad \frac{\partial^2}{\partial t_i \partial t_j}\big(\log \widetilde{f}\big)(t) = J_f(e^{t_1},\ldots,e^{t_\ell})_{ij}
$$

for all $t = (t_1,\ldots,t_\ell) \in \mathbb{R}^\ell$, $1 \leq i,j \leq \ell$. We recall the following result:

Lemma 3.4 (De Angelis [7, Theorem 6.11])**.** *Let $f(s) \in \mathbb{R}[s_1,\ldots,s_\ell]$ be a polynomial such that $f(s) > 0$ for all $s \in (\mathbb{R}^\ell_+)^\circ$. Suppose that there exists an open neighborhood V of $(\mathbb{R}^\ell_+)^\circ$ in $(\mathbb{C} \setminus \{0\})^\ell$ such that $|f(z)| \leq f(|z_1|,\ldots,|z_\ell|)$ for all $z = (z_1,\ldots,z_\ell) \in V$, and the Newton polytope $N(f)$ has affine dimension ℓ. Then the $\ell \times \ell$ matrix $J_f(s)$ is positive definite for all $s \in (\mathbb{R}^\ell_+)^\circ$.*

As before, we let $p \in \mathbb{R}[x_1,\ldots,x_n]$ be a nonconstant homogeneous polynomial. Let $\mathfrak{S}_n$ denote the group of permutations of the coordinate functions on $\mathbb{R}^n$. For each $1 \leq \ell \leq n-1$ and each $\sigma \in \mathfrak{S}_n$, we associate to p a nonhomogeneous polynomial $p_{\ell,\sigma} \in \mathbb{R}[s_1,\ldots,s_\ell]$ given by

$$
(3.5)\qquad p_{\ell,\sigma}(s_1,\ldots,s_\ell) := p(\sigma(s_1,\cdots,s_\ell,0,\ldots,0,1)).
$$

Lemma 3.5. *(i) If p satisfies* (Pos1)*, then for each $1 \leq \ell \leq n-1$ and each $\sigma \in \mathfrak{S}_n$, the Newton polytope $N(p_{\ell,\sigma})$ has affine dimension ℓ.*
(ii) If p satisfies (Pos1) *and* (Pos2)*, then for each $1 \leq \ell \leq n-1$ and each $\sigma \in \mathfrak{S}_n$, the set $S_{p_{\ell,\sigma}} := \{I - J \,\big|\, I,\, J \in \mathrm{Log}(p_{\ell,\sigma})\}$ generates $\mathbb{Z}^\ell$ as a $\mathbb{Z}$-module.*

Proof. As the proofs of the lemma for all the $p_{\ell,\sigma}$'s are the same, we will only prove the lemma for the case when σ is the identity permutation, so that $p_{\ell,\sigma}(s_1,\ldots,s_\ell) = p(s_1,\cdots,s_\ell,0,\cdots,0,1)$. Let p be of degree $d \geq 1$. If p satisfies (Pos1), then it follows readily that $\mathrm{Log}(p)$ contains the points $(d,0,\ldots,0)$, …, $(0,\ldots,0,d)$. Hence $\mathrm{Log}(p_{\ell,\sigma})(\subset \mathbb{Z}^\ell)$ contains the points $(d,0,\ldots,0)$, …, $(0,\ldots,0,d)$ and $(0,\ldots,0)$. This implies readily that $N(p_{\ell,\sigma})$ has affine dimension ℓ, and this finishes the proof of (i). We proceed to prove (ii). For each $1 \leq i \leq \ell$, one easily checks that

$$
(3.6)\qquad \frac{\partial p_{\ell,\sigma}}{\partial s_i}(0,\ldots,0) = \frac{\partial p}{\partial x_i}(0,\ldots,0,1) > 0,
$$

where the inequality holds since p satisfies (Pos2) and $(0,\ldots,0,1)\in F_i(\mathbb{R}^n_+)$. This implies that $\mathrm{Log}(p_{\ell,\sigma})(\subset\mathbb{Z}^\ell)$ contains the points $(1,0,\ldots,0)$, …, $(0,\ldots,0,1)$, and so does $S_{p_{\ell,\sigma}}$ (since $\mathrm{Log}(p_{\ell,\sigma})$ also contains $(0,\cdots,0)$ as shown in (i)). It follows that $S_{p_{\ell,\sigma}}$ generates $\mathbb{Z}^\ell$ as a $\mathbb{Z}$-module. □

