

**ON THE STRONG LEFSCHETZ PROBLEM  
FOR UNIFORM POWERS OF GENERAL LINEAR FORMS  
IN  $k[x, y, z]$**

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ABSTRACT. Schenck and Seceleanu proved that if  $R = k[x, y, z]$ , where  $k$  is an infinite field and  $I$  is an ideal generated by any collection of powers of linear forms, then multiplication by a general linear form  $L$  induces a homomorphism of maximal rank from any component of  $R/I$  to the next. That is,  $R/I$  has the *weak Lefschetz property*. Considering the more general *strong Lefschetz problem* of when  $\times L^j$  has maximal rank for  $j \geq 2$ , we give the first systematic study of this problem. We assume that the linear forms are general and that the powers are all the same, i.e. that  $I$  is generated by *uniform* powers of general linear forms. We prove that for any number of such generators,  $\times L^2$  always has maximal rank. We then specialize to almost complete intersections, i.e. to four generators, and we show that for  $j = 3, 4, 5$  the behavior depends on the uniform exponent and on  $j$  in a way that we make precise. In particular, there is always at most one degree where  $\times L^j$  fails maximal rank. Finally, we note that experimentally all higher powers of  $L$  fail maximal rank in at least two degrees.

1. INTRODUCTION

Ideals of powers of linear forms have been studied rather extensively. We can point, for example, to [5], [9], [11], [12], [18] and [21]. We take the latter as our launching point, and we consider only ideals in  $R = k[x, y, z]$ , where  $k$  is an infinite field.

If  $R/I$  is a standard graded artinian algebra and  $L$  is a general linear form, we recall that  $R/I$  is said to have the *weak Lefschetz property (WLP)* if the multiplication  $\times L : [R/I]_{\delta-1} \rightarrow [R/I]_{\delta}$  has maximal rank for all  $\delta$ . The *strong Lefschetz property (SLP)* says that for all  $j \geq 1$  the multiplication by  $L^j$  has maximal rank in all degrees. We will call the *strong Lefschetz problem* the analysis of which  $j$  and  $\delta$  provide the homomorphism  $\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$  having maximal rank.

We consider ideals of the form  $I = (L_1^{a_1}, \dots, L_r^{a_r})$  in  $R = k[x, y, z]$ , where  $k$  is an infinite field. A theorem of Stanley [22] and Watanabe [23] shows that when  $r = 3$ ,  $R/I$  has the SLP, so maximal rank always holds. Thus the problem is only of interest for  $r \geq 4$ .

The main theorem of [21] asserts that if  $I$  is *any* ideal of the stated form, then  $R/I$  has the WLP (see also [18] for a different proof). This leads naturally to the issue of what happens for higher powers of a general linear form. For  $\times L^2$  it

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was shown in [18] that for  $r = 4$ , if the linear forms are chosen generally, then  $\times L^2 : [R/I]_j \rightarrow [R/I]_{j+2}$  has maximal rank for all  $j$ . On the other hand, it was shown in [5] and [9] that if the linear forms are not required to be general, then  $\times L^2$  does not necessarily have maximal rank, and indeed the issue of maximal rank is a quite subtle one depending on the geometry of the set of points dual to the linear forms. Thus we focus on general linear forms.

So what, exactly, should we expect for  $\times L^j$  for  $L$  a general linear form and  $j \geq 2$ ? In this paper we want to begin the study of the multiplication by higher powers,  $L^j$ , of the general linear form by assuming that the exponents of the linear forms generating our ideal are all the same, i.e. that we have *uniform* powers. Our first main result, Theorem 4.4, is that for arbitrary  $r$ ,  $\times L^2 : [R/I]_{\delta-2} \rightarrow [R/I]_{\delta}$  has maximal rank for all  $\delta$ . We conjecture that in fact the result also holds for mixed powers.

For  $j \geq 3$  we already get interesting behavior for  $r = 4$  by assuming that the ideal is an almost complete intersection of uniform powers of general linear forms:  $I = (L_1^k, \dots, L_4^k)$ . We want to see if  $\times L^j$  always has maximal rank, and if not, to see how often we can expect this phenomenon to occur. We find that it is rarely the case that  $\times L^j$  has maximal rank in all degrees, in fact, but it occasionally does. In this paper we classify those values of  $j$  and  $k$  for which it does have maximal rank in all degrees and those values of  $j$  and  $k$  for which it fails maximal rank in only one degree.

More precisely, in Theorem 5.1, Theorem 5.2 and Theorem 5.3 we show that  $\times L^3$  and  $\times L^4$  sometimes have maximal rank in all degrees, depending on the congruence class of  $k$  modulo 3, and that  $\times L^5$  never has maximal rank in all degrees. However, we also show that for  $j = 3, 4, 5$ , whenever this multiplication fails maximal rank, it does so only in one spot. We note in Remark 5.5 that for higher powers of  $L$ , the multiplication fails maximal rank in more than one spot. These results show more clearly that Anick's theorem does not extend from general forms to powers of general linear forms, although this was already known. (Indeed, if  $I = (x^3, y^3, z^3, L_1^3) \subset R$ , where  $L_1$  is a general linear form, then for a general linear form  $L$ ,  $\times L^3$  fails to have maximal rank (see Proposition 6.1), while Anick's result shows that if instead we take general forms of degree 3, then maximal rank does hold.)

There are many reasons why ideals generated by powers of linear forms (often assumed to be general) and the Lefschetz properties for such ideals are of interest. The Lefschetz problem in general is ubiquitous in Commutative Algebra and Algebraic Geometry; it ties in with vector bundle problems, fat point schemes, graph theory, monomial ideals, Gorenstein algebras, etc. Ideals generated by powers of (general) linear forms and the problem of studying their Lefschetz properties are a natural extension of the monomial complete intersections considered in the Stanley-Watanabe theorem already mentioned. They also are related to Terao's conjecture ([9], [5]). They are an essential component of the well-known Waring problem. Finally, as suggested by the remarks in the previous paragraph, they are connected to the famous Fröberg conjecture – Anick's theorem proves that conjecture for the polynomial ring in three variables (see [18] for a discussion of the connection). In this paper we give the first careful analysis of the multiplication by powers of a general linear form for the quotient of an ideal generated by powers of general linear forms.

2. PRELIMINARIES

Throughout this paper we consider the homogeneous polynomial ring  $R = k[x, y, z]$ , where  $k$  is a field of characteristic zero. In this section we recall the main tools that we will use in the rest of the paper.

For any artinian ideal  $I \subset R$  and a general linear form  $L \in R$ , the exact sequence

$$\cdots \rightarrow [R/I]_{m-j} \xrightarrow{\times L^j} [R/I]_m \rightarrow [R/(I, L^j)]_m \rightarrow 0$$

gives, in particular, that the multiplication by  $L^j$  will fail to have maximal rank exactly when

$$(2.1) \quad \dim_k [R/(I, L^j)]_m \neq \max\{\dim_k [R/I]_m - \dim_k [R/I]_{m-j}, 0\};$$

in that case, we will say that  $R/I$  fails maximal rank in degree  $m$ .

