PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 2, February 2018, Pages 695–704
http://dx.doi.org/10.1090/proc/13750
Article electronically published on August 30, 2017

DISTORTION OF EMBEDDINGS OF BINARY TREES INTO DIAMOND GRAPHS

SIU LAM LEUNG, SARAH NELSON, SOFIYA OSTROVSKA, AND MIKHAIL OSTROVSKII

(Communicated by Thomas Schlumprecht)

Abstract. Diamond graphs and binary trees are important examples in the theory of metric embeddings and also in the theory of metric characterizations of Banach spaces. Some results for these families of graphs are parallel to each other; for example superreflexivity of Banach spaces can be characterized both in terms of binary trees (Bourgain, 1986) and diamond graphs (Johnson-Schechtman, 2009). In this connection, it is natural to ask whether one of these families admits uniformly bilipschitz embeddings into the other. This question was answered in the negative by Ostrovskii (2014), who left it open to determine the order of growth of the distortions. The main purpose of this paper is to get a sharp up-to-a-logarithmic-factor estimate for the distortions of embeddings of binary trees into diamond graphs and, more generally, into diamond graphs of any finite branching $k \geq 2$. Estimates for distortions of embeddings of diamonds into infinitely branching diamonds are also obtained.

1. Introduction

Binary trees and diamond graphs play an important role in the theory of metric embeddings and metric characterizations of properties of Banach spaces; see [3–8, 10, 11, 13, 18] and also presentations in the books [12, 16].

Some results for these families of graphs are parallel to each other; for example superreflexivity of Banach spaces can be characterized both in terms of binary trees (Bourgain [3]) and diamond graphs (Johnson-Schechtman [6]). In this connection, it is natural to ask whether these families of graphs admit bilipschitz embeddings with uniformly bounded distortions one into another. In one direction the answer is clear: The fact that diamond graphs do not admit uniformly bilipschitz embeddings into binary trees follows immediately from the combination of the result of Rabinovich and Raz [17, Corollary 5.3] stating that the distortion of any embedding of an n-cycle into any tree is $\geq \frac{n}{3} - 1$ and the observation that large diamond graphs contain large cycles isometrically. As for the opposite direction, it was proved in [13] that binary trees do no admit uniformly bilipschitz embeddings into diamond graphs. The goal of this paper is to get a sharp-up-to-a-logarithmic-factor estimate for the distortions of embeddings of binary trees into diamond graphs and, more generally, into diamond graphs of any finite branching $k \geq 2$. In addition, estimates for distortions of embeddings of diamonds into infinitely branching diamonds are obtained.

Received by the editors August 8, 2016, and, in revised form, March 28, 2017.

2010 *Mathematics Subject Classification.* Primary 46B85; Secondary 05C12, 30L05.

Key words and phrases. Binary tree, diamond graph, distortion of a bilipschitz embedding, Lipschitz map.

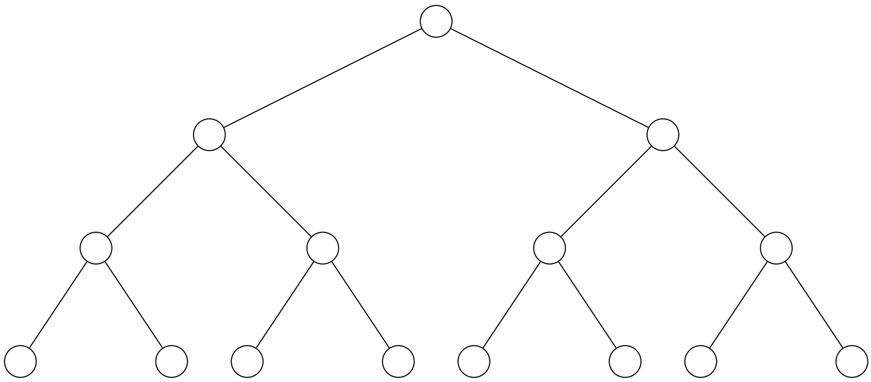

FIGURE 1. The binary tree of depth 3, that is, T_3.

2. Definitions and the main result

To begin with, let us present the necessary definitions.

Definition 2.1. A *binary tree of depth n*, denoted T_n, is a finite graph in which each vertex is represented by a finite (possibly empty) sequence of 0's and 1's, of length at most n. Two vertices in T_n are adjacent if the sequence representing one of them is obtained from the sequence representing the other by adding one term on the right. (For example, vertices corresponding to $(1,1,1,0)$ and $(1,1,1,0,1)$ are adjacent.) Vertices which correspond to sequences of length k are called vertices of k-th *generation*. The vertex corresponding to the empty sequence is called a *root*. If a sequence τ is an initial segment of the sequence σ we say that σ is a *descendant* of τ and that τ is an *ancestor* of σ. See Figure 1 for a sketch of T_3.