Proposition 3.6. *If p satisfies* (Pos1), (Pos2) *and* (Pos3), *then the domain $\Omega_{P<1}$ is strongly pseudoconvex.*

Proof. Let p be of degree $d\geq 1$. From Proposition 3.3, one knows that $P(z,\overline{z})>0$ for all $z\in\mathbb{C}^n\setminus\{0\}$. Together with the bihomogeneity of P of bidegree (d,d) with $d\geq 1$, it follows readily that $\Omega_{P<1}$ is a bounded domain in $\mathbb{C}^n$ with smooth boundary. Note that we may write $\Omega_{P<1}=\{z\in\mathbb{C}^n \mid \log P(z,\overline{z})<0\}$. To prove the proposition, it suffices to show that

$$(\sqrt{-1}\partial\overline{\partial}\log P)(v,\overline{v})>0 \quad \text{for any } z^*\in\partial\Omega_{P<1} \text{ and any } 0\neq v\in T_{z^*}(\mathbb{C}^n) \text{ satisfying } \partial(\log P)(v)=0. \tag{3.7}$$

(Here P denotes $P(z,\overline{z})$.) Regarding $\mathbb{C}^n$ as a complex manifold, it is well known that one only needs to verify (3.7) in terms of some local (possibly non-Euclidean) holomorphic coordinate system at each $z^*\in\partial\Omega_{P<1}$ (see e.g. [13, p. 66]). Take an arbitrary point $z^*=(z_1^*,\ldots,z_n^*)\in\partial\Omega_{P<1}$, so that $P(z^*,\overline{z^*})=1$ (and thus $z^*\neq 0$). By permuting the coordinate functions, we will assume without loss of generality that $z_n^*\neq 0$. Next we introduce a new local coordinate system u near z^* via the holomorphic map $\phi:\{u=(u_1,\ldots,u_n)\in\mathbb{C}^n \mid u_n\neq 0\}\to\mathbb{C}^n$ given by

$$z=\phi(u):=(u_1u_n,\ldots,u_{n-1}u_n,u_n). \tag{3.8}$$

Let $u^*=(u_1^*,\ldots,u_n^*)$ be the point such that $z^*=\phi(u^*)$, so that $u_n^*\neq 0$. Then one easily sees from (3.1), (3.8) and the homogeneity of p that $(P\circ\phi)(u,\overline{u})=p(|u_1|^2,\ldots,|u_{n-1}|^2,1)\cdot|u_n|^{2d}$, so that

$$\log(P\circ\phi)(u,\overline{u})=\log p(|u_1|^2,\ldots,|u_{n-1}|^2,1)+d\cdot\log u_n+d\cdot\log\overline{u_n} \tag{3.9}$$

near u^* (for an appropriate logarithmic branch). Hence one has

$$\frac{\partial^2(\log(P\circ\phi))}{\partial u_i\partial\overline{u_j}}(u,\overline{u}) = \begin{cases}\Big(u_j\overline{u_i}\cdot\dfrac{\partial^2(\log p)}{\partial x_i\partial x_j}+\delta_{ij}\cdot\dfrac{\partial(\log p)}{\partial x_i}\Big)(|u_1|^2,\cdots,|u_{n-1}|^2,1) & \text{if } 1\leq i,j\leq n-1,\\ 0 \quad \text{if } i=n \text{ or } j=n. & \end{cases} \tag{3.10}$$

Now we take a tangent vector $0\neq v=v_1\dfrac{\partial}{\partial u_1}+\cdots+v_n\dfrac{\partial}{\partial u_n}\in T_{u^*}(\mathbb{C}^n)$ satisfying $\partial\log(P\circ\phi)(v)=0$, or equivalently,

$$\sum_{i=1}^{n-1}v_i\cdot\overline{u_i^*}\cdot\frac{\partial(\log p)}{\partial x_i}(|u_1^*|^2,\cdots,|u_{n-1}^*|^2,1)+v_n\cdot\frac{d}{u_n^*}=0 \tag{3.11}$$