We will deeply need the following result of Emsalem and Iarrobino, which gives a duality between powers of linear forms and ideals of fat points in  $\mathbb{P}^2$ . We only quote Theorem I in [11] in the form that we need.

**Theorem 2.1** ([11]). *Let  $\langle L_1^{a_1}, \dots, L_n^{a_n} \rangle \subset R$  be an ideal generated by powers of  $n$  general linear forms. Let  $\wp_1, \dots, \wp_n$  be the ideals of  $n$  general points in  $\mathbb{P}^2$ . (Each point is actually obtained explicitly from the corresponding linear form by duality.) Choose positive integers  $a_1, \dots, a_n$ . Then for any integer  $j \geq \max\{a_i\}$ ,*

$$\dim_k [R/\langle L_1^{a_1}, \dots, L_n^{a_n} \rangle]_j = \dim_k \left[ \wp_1^{j-a_1+1} \cap \cdots \cap \wp_n^{j-a_n+1} \right]_j.$$

From now on, we will denote by

$$\mathcal{L}_2(j; b_1, b_2, \dots, b_n)$$

the linear system  $[\wp_1^{b_1} \cap \cdots \cap \wp_n^{b_n}]_j \subset [R]_j$ . Note that we view it as a vector space, not a projective space, when we compute dimensions. If necessary, in order to simplify notation, we use superscripts to indicate repeated entries. For example,  $\mathcal{L}_2(j; 5^2, 2^3) = \mathcal{L}_2(j; 5, 5, 2, 2, 2)$ .

We will always use the convention that a binomial coefficient  $\binom{a}{r}$  is zero if  $a < r$ . Notice that for every linear system  $\mathcal{L}_2(j; b_1, \dots, b_n)$ , one has

$$\dim_k \mathcal{L}_2(j; b_1, \dots, b_n) \geq \max \left\{ 0, \binom{j+2}{2} - \sum_{i=1}^n \binom{b_i+1}{2} \right\},$$

where the right-hand side is called the *expected dimension* of the linear system. If the inequality is strict, then the linear system  $\mathcal{L}_2(j; b_1, \dots, b_n)$  is called *special*. It is a difficult problem to classify the special linear systems.

Using Cremona transformations, one can relate two different linear systems (see [20], [15], or [10], Theorem 3), which we state only in the form we will need even though the cited results are more general.

**Lemma 2.2.** *Let  $n > 2$  and let  $j, b_1, \dots, b_n$  be non-negative integers, with  $b_1 \geq \cdots \geq b_n$ . Set  $m = j - (b_1 + b_2 + b_3)$ . If  $b_i + m \geq 0$  for all  $i = 1, 2, 3$ , then*

$$\dim_k \mathcal{L}_2(j; b_1, \dots, b_n) = \dim_k \mathcal{L}_2(j + m; b_1 + m, b_2 + m, b_3 + m, b_4, \dots, b_n).$$

The analogous linear systems have also been studied for points in  $\mathbb{P}^r$ . Following [8], the linear system  $\mathcal{L}_r(j; b_1, \dots, b_n)$  is said to be in *standard form* if

$$(r-1)j \geq b_1 + \cdots + b_{r+1} \quad \text{and} \quad b_1 \geq \cdots \geq b_n \geq 0.$$

In particular, for  $r = 2$ , the authors of [8] show that every linear system in standard form is non-special. (This is no longer true if  $r \geq 3$ . For example,  $\mathcal{L}_3(6; 3^9)$  is in standard form and special.)

*Remark 2.3.* Bézout's theorem also provides a useful simplification. Again, we only state the result we need in this paper. Assume the points  $P_1, \dots, P_n$  are general. If  $2j < b_1 + \dots + b_5$ , then

$$\dim \mathcal{L}_2(j; b_1, \dots, b_n) = \dim \mathcal{L}_2(j - 2; b_1 - 1, \dots, b_5 - 1, b_6, \dots, b_n).$$

If  $j < b_1 + b_2$ , then

$$\dim \mathcal{L}_2(j; b_1, \dots, b_n) = \dim \mathcal{L}_2(j - 1; b_1 - 1, b_2 - 1, b_3, \dots, b_n).$$

**Lemma 2.4.** *Let  $P_1, \dots, P_4$  be general points in  $\mathbb{P}^2$  with homogeneous ideals  $\wp_1, \dots, \wp_4$  respectively, and let  $X = \{P_1, \dots, P_4\}$ . Let  $m \geq 1$  be an integer. Then  $I_X^m$  is a saturated ideal, and the minimal free resolution of  $I_X^m$  has the form*

$$0 \rightarrow R(-2m - 2)^m \rightarrow R(-2m)^{m+1} \rightarrow I_X^m \rightarrow 0.$$

In particular,  $I_X^m = \wp_1^m \cap \dots \cap \wp_4^m = I_X^{(m)}$ .

*Proof.* This is well known, since  $X$  is the reduced complete intersection of two conics. See for instance [14], Theorem 2.8, or [6], Corollary 2.10.  $\square$

### 3. PREPARATION

From now on we will consider quotients of the form  $R/I$ , where  $R = k[x, y, z]$ ,  $I = (L_1^k, \dots, L_r^k)$  and  $L_1, \dots, L_r$  are general linear forms. Specifically, we are interested in whether

$$\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$$

has maximal rank for all  $\delta$ , for  $j = 2, 3, 4$  and  $5$ , where  $L$  is a general linear form.

We first observe that the case  $k = 2$  is trivial. Indeed:

- When  $r \leq 2$  the ideal is saturated; hence the map is always an injection.
- When  $r = 3$  it is a complete intersection with  $h$ -vector  $(1, 3, 3, 1)$ . Theorem 2.3 of [13] then gives that  $\times L$  always has maximal rank. From this it follows that  $\times L^2$  is injective when  $\delta = 2$  and surjective when  $\delta = 3$ . Since  $L$  is general, the map  $\times L^3$  is an isomorphism from degree 0 to degree 3.
- When  $r = 4, 5, 6$ , the  $h$ -vector is (respectively)  $(1, 3, 2)$ ,  $(1, 3, 1)$  and  $(1, 3)$ , and all the maximal rank properties follow from the case  $r = 3$  and the fact that  $L$  is general.
- When  $r > 6$  the new generators are not minimal.

Thus from now on we assume  $k \geq 3$ . In section 4 we work with arbitrary  $r$ , but in section 5 we restrict to  $r = 4$ . In this section we give technical preparatory results that will be central to our proofs in section 5. Thus from now on in this section we assume  $r = 4$  (and we return to arbitrary  $r$  in section 4). However, the general approach used in section 4 will also be reflected in our preparation in this section.

We first compute the socle degree (i.e. the last non-zero component) of  $R/I$ . Since  $L_1, L_2, L_3$  are general, without loss of generality we can assume that  $L_1 = x, L_2 = y, L_3 = z$ . Then by a well-known result of Stanley [22] and Watanabe [23],  $\times L_4^k$  has maximal rank in all degrees. The socle degree of  $R/I$  is the last degree where  $\times L_4^k$  is not surjective. Since  $R/(L_1^k, L_2^k, L_3^k)$  has socle degree  $3k - 3$ , one

checks that the socle degree of  $R/I$  is  $2k - 2$ . (This also follows from [16], Lemma 2.5.)