Definition 2.2 ([5])**.** Diamond graphs $\{D_n\}_{n=0}^\infty$ are defined inductively as follows: The *diamond graph* of level 0 is denoted by D_0. It has two vertices joined by an edge. The *diamond graph D_n* is obtained from D_{n-1} as follows. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. See Figure 2 for a sketch of D_2.

All graphs considered in this paper are endowed with the shortest path distance: the distance between any two vertices is the number of edges in a shortest path between them.

Definition 2.3. Let M be a finite metric space and $\{R_n\}_{n=1}^\infty$ be a sequence of finite metric spaces with increasing cardinalities. The *distortion* $c_R(M)$ of embeddings of M into $\{R_n\}_{n=1}^\infty$ is defined as the infimum of $C \geq 1$ for which there is $n \in \mathbb{N}$, a map $f : M \to R_n$, and a number $r = r(f) > 0$ — called the *scaling factor* — satisfying the condition:

$$\forall u, v \in M \quad r d_M(u,v) \leq d_{R_n}(f(u), f(v)) \leq rC d_M(u,v). \tag{1}$$

Therefore, $c_D(T_n)$ is the infimum of distortions of embeddings of the binary tree T_n into diamond graphs. Our main result is expressed by the following assertion:

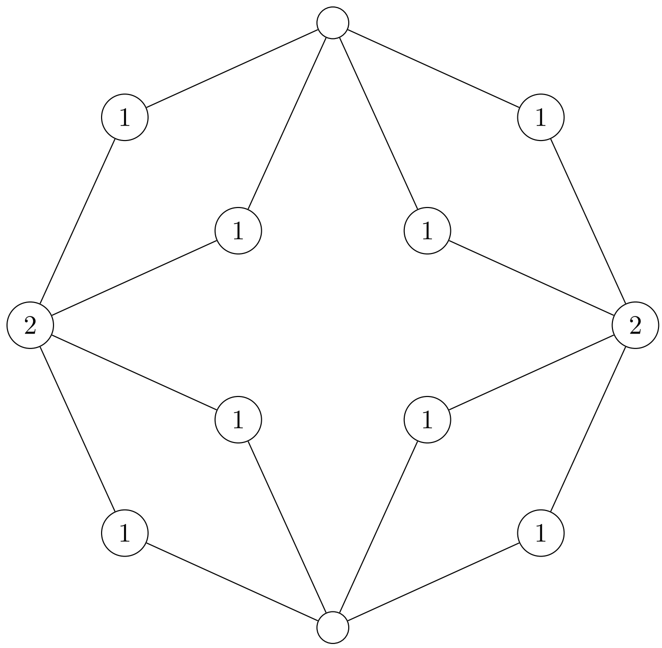

FIGURE 2. Diamond D_2 in which generations of vertices are shown.

Theorem 2.4. *There exists a constant* $c > 0$ *such that*

$$c\,\frac{n}{\log_2 n} \leq c_D(T_n) \leq 2n$$

for all $n \geq 2$.

In recent years [1, 9, 14, 15] we see an increasing interest in diamonds of high branching (see Definition 2.5). In view of this, we prove versions of Theorem 2.4 for such graphs (Theorems 2.6 and 2.7).

Definition 2.5. Fix $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$. Let $D_{0,k}$ be a graph consisting of two vertices joined by one edge. The graph $D_{n+1,k}$ is obtained from $D_{n,k}$ if we replace each edge uv in $D_{n,k}$ by a set of k paths of length 2 joining u and v. We call the graphs $D_{n,k}$ *diamonds of branching* k if k is finite and *diamonds of infinite branching* if $k = \infty$.

Call one of the vertices of $D_{0,k}$ the *top* and the other the *bottom*. Define the *top* and the *bottom* of $D_{n,k}$ as vertices which evolved from the top and the bottom of $D_{0,k}$, respectively. A *subdiamond* of $D_{n,k}$ is a subgraph which evolved from an edge of some $D_{m,k}$ for $0 \leq m \leq n$. The *top* and *bottom* of a subdiamond of $D_{n,k}$ are defined as the vertices of the subdiamond which are the closest to the top and bottom of $D_{n,k}$, respectively.

It can be noticed that $D_n = D_{n,2}$. Let $c_{(D,k)}(M)$ denote the distortion of embeddings of a finite metric space M into $\{D_{n,k}\}$, as in Definition 2.3. The next generalization of Theorem 2.4 holds.

Theorem 2.6. *If k is finite, then there exists $c(k) > 0$ such that*

$$c(k)\frac{n}{\log_2 n} \leq c_{(D,k)}(T_n) \leq 2n$$

for all $n \geq 2$.