(cf. (3.9)). Together with the condition that $v\neq 0$, it follows readily that

$$(v_1,\ldots,v_{n-1})\neq(0,\ldots,0). \tag{3.12}$$

Let ℓ be the number of nonzero u_i^*'s for $1\leq i\leq n-1$ (so that $0\leq\ell\leq n-1$). By permuting the first $n-1$ coordinate functions, we will assume without loss of generality that $u_i^* \neq 0$ for each $1 \leq i \leq \ell$ and $u_{\ell+1}^* = \cdots = u_{n-1}^* = 0$. By using (3.10) and (3.2) (with $f = p_{\ell,\mathrm{Id}}$ where Id denotes the identity permutation, and $s = (|u_1^*|^2, \ldots, |u_\ell^*|^2, 0, \ldots, 0, 1)$), one easily checks that

$$\sum_{1\leq i,j\leq n} \overline{v_i} \cdot \frac{\partial^2(\log(P\circ\phi))}{\partial u_i \partial \overline{u_j}}(u^*, \overline{u^*}) \cdot v_j = A_1 + A_2, \quad \text{where} \tag{3.13}$$

$$A_1 := \sum_{1\leq i,j\leq \ell} \frac{\overline{v_i}}{u_i^*} \cdot J_{p_{\ell,\mathrm{Id}}}(|u_1^*|^2, \ldots, |u_\ell^*|^2)_{ij} \cdot \frac{v_j}{\overline{u_j^*}} \quad \text{and}$$

$$A_2 := \sum_{\ell+1\leq i\leq n-1} |v_i|^2 \cdot \frac{\partial(\log p)}{\partial x_i}(|u_1^*|^2, \ldots, |u_\ell^*|^2, 0, \cdots, 0, 1).$$

Here A_1 (resp. A_2) is taken to be zero if $\ell = 0$ (resp. $\ell = n-1$). Note that $(|u_1^*|^2, \ldots, |u_\ell^*|^2, 0, \cdots, 0, 1) \in F_i(\mathbb{R}_+^n)$ for each $\ell+1 \leq i \leq n-1$. Hence from (Pos2), we see that $A_2 > 0$ whenever $\ell < n-1$ and $(v_{\ell+1}, \ldots, v_{n-1}) \neq (0, \ldots, 0)$. From (Pos3) (for the set $\mathbb{C}^n \setminus (U(1)\cdot\mathbb{R}_+^n)$) and the homogeneity of p (for the set $U(1)\cdot\mathbb{R}_+^n$), one easily sees that

$$|p(z)| \leq p(|z_1|, \ldots, |z_n|) \quad \text{for all } z = (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{3.14}$$

Together with Lemma 3.5, it follows that one can apply Lemma 3.4 to conclude that $A_1 > 0$ whenever $\ell > 0$ and $(v_1, \ldots, v_\ell) \neq (0, \ldots, 0)$. Since $n \geq 2$, by using (3.12), one easily concludes that $A_1 + A_2 > 0$ in each of the three cases when $\ell = 0$, $1 \leq \ell < n-1$ or $\ell = n-1$. This finishes the proof of (3.7). $\square$

Proposition 3.7. *If p satisfies* (Pos1), (Pos2) *and* (Pos3)*, then*
(i) for all $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$, we have

$$p(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n})^2 \leq p(x)\cdot p(y); \quad \textit{and} \tag{3.15}$$

(ii) P satisfies the SGCS inequality.

Proof. First we recall from Proposition 3.3 that $p(x) > 0$ for all $x \in (\mathbb{R}_+^n)^\circ$. Write $f := p_{n-1,\mathrm{Id}}$ where $p_{n-1,\mathrm{Id}}$ is as in the proof of Proposition 3.6 (cf. also (3.5)), so that $f(s_1, \ldots, s_{n-1}) = p(s_1, \ldots, s_{n-1}, 1)$. As in (3.3), we consider the associated function $\tilde{f} : \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$\tilde{f}(t_1, \ldots, t_{n-1}) := f(e^{t_1}, \ldots, e^{t_{n-1}}) = p(e^{t_1}, \ldots, e^{t_{n-1}}, 1). \tag{3.16}$$