More precisely, we make the following Hilbert function calculation, also using the fact that the Hilbert function of  $R/(x^k, y^k, z^k)$  is symmetric and that of  $R/I$  ends in degree  $2k - 2$ .

degree	0	1	2	...	$k - 2$	$k - 1$	$k$	$k + 1$	...
$R/(x^k, y^k, z^k)$	1	3	6	...	$\binom{k}{2}$	$\binom{k+1}{2}$	$\binom{k+2}{2} - 3$	$\binom{k+3}{2} - 9$	...
$R/I$	1	3	6	...	$\binom{k}{2}$	$\binom{k+1}{2}$	$\binom{k+2}{2} - 4$	$\binom{k+3}{2} - 12$	...
							$2k - 4$	$2k - 3$	$2k - 2$
							$\binom{k+3}{2} - 9$	$\binom{k+2}{2} - 3$	$\binom{k+1}{2}$
							$5k - 9$	$3k - 3$	$k$

In [17, Proposition 2.1] it was observed that for any standard graded algebra  $R/I$ , if  $\times L : [R/I]_{\delta-1} \rightarrow [R/I]_{\delta}$  is surjective, then so is  $\times L : [R/I]_{\delta+i} \rightarrow [R/I]_{\delta+i+1}$  for all  $i \geq 0$ . The same clearly holds for  $\times L^j$  (after adjusting the indices). Furthermore, if  $R/I$  is level and  $\times L : [R/I]_{\delta-1} \rightarrow [R/I]_{\delta}$  is injective, then so is  $\times L : [R/I]_{\delta-i} \rightarrow [R/I]_{\delta-i+1}$  for all  $i \geq 2$ . In our present situation, we conjecture that  $R/I$  is always level:

**Conjecture 3.1.** *If  $R = k[x, y, z]$  and  $I = (L_1^k, \dots, L_4^k)$  with  $L_1, \dots, L_4$  general, then  $R/I$  is level with Cohen-Macaulay type  $k$ .*

However, since we are interested in multiplication by higher powers of  $L$ , it turns out that we do not need  $R/I$  to be level, as we now show.

**Lemma 3.2.** *Let  $M$  be a graded module generated in the first  $m$  degrees, say  $b, b + 1, \dots, b + m - 1$ , for some  $m \geq 1$ . Let  $L$  be a general linear form. If  $j \geq m$  and  $\times L^j : [M]_b \rightarrow [M]_{b+j}$  is surjective, then  $\times L^j : [M]_{b+i} \rightarrow [M]_{b+i+j}$  is also surjective, for all  $i \geq 0$ .*

*Proof.* The module  $M/(L^j M)$  is generated in degree  $\leq b + m - 1$  and is zero in degree  $b + j \geq b + m$ , hence is zero thereafter. □

**Lemma 3.3.** *Let  $I = (L_1^k, \dots, L_4^k)$ , where  $L_1, \dots, L_4$  are general linear forms. Then the socle of  $R/I$  occurs in degree  $2k - 2$  and possibly in degree  $2k - 3$ .*

*Proof.* Since the socle degree of  $R/I$  is  $2k - 2$ , we just have to show that  $R/I$  has no socle in degree  $\leq 2k - 4$ . The ideal  $(L_1^k, L_2^k, L_3^k)$  is a complete intersection, linking the almost complete intersection  $I$  to a Gorenstein ideal  $J$ . Using the formula for the Hilbert function of artinian algebras under liaison (see [7]) and the above Hilbert function calculation, we see that  $R/J$  has socle degree  $(3k - 3) - k = 2k - 3$  and Hilbert function

$$\left( 1, 3, 6, \dots, \binom{k-2}{2}, \binom{k-1}{2}, \binom{k}{2}, \binom{k}{2}, \binom{k-1}{2}, \binom{k-2}{2}, \dots, 6, 3, 1 \right).$$

Let us consider the minimal free resolutions. That of  $I$  has the form

$$\begin{array}{ccccccc}
 & & R(-2k-1)^k & & & & \\
 & & \oplus & & & & \\
 0 & \rightarrow & R(-2k)^a & \rightarrow & G & \rightarrow & R(-k)^4 \rightarrow I \rightarrow 0 \\
 & & \oplus & & & & \\
 & & F & & & & 
 \end{array}$$

where

$$a \geq 0;$$

$$F = \bigoplus_{k+2 \leq i \leq 2k-1} R(-i)^\bullet,$$

$$G = \bigoplus_{k+1 \leq i \leq 2k} R(-i)^\bullet$$

(we do not care what the exponents of the components of  $F$  are because we will show  $F = 0$ , nor do we care what the exponents of the components of  $G$  are).

Linking  $I$  by the complete intersection, the standard mapping cone construction (splitting three copies of  $R(-k)$ ) gives a free resolution for  $J$ :

$$0 \rightarrow R(-2k) \rightarrow G^\vee(-3k) \rightarrow \begin{array}{c} R(1-k)^k \\ \oplus \\ R(-k)^a \\ \oplus \\ F^\vee(-3k) \end{array} \rightarrow J \rightarrow 0$$

where

$$G^\vee(-3k) = R(1-2k)^\bullet \oplus R(2-2k)^\bullet \oplus \cdots \oplus R(-k)^\bullet$$

and

$$F^\vee(-3k) = R(2-2k)^\bullet \oplus \cdots \oplus R(-k-1)^\bullet.$$

Now, any summand of  $G^\vee(-3k)$  of the form  $R(-i)$  for  $i \geq k+2$  must correspond, by the duality of the resolution, to a minimal generator of degree  $2k-i \leq k-2$ , which is forbidden by the Hilbert function. But any minimal generator of  $J$  must be represented in this way, so  $J$  only has generators of degrees  $k-1$  and  $k$ , and  $F = 0$  as desired. But returning to the minimal free resolution of  $I$ , this means that the socle of  $R/I$  is as claimed.  $\square$

The following consequence allows us to confirm the maximal rank property for  $\times L^j$  by checking only two degrees (which sometimes coincide).

**Corollary 3.4.** *Let  $I = (L_1^k, \dots, L_4^k)$  as above. Let  $j \geq 2$ . Let*

$$\begin{aligned} a &:= \max\{\delta \mid h_{R/I}(\delta-j) \leq h_{R/I}(\delta)\}, \\ b &:= \min\{\delta \mid h_{R/I}(\delta-j) \geq h_{R/I}(\delta)\}. \end{aligned}$$

*If  $\times L^j : [R/I]_{a-j} \rightarrow [R/I]_a$  is injective and  $\times L^j : [R/I]_{b-j} \rightarrow [R/I]_b$  is surjective, then  $\times L^j$  has maximal rank in all degrees.*

*Proof.* The fact that  $\times L^j$  is surjective in all degrees  $\geq b$  was noted above and is standard. We have to show the analogous result for injectivity of  $\times L^j$  for all degrees  $\leq a$ .

Consider the canonical module,  $M$ , of  $R/I$ . Since  $R/I$  is artinian,  $M$  is isomorphic to a shift of the  $k$ -dual of  $R/I$ . The injectivity of  $\times L^j : [R/I]_a \rightarrow [R/I]_{a+j}$  is equivalent to the surjectivity of the dual homomorphism on  $M$ , say from  $[M]_{a'}$  to  $[M]_{a'+j}$ . By abuse of notation we continue to write this as  $\times L^j$ .