For infinitely branching diamonds, the following weaker version of Theorem 2.4 is valid:

Theorem 2.7. *There exists constant $c(\infty) > 0$ such that*

$$c(\infty)\sqrt{n} \leq c_{(D,\infty)}(T_n) \leq 2n.$$

We refer to [2] for graph-theoretical terminology and to [12] for terminology of the theory of metric embeddings.

3. Estimates from above

Since D_n is isometric to a subset of $D_{n,k}$ whenever $k \geq 2$, it suffices to prove the estimate from above for the binary diamonds $\{D_n\}$.

Proof of $c_D(T_n) \leq 2n$. Observe that the diamond D_k contains isometrically the tree which is customarily denoted $K_{1,2^k}$. This tree has $2^k + 1$ vertices, and one of the vertices is incident to the remaining 2^k vertices. In fact, one can easily establish by induction that the bottom of the diamond D_k has degree 2^k, and the bottom together with all of its neighbors forms the desired tree.

Choose k in such a way that $2^k + 1 \geq 2^{n+1} - 1$, where $2^{n+1} - 1$ is the number of vertices in T_n.

Now, map the root of T_n to the bottom of D_k and all of the other vertices of T_n to distinct vertices adjacent to the bottom. Denote the obtained map by F_n and the vertex set of T_n by $V(T_n)$. We claim that the following inequalities are true:

$$\forall u, v \in V(T_n) \quad \frac{1}{n} d_{T_n}(u, v) \leq d_{D_k}(F_n(u), F_n(v)) \leq 2 d_{T_n}(u, v), \tag{2}$$

yielding $c_D(T_n) \leq 2n$.

Indeed, the right-hand side inequality follows from the fact that any distance between two distinct vertices in $K_{1,2^k}$ does not exceed 2.

To justify the left-hand side inequality consider the two cases:

(1) One of the vertices, say u, is the root of T_n. Then $d_{D_k}(F_n(u), F_n(v)) = 1$ and $d_{T_n}(u, v) \leq n$. The left-hand side inequality in this case follows.

(2) Neither u nor v is the root of T_n. Then $d_{D_k}(F_n(u), F_n(v)) = 2$ and $d_{T_n}(u, v) \leq 2n$. The left-hand side inequality follows in this case, too. □

4. Estimates from below

4.1. **Diamonds of finite branching.** Observe that Theorem 2.4 is a special case of Theorem 2.6. For this reason, only the lower estimate of Theorem 2.6 has to be proved.

Proof of $c_{(D,k)}(T_n) \geq c(k)\,\frac{n}{\log_2 n}$. Fix an integer k ($2 \leq k < \infty$) for the whole proof and omit from most of our notation dependence on k, as it is clear that almost all of the introduced objects depend on k. If $\alpha_n = 3c_{(D,k)}(T_n)$, then there exists a map F_n of $V(T_n)$ into $V(D_{m(n),k})$ for some $m(n) \in \mathbb{N}$ satisfying (1) with $C = \alpha_n$ and the scaling factor being an integer power of 2, that is,

$$(3) \quad \forall u, v \in V(T_n) \quad 2^{p(n)} d_{T_n}(u,v) \le d_{D_{m(n),k}}(F_n(u), F_n(v)) \le \alpha_n 2^{p(n)} d_{T_n}(u,v)$$

for some $p(n) \in \mathbb{Z}$. If $p(n) < 0$, we compose the map F_n with the natural map of $D_{m(n),k}$ into $D_{m(n)-p(n),k}$. As the latter map increases all distances into $2^{-p(n)}$ times, the resulting map has scaling factor equal to 1. Therefore, one may assume without loss of generality that $p(n) \ge 0$.

Now our goal is to show that the existence of $n, r, d \in \mathbb{N}$, such that the condition $1 \le r < n$ is satisfied simultaneously with the following three inequalities:

$$(4) \qquad 2^{d-1} > \alpha_n 2^{p(n)}(r+1),$$

$$(5) \qquad (2k)^{d-p(n)} < 2^r,$$

$$(6) \qquad 2^d < 2^{p(n)}(n-r),$$

leads to a contradiction.

We introduce generations of vertices in diamonds, including the case $k = \infty$, as follows. Generations are labelled recursively from the end in the following way. *Generation number* 1 in $D_{m,k}$ is the set of vertices which appeared in the last step of the construction of $D_{m,k}$. Further, *generation number* 2 is the set of vertices which appeared in the previous step of the construction, and so on. In this way, one obtains m *generations*, while the two original vertices do not belong to any of the generations. See Figure 2 for generations in D_2. This definition leads to the following:

Observation 4.1. Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, $k \ge 2$.

(1) Let v be a vertex of generation number d in $D_{m,k}$, where $d \in \{1, \dots, m\}$. Then the 2^{d-1}-neighborhood of v consists of two subdiamonds of diameter 2^{d-1} each, pasted together at v.