By Lemma 3.5(i), $N(f)$ has affine dimension $n-1$. It also follows from (3.14) that $|f(z_1, \ldots, z_{n-1})| \leq f(|z_1|, \ldots, |z_{n-1}|)$ for all $(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$. Hence, by Lemma 3.4 and (3.4), the Hessian matrix $\left(\frac{\partial^2}{\partial t_i \partial t_j} \log \tilde{f}(t)\right)_{1\leq i,j\leq n-1}$ is positive definite for all $t \in \mathbb{R}^{n-1}$, and it follows that $\log \tilde{f}$ is a convex function on $\mathbb{R}^{n-1}$ (see e.g. [2, p. 37]). In particular, we have $\tilde{f}(\frac{t+t'}{2}) \leq \frac{1}{2}(\tilde{f}(t) + \tilde{f}(t'))$ for all $t = (t_1, \ldots, t_{n-1}),\ t' = (t_1', \ldots, t_{n-1}') \in \mathbb{R}^{n-1}$. By letting $t_i = \log s_i,\ t_i' = \log s_i'$ for each i, it follows that we have

$$\log p(\sqrt{s_1 s_1'}, \ldots, \sqrt{s_{n-1}s_{n-1}'}, 1) \leq \frac{1}{2}(\log p(s,1) + \log p(s',1)) \tag{3.17}$$

for all $s = (s_1, \ldots, s_{n-1}), s' = (s_1', \ldots, s_{n-1}') \in (\mathbb{R}_+^{n-1})^\circ$. For any given $x, y \in (\mathbb{R}_+^n)^\circ$, by setting $s = (x_1/x_n, \ldots, x_{n-1}/x_n)$ and $s' = (y_1/y_n, \ldots, y_{n-1}/y_n)$ in (3.17), and using the homogeneity of p, one easily sees that the inequality in (3.15) holds for such $x, y \in (\mathbb{R}_+^n)^\circ$. Together with the continuity of p, it follows that the inequality in (3.15) actually holds for all $x, y \in \mathbb{R}^n_+$, and this finishes the proof of (i). For (ii), we let $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ be linearly independent, which implies readily that $(z_1\overline{w_1}, \dots, z_n\overline{w_n}) \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}^n_+)$. Hence it follows from (Pos3) and (i) that

$$\begin{aligned} |p(z_1\overline{w_1}, \dots, z_n\overline{w_n})|^2 &< p(|z_1||w_2|, \dots, |z_n||w_n|)^2 \\ &\leq p(|z_1|^2, \dots, |z_n|^2) \cdot p(|w_1|^2, \dots, |w_n|^2), \end{aligned} \tag{3.18}$$

which, together with (3.1), imply that $|P(z, \overline{w})|^2 < P(z, \overline{z}) \cdot P(w, \overline{w})$, and this finishes the proof of (ii). ☐

We conclude this section with the following

Proof of Proposition 3.2. Proposition 3.2 follows directly from Proposition 3.3, Proposition 3.6 and Proposition 3.7. ☐

4. Proof of Theorem 1.1

Lemma 4.1. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a nonconstant homogeneous polynomial which has all positive coefficients. Then*

$$f(x) > 0 \quad \textit{for all } x \in \mathbb{R}^n_+ \setminus \{0\}, \tag{4.1}$$

and f satisfies (Pos1), (Pos2) *and* (Pos3).

Proof. Write $f = \sum_{|I|=d} b_I x^I$, so that $d \geq 1$ and $b_I > 0$ for all $|I| = d$. Then one easily sees that (4.1) holds, which, in turn, implies that f satisfies (Pos1). For (Pos2), we first consider the facet $F_1(\mathbb{R}^n_+)$. Let $x = (x_1, \dots, x_n) \in F_1(\mathbb{R}^n_+) \setminus \{0\}$ be given, so that $x_1 = 0$ and $x_j > 0$ for some $1 < j \leq n$. Assume without loss of generality that $j = 2$. Then

$$\frac{\partial f}{\partial x_1}(x) = \sum_{|I|=d} b_I I_1 x_1^{I_1 - 1} x_2^{I_2} \cdots x_n^{I_n} \geq b_{(1,d-1,0,\dots,0)} \cdot 1 \cdot x_2^{d-1} > 0. \tag{4.2}$$

The same argument yields the desired inequality on the other $F_k(\mathbb{R}^n_+)$'s, and this finishes the proof of (Pos2). For (Pos3), we let $z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}^n_+)$ be given, so that $z_k \neq 0$ for some $1 \leq k \leq n$. Since $b_I > 0$ for all $|I| = d$, we have

$$|f(z)| = \Big|\sum_{|I|=d} b_I z^I\Big| \leq \sum_{|I|=d} b_I |z_1|^{I_1} \cdots |z_n|^{I_n} = f(|z_1|, \dots, |z_n|). \tag{4.3}$$

If the inequality in (4.3) is in fact an equality, then it is easy to see that all the $b_I z^I$'s (and thus all the z^I's) will have the same argument. By comparing the arguments of z_k^d and $z_k^{d-1} z_j$ for each $1 \leq j \leq n$, one sees that all the z_j's have the same argument, contradicting the assumption that $z \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}^n_+)$. Hence the inequality in (4.3) is strict. Thus f satisfies (Pos3). ☐

We are ready to give the proof of Theorem 1.1 as follows:

Proof of Theorem 1.1. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a nonconstant homogeneous polynomial. In the case when $n = 1$, it is easy to see that p satisfies (a) (resp. (b)) (in Theorem 1.1) if and only if p is a monomial with positive coefficient. Hence we only need to consider the case when $n \geq 2$.