By Lemma 3.3,  $M$  is generated in the first two degrees, at most. If  $a'$  is not the initial degree of  $M$ , let  $N$  be the truncation of  $M$  in degree  $a'$ , i.e.  $N = \bigoplus_{i \geq a'} [M]_i$ .  $N$  is generated in the first degree, unless  $N = M$ , in which case it may be generated

in the first two degrees. Either way, Lemma 3.2 gives that  $\times L^j$  is surjective in all degrees  $\geq a'$ . Then by duality,  $\times L^j$  is injective in all degrees  $\leq a$ .  $\square$

*Remark 3.5.* Of course it is important to determine the values of  $a$  and  $b$  in order to be able to apply Corollary 3.4. Our method will be to take advantage of the fact that four general points in  $\mathbb{P}^2$  are a complete intersection and use Lemma 2.4. We note here that we will implicitly use the fact that the Hilbert function is unimodal (a fact that is true not just for four powers of linear forms but in fact for any ideal generated by powers of linear forms in  $k[x, y, z]$ ), which is an immediate consequence of the fact that the algebra has the weak Lefschetz property [21], so the unimodality follows from [13], Remark 3.3.

The following is central to determining the values of  $a$  and  $b$  in Corollary 3.4. For any  $\delta$  we have the exact sequence

$$(3.1) \quad [R/(L_1^k, \dots, L_4^k)]_{\delta-j} \xrightarrow{\times L^j} [R/(L_1^k, \dots, L_4^k)]_{\delta} \rightarrow [R/(L_1^k, \dots, L_4^k, L^j)]_{\delta} \rightarrow 0.$$

Let  $\wp_i$  be the point dual to  $L_i$  and let  $\wp_1 \cap \dots \cap \wp_4 = I_X$ . We will define the following functions of  $\delta, j$  and  $k$ :

$$C_1 = \dim[R/(L_1^k, \dots, L_4^k)]_{\delta-j} \quad \text{and} \quad C_2 = \dim[R/(L_1^k, \dots, L_4^k)]_{\delta}.$$

We would like to apply Theorem 2.1. It is certainly no loss of generality to assume that  $\delta \geq k$  since in smaller degrees  $R/I$  coincides with the polynomial ring, where maximal rank holds. Thus we have

$$C_2 = \dim[R/(L_1^k, \dots, L_4^k)]_{\delta} = \dim[\wp_1^{\delta-k+1} \cap \dots \cap \wp_4^{\delta-k+1}]_{\delta} = \dim[I_X^{\delta-k+1}]_{\delta}.$$

We also have

$$C_1 = \dim[R/(L_1^k, \dots, L_4^k)]_{\delta-j} = \begin{cases} \dim[R]_{\delta-j}, & \text{if } \delta \leq j+k-1; \\ \dim[\wp_1^{\delta-j-k+1} \cap \dots \cap \wp_4^{\delta-j-k+1}]_{\delta-j} = [I_X^{\delta-j-k+1}]_{\delta-j}, & \text{if } \delta \geq j+k-1. \end{cases}$$

Since the Hilbert function of  $R/I$  is unimodal (Remark 3.5), we simply need to set  $C_1 - C_2$  equal to zero and find the nearest integer values for  $\delta$ , as we make precise now. We make use of Lemma 2.4.

*Case 1.* First we assume that  $\delta \geq j+k-1$ . Notice that we adopt the convention that  $I_X^0 = R$ . We have the resolutions

$$0 \rightarrow R(-2\delta+2j+2k-4)^{\delta-j-k+1} \rightarrow R(-2\delta+2j+2k-2)^{\delta-j-k+2} \rightarrow I_X^{\delta-j-k+1} \rightarrow 0$$

and

$$0 \rightarrow R(-2\delta+2k-4)^{\delta-k+1} \rightarrow R(-2\delta+2k-2)^{\delta-k+2} \rightarrow I_X^{\delta-k+1} \rightarrow 0.$$

Using this, we have

$$C_1 - C_2 = (\delta - j - k + 2) \binom{-\delta + j + 2k}{2} - (\delta - j - k + 1) \binom{-\delta + j + 2k - 2}{2} - (\delta - k + 2) \binom{-\delta + 2k}{2} + (\delta - k + 1) \binom{-\delta + 2k - 2}{2},$$

from which an elementary but tedious calculation gives

$$(3.2) \quad C_1 - C_2 = 3j\delta - 4kj - 3\binom{j-1}{2} + 3.$$

In particular, we have the following values:

$$(3.3) \quad \begin{array}{c|c} j & C_1 - C_2 \\ \hline 2 & 6\delta - 8k + 3 \\ 3 & 9\delta - 12k \\ 4 & 12\delta - 16k - 6 \\ 5 & 15\delta - 20k - 15 \end{array}$$

*Remark 3.6.* Notice that when  $\delta \geq j + k - 1$  we have

$$\begin{aligned} a &= \max\{\delta \in \mathbb{Z} \mid C_1 - C_2 \leq 0\}, \\ b &= \min\{\delta \in \mathbb{Z} \mid C_1 - C_2 \geq 0\}. \end{aligned}$$

More precisely, using (3.2), an easy calculation gives that for  $\delta \geq j + k - 1$  we get:

- If  $j$  is odd, then

$$a = \begin{cases} 4k_0 + \frac{j-1}{2} - 1, & \text{if } k = 3k_0; \\ 4k_0 + \frac{j-1}{2}, & \text{if } k = 3k_0 + 1; \\ 4k_0 + \frac{j-1}{2} + 1, & \text{if } k = 3k_0 + 2. \end{cases}$$

$$b = \begin{cases} 4k_0 + \frac{j-1}{2} - 1, & \text{if } k = 3k_0; \\ 4k_0 + \frac{j-1}{2} + 1, & \text{if } k = 3k_0 + 1; \\ 4k_0 + \frac{j-1}{2} + 2, & \text{if } k = 3k_0 + 2. \end{cases}$$

- If  $j$  is even, then

$$a = \begin{cases} 4k_0 + \frac{j}{2} - 2, & \text{if } k = 3k_0; \\ 4k_0 + \frac{j}{2} - 1, & \text{if } k = 3k_0 + 1; \\ 4k_0 + \frac{j}{2} + 1, & \text{if } k = 3k_0 + 2. \end{cases}$$

$$b = \begin{cases} 4k_0 + \frac{j}{2} - 1, & \text{if } k = 3k_0; \\ 4k_0 + \frac{j}{2}, & \text{if } k = 3k_0 + 1; \\ 4k_0 + \frac{j}{2} + 2, & \text{if } k = 3k_0 + 2. \end{cases}$$

*Case 2.* Now we assume that  $k \leq \delta \leq j + k - 2$ . Then

$$\begin{aligned} C_1 - C_2 &= \binom{\delta - j + 2}{2} - (\delta - k + 2)\binom{-\delta + 2k}{2} + (\delta - k + 1)\binom{-\delta + 2k - 2}{2} \\ &= \binom{\delta - j + 2}{2} - \binom{-\delta + 2k}{2} + (2\delta - 4k + 3)(\delta - k + 1). \end{aligned}$$

*Remark 3.7.* In proving our main results in the next section, an important issue is that we have two formulas for the value of  $C_1 - C_2$ , depending on the relation between  $\delta$ ,  $j$  and  $k$ . The value of  $C_2$  is not at issue, but the value of  $C_1$  is. We would like to use our formulas from Remark 3.6 to make our calculations in the proofs given in the next section. However, we sometimes need to use values of  $\delta$  as low as  $a - 1$ , and we need to understand which values of  $\delta$ ,  $j$  and  $k$  force us to use Case 2 above instead of Case 1.



4. MULTIPLICATION BY  $L^2$

For ideals generated by powers of linear forms in  $k[x, y, z]$ , the following two results are known.

**Theorem 4.1** ([21], Main Theorem). *An artinian quotient of  $k[x, y, z]$  by powers of (arbitrary) linear forms has WLP.*

In the following sections we will see that multiplication by  $L^j$  for  $j \geq 3$  does not necessarily have maximal rank, even when  $I$  is an almost complete intersection. This leaves the issue of maximal rank for  $\times L^2$ . In [9] and [5] it is shown that there exist ideals generated by powers of linear forms for which  $\times L^2$  does not have maximal rank in all degrees, so the remaining question is what happens for powers of general linear forms. When  $I$  is an almost complete intersection we have the following result:

**Theorem 4.2** ([18], Proposition 4.7). *Let  $L_1, \dots, L_4, L$  be five general linear forms of  $R = k[x, y, z]$ . Let  $I$  be the ideal  $(L_1^{a_1}, \dots, L_4^{a_4})$ . Let  $A = R/I$ . Then, for each integer  $j$ , the multiplication map  $\times L^2 : [A]_{j-2} \rightarrow [A]_j$  has maximal rank.*

Improving on this result, our next goal will be to prove that if  $I$  is generated by any number,  $r$ , of uniform powers of general linear forms, then  $R/I$  has the property that  $\times L^2$  has maximal rank in all degrees. Since the case  $r \leq 4$  is already known, we will assume that  $r \geq 5$ . Recall that any ideal generated by uniform powers of general linear forms has WLP. In particular, its Hilbert function is unimodal, and we will now determine its peak(s).

**Lemma 4.3.** *Let  $L_1, \dots, L_r \in k[x, y, z]$  be  $r \geq 5$  general linear forms. Let  $I$  be the ideal  $(L_1^k, \dots, L_r^k)$ . Write  $k = (r - 1)k_0 + e$  with  $0 \leq e \leq r - 2$ . The following hold:*

- (i) *If  $2 \leq k \leq r - 2$ , then  $R/I$  has exactly one peak at  $k - 1$ .*
- (ii) *If  $k_0 \geq 1$  and  $1 \leq e \leq r - 2$ , then  $R/I$  has exactly one peak at  $rk_0 + e - 1$ .*
- (iii) *If  $k_0 \geq 1$  and  $e = 0$ , then  $R/I$  has exactly two peaks at  $rk_0 - 2$  and  $rk_0 - 1$ .*

*Proof.* (i) For  $2 \leq k \leq r - 2$ , we have

$$\begin{aligned} \dim[R/I]_{k-2} &= \dim R_{k-2} = \binom{k}{2}, \\ \dim[R/I]_{k-1} &= \dim R_{k-1} = \binom{k+1}{2} \text{ and} \\ \dim[R/I]_k &= \binom{k+2}{2} - r. \end{aligned}$$

Hence,  $R/I$  has a peak at  $k - 1$ .

(ii) Let us first assume that  $k_0 \geq 2$ . We say  $A_i = \dim[R/I]_{rk_0+e-1+i}$  with  $i = -1, 0, 1$ . We have to prove that  $A_0 - A_i > 0$  for  $i = -1, 1$ . Let us compute  $A_i$  for  $i = -1, 0, 1$ . Since  $k_0 \geq 2$  we can apply Theorem 2.1 and get

$$\begin{aligned} A_i &= \dim[R/I]_{rk_0+e-1+i} = \dim_k \mathcal{L}_2(rk_0 + e - 1 + i; (k_0 + i)^r) \\ &= \binom{rk_0 + e + 1 + i}{2} - r \binom{k_0 + i + 1}{2} \end{aligned}$$

where the last equality follows from the fact that the linear system

$$\mathcal{L}_2(rk_0 + e - 1 + i; k_0 + i^r)$$

is in standard form and, hence, it is non-special. Now, we easily check that  $A_0 - A_{-1} = e > 0$  and  $A_0 - A_1 = r - 1 - e > 0$ . Therefore,  $R/I$  has a peak at  $rk_0 + e - 1$ .

For  $k_0 = 1$  we have

$$\dim[R/I]_{k-1} = \dim R_{k-1} = \binom{k+1}{2},$$

$$\dim[R/I]_k = \binom{k+2}{2} - r \text{ and}$$

$$\dim[R/I]_{k+1} = \dim \mathcal{L}_2(k+1; 2^r) = \binom{k+3}{2} - 3r.$$

Since  $\dim[R/I]_k - \dim[R/I]_{k-1} = k + 1 - r > 0$  and  $\dim[R/I]_k - \dim[R/I]_{k+1} = 2r - (k + 2) > 0$ ,  $R/I$  has a peak at  $k$ .

(iii) First we assume that  $k_0 \geq 2$ . Denote  $B_i = \dim[R/I]_{rk_0-2+i}$  with  $i = -1, 0, 1, 2$ . We have

$$B_i = \dim[R/I]_{rk_0-2+i} = \dim_k \mathcal{L}_2(rk_0-2+i; (k_0-1+i)^r) = \binom{rk_0+i}{2} - r \binom{k_0+i}{2}.$$

Notice that if  $k_0 = 2$ , we have  $B_{-1} = \binom{2r-1}{2}$ . In all cases, we get  $B_0 = B_1$ ,  $B_0 - B_2 = r - 1$  and  $B_0 - B_{-1} = r - 1$  and we conclude that  $R/I$  has exactly two peaks, at  $rk_0 - 2$  and  $rk_0 - 1$ .

Finally for  $k_0 = 1$ , we have  $B_{-1} = \dim[R]_{r-3} = \binom{r-1}{2}$ ,  $B_0 = \dim[R]_{r-2} = \binom{r}{2}$  and

$$B_i = \dim[R/I]_{r-2+i} = \dim_k \mathcal{L}_2(r-2+i; i^r) = \binom{r+i}{2} - r \binom{i+1}{2}$$

for  $i = 1, 2$ . (This follows either from the fact that it is in standard form or from the Alexander-Hirschowitz theorem [1].) Again  $B_0 = B_1$ ,  $B_0 - B_2 = r - 1$ ,  $B_0 - B_{-1} = r - 1$  and  $R/I$  has exactly two peaks, at  $r - 2$  and  $r - 1$ .  $\square$

**Theorem 4.4.** *Let  $L_1, \dots, L_r, L \in k[x, y, z]$  be  $r + 1$  general linear forms. Let  $I$  be the ideal  $(L_1^k, \dots, L_r^k)$ . Then, for each integer  $j$ , the multiplication map*

$$\times L^2 : [R/I]_{j-2} \longrightarrow [R/I]_j$$

*has maximal rank.*

*Proof.* We write  $k = (r - 1)k_0 + e$  with  $0 \leq e \leq r - 2$  and we distinguish two cases.