<u>(a) $\Longrightarrow$ (b)</u>: Suppose p satisfies (Pos1), (Pos2) and (Pos3), and $q \in \mathbb{R}[x_1, \dots, x_n]$ is a homogeneous polynomial such that $q(x) > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$. Let P and Q be the Hermitian bihomogeneous polynomial associated to p and q respectively.

Then from Proposition 3.2 and Theorem 2.1, one knows that there exists $m_o > 0$ such that for each integer $m \geq m_o$, $P^m \cdot Q$ is a maximal squared norm. By Proposition 3.1, it follows that $p^m q$ has all positive coefficients for each such m.
$\underline{\text{(b)} \implies \text{(a)}}$: By setting $q = 1$ in (b), one knows that p^m has all positive coefficients for some odd integer m. From Lemma 4.1 (with $f = p^m$), one knows that $p(x)^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$, and p^m satisfies (Pos1), (Pos2) and (Pos3). By taking the m-th root, one immediately sees that $p(x) > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$, and p also satisfies (Pos1) and (Pos3). Since $p(x) > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$ and we have

$$\frac{\partial(p^m)}{\partial x_k}(x) = m \cdot p(x)^{m-1} \cdot \frac{\partial p}{\partial x_k}(x) \quad \text{for each } 1 \leq k \leq n, \tag{4.4}$$

it follows readily from (Pos2) for p^m that p also satisfies (Pos2). $\square$

5. Application to polynomial spectral radius functions

In this section, we apply Theorem 1.1 to prove Corollary 1.2, and interpret Corollary 1.2 in terms of Markov chains.

Let $B = (B_{ij})$ be a square matrix with $B_{ij} \in \mathbb{R}_+$ for all i, j. We recall that B is said to be *irreducible* if, for each pair of indices i and j, there exists an integer $k \geq 1$ such that $(B^k)_{ij} > 0$. B is said to be *aperiodic* if for each i, we have $\gcd\{k \in \mathbb{Z}_+ \,|\, (B^k)_{ii} > 0\} = 1$. By the Perron-Frobenius theorem (see e.g. [23]), if the matrix B is irreducible or aperiodic, then its spectral radius $\beta(B)$ is positive.

Next we recall that an *irreducible* (resp. *aperiodic*) *square matrix* $A = (A_{ij})$ *over* $\mathbb{Z}_+[x_1, \ldots, x_n]$ means that $A_{ij} \in \mathbb{Z}_+[x_1, \ldots, x_n]$ for all i, j, and for some (and hence all) $x \in (\mathbb{R}^n_+)^\circ$, the corresponding matrix $A(x)$ (with entries in $\mathbb{R}_+$) is irreducible (resp. aperiodic). In particular, for such A, we obtain its spectral radius function $\beta_A : (\mathbb{R}^n_+)^\circ \to (0, \infty)$ given by $\beta_A(x) := \beta(A(x))$ for $x \in (\mathbb{R}^n_+)^\circ$.

In the probability theory of stochastic processes, an irreducible (resp. aperiodic) square matrix A over $\mathbb{Z}[x_1, \ldots, x_n]$ defines an irreducible (resp. aperiodic) Markov chain Σ_A. The spectral radius function β_A is also known as the *beta function* of Σ_A in Tuncel's paper [24] (see also [8]). The beta function is an important topological invariant in Markov shifts (see e.g. [16, 17] and the references therein). In such context, Corollary 1.2 may be interpreted as a characterization of certain polynomials as the beta functions of some irreducible or aperiodic Markov chains.