*Case 1.*  $k_0 \geq 1$ . We distinguish 3 subcases:

1.1. Assume  $e = 0$ . In this case the result follows from Lemma 4.3 and the fact that, for any integer  $j$ , the multiplication map  $\times L : [R/I]_{j-1} \longrightarrow [R/J]_j$  has maximal rank.

1.2. Assume  $1 \leq e \leq \frac{r-1}{2}$ . By Lemma 4.3,  $R/I$  has exactly one peak, at  $rk_0 + e - 1$ , and, moreover,  $A_{-1} = \dim[R/I]_{rk_0+e-2} \geq A_1 = \dim[R/I]_{rk_0+e}$ . So, we only need to check that  $[R/(I, L^2)]_{rk_0+e} = 0$  since this will imply the surjectivity of  $\times L^2 : [R/I]_{rk_0+e-2} \rightarrow [R/I]_{rk_0+e}$ . We have

$$\begin{aligned} \dim \operatorname{coker}(\times L^2)_{rk_0+e} &= \dim[R/(L_1^{(r-1)k_0+e}, \dots, L_r^{(r-1)k_0+e}, L^2)]_{rk_0+e} && \text{(by (3.1))} \\ &= \dim[\wp_1^{k_0+1} \cap \dots \cap \wp_r^{k_0+1} \cap \wp^{rk_0+e-1}]_{rk_0+e} && \text{(by Thm 2.1)} \\ &= \dim \mathcal{L}_2(rk_0 + e; (k_0 + 1)^r, rk_0 + e - 1) \\ &= \dim \mathcal{L}_2(rk_0 + e - r; k_0^r, rk_0 + e - 1 - r) && \text{(by Remark 2.3)} \\ &= \dots && \text{(by Remark 2.3)} \\ &= \dim \mathcal{L}_2(e; 1^r, e - 1) && \text{(by Remark 2.3)} \\ &= 0 && \text{(since } 2e+1 \leq r \text{)}. \end{aligned}$$

1.3. Assume  $\frac{r-1}{2} < e \leq r - 2$ . By Lemma 4.3,  $R/I$  has exactly one peak at  $rk_0 + e - 1$  and, moreover,  $\dim[R/I]_{rk_0+e} - \dim[R/I]_{rk_0+e-2} = A_1 - A_{-1} = (A_1 - A_0) - (A_{-1} - A_0) = 2e - r + 1 > 0$ . Hence we have to show that  $\times L^2$  is injective, with cokernel of dimension  $2e - r + 1$ . Let us compute  $\dim[R/(I, L^2)]_{rk_0+e}$ . As above we have

$$\begin{aligned} \dim \operatorname{coker}(\times L^2)_{rk_0+e} &= \dim[R/(L_1^{(r-1)k_0+e}, \dots, L_r^{(r-1)k_0+e}, L^2)]_{rk_0+e} && \text{(by (3.1))} \\ &= \dim[\wp_1^{k_0+1} \cap \dots \cap \wp_r^{k_0+1} \cap \wp^{rk_0+e-1}]_{rk_0+e} && \text{(by Thm 2.1)} \\ &= \dim \mathcal{L}_2(rk_0 + e; (k_0 + 1)^r, rk_0 + e - 1) \\ &= \dim \mathcal{L}_2(rk_0 + e - r; k_0^r, rk_0 + e - 1 - r) && \text{(by Remark 2.3)} \\ &= \dots && \text{(by Remark 2.3)} \\ &= \dim \mathcal{L}_2(e; 1^r, e - 1) && \text{(by Remark 2.3)} \\ &= \binom{e+2}{2} - \binom{e}{2} - r \\ &= 2e - r + 1 \end{aligned}$$

as expected.

Case 2 ( $k_0 = 0$ ). In this case we have  $k = e$ . It immediately follows from the equalities

$$\dim[R/(I, L^2)]_k = \dim \mathcal{L}_2(k; 1^r, k - 1) = \max \left\{ 0, \binom{k+2}{2} - \binom{k}{2} - r \right\}$$

and

$$\dim[R/I]_k - \dim[R/I]_{k-2} = \binom{k+2}{2} - r - \binom{k}{2}.$$

□

Experimentally it seems that an even more general result is true, namely, to remove the assumption of uniform powers, but we have not been able to prove it apart from the case of almost complete intersections mentioned earlier:

**Conjecture 4.5.** *For any artinian quotient of  $k[x, y, z]$  generated by powers of general linear forms and for a general linear form  $L$ , multiplication by  $L^2$  has maximal rank in all degrees.<sup>1</sup>*

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<sup>1</sup>This conjecture was recently proven simultaneously by C. Almeida and A. Andrade [2], and by J. Migliore and U. Nagel [19].

5. MULTIPLICATION BY  $L^j$  FOR  $3 \leq j \leq 5$

Let  $L \in k[x, y, z]$  be a general linear form. As noted in the introduction, the main result of [21] shows that for any ideal generated by powers of linear forms,  $\times L$  has maximal rank in all degrees. As we look to multiplication by successively larger powers of  $L$ , we will see that the maximal rank property in all degrees quickly erodes away. For uniform powers it is already known that if the linear forms are not general, then  $\times L^2$  does not necessarily always have maximal rank ([9], [5]). On the other hand, for ideals generated by arbitrary powers of four general linear forms ([18]) and for ideals generated by uniform powers of any number of general linear forms (Theorem 4.4),  $\times L^2$  does have maximal rank.

In this section we study what happens for ideals of uniform powers of four general linear forms under multiplication by  $L^3, L^4$  and  $L^5$ . The problem is trivial for  $k \leq 2$ , so we assume  $k \geq 3$ . In Theorem 5.3 we assume  $k \geq 4$  because the socle degree is too small when  $k = 3$ ; maximal rank holds trivially in all degrees in this case.

In this section we prove our main results, which we separate into the following theorems. A good part of the proofs will be merged using the setup from section 3.

**Theorem 5.1** (Multiplication by  $L^3$ ). *Let  $I = (L_1^k, \dots, L_4^k)$ , where  $L_1, \dots, L_4$  are general linear forms and  $k \geq 3$ .*

- (i) *If  $k \cong 0 \pmod{3}$ , set  $k = 3k_0$ . Then for  $\delta = 4k_0$  we have  $\dim[R/I]_{\delta-3} = \dim[R/I]_{\delta}$ , and  $\times L^3$  fails by exactly one to be an isomorphism between these components. In all other degrees,  $\times L^3$  has maximal rank.*
- (ii) *If  $k \not\cong 0 \pmod{3}$ , then  $\times L^3$  has maximal rank in all degrees.*

**Theorem 5.2** (Multiplication by  $L^4$ ). *Let  $I = (L_1^k, \dots, L_4^k)$ , where  $L_1, \dots, L_4$  are general linear forms and  $k \geq 3$ .*

- (i) *If  $k \cong 0 \pmod{3}$ , then  $\times L^4$  has maximal rank in all degrees.*
- (ii) *If  $k \cong 1 \pmod{3}$ , set  $k = 3k_0 + 1$ . Then  $\times L^4$  fails surjectivity by 1 from degree  $4k_0 - 2$  to degree  $4k_0 + 2$ . In all other degrees  $\times L^4$  has maximal rank.*
- (iii) *If  $k \cong 2 \pmod{3}$ , set  $k = 3k_0 + 2$ . Then  $\times L^4$  fails injectivity by 1 from degree  $4k_0 - 1$  to degree  $4k_0 + 3$ . In all other degrees,  $\times L^4$  has maximal rank.*

**Theorem 5.3** (Multiplication by  $L^5$ ). *Let  $I = (L_1^k, \dots, L_4^k)$ , where  $L_1, \dots, L_4$  are general linear forms and  $k \geq 4$ .*

- (i) *If  $k \cong 0 \pmod{3}$ , set  $k = 3k_0$ . Then  $\dim[R/I]_{4k_0-4} = \dim[R/I]_{4k_0+1}$  and  $\times L^5$  fails by 3 to be an isomorphism. In all other degrees,  $\times L^5$  has maximal rank.*
- (ii) *If  $k \cong 1 \pmod{3}$ , set  $k = 3k_0 + 1$ . Then  $\times L^5$  fails injectivity by 1 from degree  $4k_0 - 3$  to degree  $4k_0 + 2$ . In all other degrees,  $\times L^5$  has maximal rank.*
- (iii) *If  $k \cong 2 \pmod{3}$ , set  $k = 3k_0 + 2$ . Then  $\times L^5$  fails surjectivity by 1 from degree  $4k_0 - 1$  to degree  $4k_0 + 4$ . In all other degrees,  $\times L^5$  has maximal rank.*

The arguments for all of the cases of Theorems 5.1, 5.2 and 5.3 are more or less the same, using Theorem 2.1, Lemma 2.2, Remark 2.3 and Lemma 2.4. We will carefully explain one case and leave the rest to the reader.

*Remark 5.4.* The multiplication by  $L^j$  is reflected in the exact sequence (3.1). We have four scenarios.

- To prove that  $\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$  is surjective, we have to prove that  $\dim[R/(I, L^j)]_{\delta} = 0$ .
- To prove that  $\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$  fails surjectivity by  $\epsilon$ , we have to prove that we expect surjectivity (i.e. that  $C_1 - C_2 \geq 0$  in Remark 3.5) and that  $\dim[R/(I, L^j)]_{\delta} = \epsilon$ .
- To prove that  $\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$  is injective, we have to prove that  $\dim[R/(I, L^j)]_{\delta} = -(C_1 - C_2)$ .
- To prove that  $\times L^j : [R/I]_{\delta-j} \rightarrow [R/I]_{\delta}$  fails injectivity by  $\epsilon$ , we have to prove that we expect injectivity (i.e. that  $C_1 - C_2 \leq 0$  in Remark 3.5) and that  $\dim[R/(I, L^j)]_{\delta} = -(C_1 - C_2) + \epsilon$ .

The issue of the value of  $C_1 - C_2$  was discussed at the end of section 3. The mere fact that  $\dim[R/(I, L^j)]_{\delta} > 0$  does not tell us which case we are in.

With minor differences, the proofs of all parts of these theorems follow the same lines. We will prove Theorem 5.1(i) and Theorem 5.3(i) here. ‘‘Low’’ values of  $k$  have to be dealt with separately, in keeping with Remark 3.7.

Assume that  $k = 3k_0$  and that  $j$  is odd. Then by inspection for  $k = 3, 6, 9$  and by Remark 3.6 for  $k \geq 12$ , we have

$$a = b = 4k_0 + \frac{j-1}{2} - 1 = 4k_0 + \frac{j-3}{2},$$

and when  $\delta = a$  we get

$$C_1 - C_2 = 0$$

for both  $j = 3$  and  $j = 5$ . This proves the equality of the dimensions, asserted in Theorem 5.1(i) and Theorem 5.3(i). We compute

$$\begin{aligned} \dim \operatorname{coker}(\times L^j)_{4k_0 + \frac{j-3}{2}} &= \dim[R/(L_1^{3k_0}, \dots, L_4^{3k_0}, L^j)]_{4k_0 + \frac{j-3}{2}} \\ &= \dim[\wp_1^{k_0 + \frac{j-1}{2}} \cap \dots \cap \wp_4^{k_0 + \frac{j-1}{2}} \cap \wp^{4k_0 - \frac{j+1}{2}}]_{4k_0 + \frac{j-3}{2}} \\ &= \dim[\wp_1^{k_0 + \frac{j-1}{2} - 1} \cap \dots \cap \wp_4^{k_0 + \frac{j-1}{2} - 1} \cap \wp^{4k_0 - \frac{j+3}{2}}]_{4k_0 + \frac{j-7}{2}} \end{aligned}$$

Note that the first equality follows from (3.1), the second from Theorem 2.1 and the third from Remark 2.3. If  $j = 3$ , this is

$$\dim[\wp_1^{k_0} \cap \dots \cap \wp_4^{k_0} \cap \wp^{4k_0-3}]_{4k_0-2} = \dim[\wp_1^{k_0-1} \cap \dots \cap \wp_4^{k_0-1} \cap \wp^{4k_0-4}]_{4k_0-4}$$

(by Remark 2.3). If we denote by  $\ell_i$  the equation of the line joining the point corresponding to  $\wp_i$  to the point corresponding to  $\wp$ , then  $(\ell_1 \ell_2 \ell_3 \ell_4)^{k_0-1}$  defines the unique non-zero element (up to scalar multiplication) in this vector space, so this dimension is 1 as claimed, proving the failure of isomorphism claimed in Theorem 5.1(i).

Now assume  $j = 5$ . Since  $k \geq 4$  (else the socle degree is too small for  $\times L^5$  to be non-trivial), we have  $k_0 \geq 2$  and

$$\begin{aligned} \dim[\wp_1^{k_0+1} \cap \dots \cap \wp_4^{k_0+1} \cap \wp^{4k_0-4}]_{4k_0-1} &= \dim[\wp_1^{k_0} \cap \dots \cap \wp_4^{k_0} \cap \wp^{4k_0-5}]_{4k_0-3} \\ &= \dim[\wp_1^{k_0-1} \cap \dots \cap \wp_4^{k_0-1} \cap \wp^{4k_0-6}]_{4k_0-5}. \end{aligned}$$

Note that the equalities follow from Remark 2.3. If  $k_0 = 2$  this is easily computed to be 3, as claimed. If  $k_0 \geq 3$ , we use the other part of Remark 2.3 to obtain

$$\dim[\wp_1^{k_0-2} \cap \dots \cap \wp_4^{k_0-2} \cap \wp^{4k_0-10}]_{4k_0-9}.$$

We can continue to apply this remark until we obtain

$$\dim[\wp_1 \cap \dots \cap \wp_4 \cap \wp^2]_3 = 3$$

as desired. This proves the failure of isomorphism claimed in Theorem 5.3(i).