First we recall some results of De Angelis:

Lemma 5.1 ([8, Theorem 3.3(i) (resp. Theorem 3.5)])**.** *Let $p \in \mathbb{Z}[x_1, \ldots, x_n]$. If there exists $m > 0$ and an irreducible (resp. aperiodic) square matrix B over $\mathbb{Z}_+[x_1, \ldots, x_n]$ such that $p^m = \beta_B$, then $p = \beta_A$ for some irreducible (resp. aperiodic) square matrix A over $\mathbb{Z}_+[x_1, \ldots, x_n]$.*

Lemma 5.2 ([8, Theorem 6.6])**.** *Let $q \in \mathbb{R}[s_1, \ldots, s_\ell]$. Suppose that the set $S_q := \{I - J \,|\, I, J \in \operatorname{Log}(q)\}$ generates $\mathbb{Z}^\ell$ as a $\mathbb{Z}$-module, and $q = \beta_A$ for some square matrix A over $\mathbb{Z}_+[s_1, \ldots, s_\ell]$. Then $|q(z)| < q(|z_1|, \ldots, |z_\ell|)$ for all $z = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \setminus \mathbb{R}^\ell_+$.*

Finally we give the deduction of Corollary 1.2 as follows:

Proof of Corollary 1.2. Let $p \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous polynomial which satisfies (Pos1) and (Pos2).

(ii) $\implies$ (i) (resp. (iii) $\implies$ (i)): Suppose that there exists an irreducible (resp. aperiodic) square matrix A over $\mathbb{Z}_+[x_1, \ldots, x_n]$ such that $p = \beta_A$. Let $p_{n-1,\mathrm{Id}}$ be as in (3.5), so that

$$p_{n-1,\mathrm{Id}}(s_1, \ldots, s_{n-1}) = p(s_1, \ldots, s_{n-1}, 1). \tag{5.1}$$

Then $p_{n-1,\mathrm{Id}} \in \mathbb{Z}[s_1, \ldots, s_{n-1}]$ and $p_{n-1,\mathrm{Id}} = \beta_B$, where B is the matrix over $\mathbb{Z}_+[s_1, \ldots, s_{n-1}]$ given by $B(s_1, \ldots, s_{n-1}) = A(s_1, \ldots, s_{n-1}, 1)$. Furthermore, it follows from Lemma 3.5(ii) that $S_{p_{n-1,\mathrm{Id}}}$ generates $\mathbb{Z}^{n-1}$ as a $\mathbb{Z}$-module. Thus by Lemma 5.2 and (5.1), we have

$$|p(z_1, \ldots, z_{n-1}, 1)| < p(|z_1|, \ldots, |z_{n-1}|, 1) \quad \text{for all } (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \setminus \mathbb{R}_+^{n-1}. \tag{5.2}$$

Next, we let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus (U(1) \cdot \mathbb{R}_+^n)$, so that the z_i's do not have the same argument. By permuting the coordinate functions, we may assume without loss of generality that $z_n \neq 0$, so that $(z_1/z_n, \ldots, z_{n-1}/z_n) \in \mathbb{C}^{n-1} \setminus \mathbb{R}_+^{n-1}$. Then it follows from (5.2) (with z_i there replaced by z_i/z_n) that

$$|p(\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}, 1)| < p(|\frac{z_1}{z_n}|, \ldots, |\frac{z_{n-1}}{z_n}|, 1) \implies |p(z)| < p(|z_1|, \ldots, |z_n|), \tag{5.3}$$

where the implication follows from the homogenity of p. Hence p satisfies (Pos3).

(i) $\implies$ (ii) (resp. (i) $\implies$ (iii)): Suppose that p also satisfies (Pos3). Then by Theorem 1.1 (with $q = 1$ in (b)), there exists $m > 0$ such that p^m has all positive coefficients. Since p^m is nonzero, the 1×1 matrix $B := (p^m)$ is irreducible (resp. aperiodic) over $\mathbb{Z}_+[x_1, \ldots, x_n]$, and $\beta_B = p^m$. Hence by Lemma 5.1, there exists an irreducible (resp. aperiodic) square matrix A over $\mathbb{Z}_+[x_1, \ldots, x_n]$ such that $p = \beta_A$. □

Acknowledgments

The first author would like to thank Ser Peow Tan, Yan Loi Wong and Xingwang Xu for suggesting to work in the direction of this problem and is grateful for his wife's encouragement to complete this article. The authors also acknowledge John P. D'Angelo, Valerio De Angelis, Alexandre Eremenko, David Handelman, and John Jiang for sharing their work and for helpful discussions. The authors are grateful to the referee for numerous comments and suggestions leading to the paper in its present form.

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