To finish Theorem 5.1(i) and Theorem 5.3(i), we have to prove surjectivity when  $\delta \geq b + 1$  and injectivity for  $\delta \leq a - 1$ . The proof of Corollary 3.4 shows that it is enough to prove surjectivity for  $\delta = b + 1$  and injectivity for  $\delta = a - 1$ .

Note that if we replace  $b$  by  $b + 1$  in the calculations above, then the same argument will yield

$$\dim[\wp_1^{k_0 + \frac{j-1}{2}} \cap \dots \cap \wp_4^{k_0 + \frac{j-1}{2}} \cap \wp^{4k_0 - \frac{j+3}{2} + 1}]_{4k_0 + \frac{j-7}{2} + 1},$$

which is

$$\begin{cases} \dim[\wp_1^{k_0+1} \cap \dots \cap \wp_4^{k_0+1} \cap \wp^{4k_0-2}]_{4k_0-1} & \text{if } j = 3, \\ \dim[\wp_1^{k_0+2} \cap \dots \cap \wp_4^{k_0+2} \cap \wp^{4k_0-3}]_{4k_0} & \text{if } j = 5, \end{cases}$$

which reduces to

$$\begin{cases} \dim[\wp_1^{k_0} \cap \dots \cap \wp_4^{k_0} \cap \wp^{4k_0-3}]_{4k_0-3} & \text{if } j = 3, \\ \dim[\wp_1^{k_0-1} \cap \dots \cap \wp_4^{k_0-1} \cap \wp^{4k_0-6}]_{4k_0-6} & \text{if } j = 5, \end{cases}$$

and these are both clearly 0, since for a curve of degree  $d$  to have a singularity of degree  $d$  at a point  $p$ , it must be a union of lines through  $p$  (up to multiplicity), and in both cases such a union of lines cannot account for the remaining singularities. This gives our surjectivity.

Now let  $\delta = a - 1$ . We want to show that

$$\dim[R/I]_{a-1} - \dim[R/I]_{a-1-j} = \dim[R/(I, L^j)]_{a-1}.$$

For  $j = 3$ , we just have to show the result for  $k = 3$  and  $k = 6$  separately since we can use Remark 3.6 (adjusted so that  $a - 1 \geq j + k - 1$ ) for larger  $k$ . For  $j = 5$ , similarly we only have to show the cases  $k = 3, 6, 9$  separately. These are tedious but easy calculations using the above methods (or on a computer). So we can also compute the case  $k_0 = 3$  when  $j = 3$ , and now assume  $k_0 \geq 4$  for both values of  $j$ .

So instead assume that  $a - 1 = 4k_0 + \frac{j-1}{2} - 2$  and let  $C_1$  and  $C_2$  be the values obtained from setting  $\delta = a - 1$ . We get from (3.2) that

$$C_1 - C_2 = 3j \left( 4k_0 + \frac{j-1}{2} - 2 \right) - 4(3k_0)j - 3 \binom{j-1}{2} + 3 = -3j.$$

Now we compute

$$\begin{aligned} \dim[R/(I, L^j)]_{a-1} &= \dim[\wp_1^{a-k} \cap \dots \cap \wp_4^{a-k} \cap \wp^{a-j}]_{a-1} \\ &= \dim[\wp_1^{k_0 + \frac{j-1}{2} - 1} \cap \dots \cap \wp_4^{k_0 + \frac{j-1}{2} - 1} \cap \wp^{4k_0 - \frac{j+1}{2} - 1}]_{4k_0 + \frac{j-1}{2} - 2}. \end{aligned}$$

As long as  $k_0 > \frac{j-1}{2} + 1$ , which is true in our case, we can split off four lines.

First assume  $j = 3$ . We have

$$\dim[\wp_1^{k_0} \cap \dots \cap \wp_4^{k_0} \cap \wp^{4k_0-3}]_{4k_0-1}.$$

We continue to split off four lines at a time,  $n$  times. We can do this as long as

$$(k_0 - n) + (4k_0 - 3 - 4n) > 4k_0 - 1 - 4n;$$

i.e. we can do this  $k_0 - 2$  times. We reduce to

$$\dim[\wp_1^2 \cap \dots \cap \wp_4^2 \cap \wp^5]_7.$$

Applying Lemma 2.2 twice, we see this is equal to  $9 = 3j$ , as desired.

When  $j = 5$ , we have

$$\dim[\wp_1^{k_0+1} \cap \dots \cap \wp_4^{k_0+1} \cap \wp^{4k_0-4}]_{4k_0}.$$

We can split off four lines at a time, arriving at

$$\dim[\wp_1^4 \cap \dots \cap \wp_4^4 \cap \wp^8]_{12}.$$

Again applying Lemma 2.2 twice, we obtain  $15 = 3j$  as desired.

*Remark 5.5.* For  $\times L^j$  with  $j \geq 6$ , we have checked using [4] that failure of maximal rank occurs in more than one degree for all sufficiently large values of  $k$ . Of course this can be confirmed with our methods.

### 6. FINAL REMARKS

As a nice application of our approach to analyze whether ideals generated by powers of linear forms have SLP we have the following result.

**Proposition 6.1.** *Let  $L_1, \dots, L_5 \in R$  be general linear forms and  $k \geq 3$ . Consider the ideals  $I = (L_1^k, \dots, L_4^k)$  and  $J = (L_1^k, \dots, L_4^k, L_5^k)$ . Then,  $R/I$  and  $R/J$  have the same socle degree, namely,  $2k - 2$ .*

*Proof.* As we pointed out at the beginning of section 3 the socle degree of  $R/I$  is  $2k - 2$ . To prove that  $R/J$  has socle degree  $2k - 2$  it is enough to check that

$$\times L_5^k : [R/I]_{k-2} \longrightarrow [R/I]_{2k-2}$$

is not surjective or, equivalently, that  $[R/J]_{2k-2} \neq 0$ . Arguing as in the previous sections, we have

$$\dim[R/J]_{2k-2} = \dim[\wp_1^{k-1} \cap \dots \cap \wp_5^{k-1}]_{2k-2} = 1,$$

which proves what we want. □

*Remark 6.2.* It is natural to ask what happens for ideals generated by uniform powers of more than four general linear forms. Here is what happens for 6 general linear forms, at least experimentally using [4]. Here,  $\times L^j$  fails to have maximal rank in all degrees for the following values of  $k$  (taking  $3 \leq j \leq 10, 3 \leq k \leq 30$ ).

$j$	$k$
3	5, 10, 15, 20, 25, 30
4	7, 8, 12, 13, 17, 18, 22, 23, 27, 28
5	9, 10, 11, 14, 15, 16, 19, 20, 21, 24, 25, 26, 29, 30
6	9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28, 29
7	11, ..., 30
8	10, 13, ..., 30
9	12, 13, 15, ..., 30
10	14, ..., 30

It would be very interesting to extend the approach of Theorem 4.4 to handle more than four general linear forms and prove an asymptotic result following the patterns visible here.

*Remark 6.3.* The reader will note that some care was taken in this paper to precisely identify various important degrees and to compute Hilbert functions in those degrees in order to obtain our results. Very recently, the paper [19] appeared, in which it was shown that often one can find shortcuts to simplify some of these arguments.

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