(2) Let Z_d be the set of all vertices of generation number d in $D_{m,k}$. Then the connected components of $D_{m,k} \backslash Z_d$ have diameters strictly less than 2^d.

Recall that generations for vertices of T_n are defined in the standard way: the *generation* of a vertex in T_n is its distance to the root.

Let n and r satisfy the conditions above, so $1 \le r < n$. Consider any vertex τ_{n-r} of generation $n-r$ in T_n. For a path $\tau_0, \dots, \tau_{n-r}$ joining the root τ_0 and τ_{n-r}, inequality (3) implies that $d_{D_{m(n),k}}(F_n(\tau_i), F_n(\tau_{i+1})) \le \alpha_n 2^{p(n)}$ and $d_{D_{m(n),k}}(F_n(\tau_0), F_n(\tau_{n-r})) \ge 2^{p(n)}(n-r)$. Combining these inequalities with condition (6) and Observation 4.1(2), one concludes that there exists $i \in \{0, \dots, n-r\}$ such that $d_{D_{m(n),k}}(F_n(\tau_i), v) \le \alpha_n 2^{p(n)}$ for some v of generation d in $D_{m(n),k}$.

By inequalities (3), (4) and Observation 4.1(1), F_n maps descendants of τ_i (in T_n) of generations $i+1, \dots, i+r$ (note that $i+r \le n$) into the union of two subdiamonds of diameter 2^{d-1} each, pasted together at v. To obtain a contradiction to (5) we need the following lemma:

Lemma 4.2. *The cardinality of a $2^{p(n)}$-separated set — i.e., a set satisfying $d(u,v) \ge 2^{p(n)}$ for any $u \ne v$ — in a subdiamond of $D_{m,k}$ of diameter 2^q does not exceed $k \cdot (2k)^{q-p(n)}$ if $q \ge p(n)$.*

Proof. It is easy to see that each subdiamond of $D_{m,k}$ of diameter $2^{p(n)}$ contains at most k vertices out of each $2^{p(n)}$-separated set. The number of subdiamonds of diameter $2^{p(n)}$ in a diamond of diameter 2^q is equal to the number of edges in the diamond of diameter $2^{q-p(n)}$. This number of edges is $(2k)^{q-p(n)}$, because in each step of the construction of diamonds the diameter doubles and the number of edges is multiplied by $(2k)$. □

This contradicts (5) because, on one hand, the vertex τ_i has more than 2^r descendants in the next r generations, and the images of these descendants, by the bilipschitz condition (3), should form a $2^{p(n)}$-separated set. On the other hand, Lemma 4.2 implies that a $2^{p(n)}$-separated set in a union of two diamonds of diameters 2^{d-1} does not exceed $2 \cdot k \cdot (2k)^{d-1-p(n)} = (2k)^{d-p(n)}$.

Since $\{c_{(D,k)}(T_n)\}_{n=2}^{\infty}$ and $\{\frac{n}{\log_2 n}\}_{n=2}^{\infty}$ are sequences of positive numbers and $\alpha_n = 3c_{(D,k)}(T_n)$, to prove the existence of a constant $c(k) > 0$ such that $c_{(D,k)}(T_n) \geq c(k)\,\frac{n}{\log_2 n}$, it suffices to show that the existence of the subsequence of values of n for which $\alpha_n = o(\frac{n}{\log_2 n})$ leads to a contradiction. This will be done by demonstrating that the existence of such a subsequence implies the existence of n, r and d satisfying $1 \leq r < n$ and (4)–(6). Let us rewrite inequalities (4)–(6) as

$$2^{d-p(n)} > 2\alpha_n(r+1), \tag{7}$$

$$(2^{(d-p(n))})^{\log_2(2k)} < 2^r, \tag{8}$$

$$2^{d-p(n)} < n - r. \tag{9}$$

Set $r = r(n) = \lceil \log_2(2k) \cdot \log_2 n \rceil$. Since k is fixed, for sufficiently large n, one has $n - r > 2$. Define $d = d(n) \in \mathbb{N}$ to be the largest integer for which (9) holds. It has to be pointed out that with this choice of d, the inequality $2^{d-p(n)} > \frac{n}{4}$ holds when n is sufficiently large. Since for our choice of r we have $2\alpha_n(r(n)+1) = o(n)$ for the corresponding subsequence of values of n, it is clear that, for sufficiently large n in the subsequence, the condition (7) is also satisfied. It remains to observe that, with the described choice of r, the inequality (8) follows from

$$(2^{d-p(n)})^{\log_2(2k)} < n^{\log_2(2k)}.$$

Since by virtue of (9), $2^{d-p(n)} < n$, the last inequality is obvious. □

4.2. **Diamonds of infinite branching.** In this case, the methods based on the upper bounds for cardinalities of $2^{p(n)}$-separated sets in subdiamonds are not applicable since the cardinalities are infinite. Consequently, the method of [13], which gives weaker estimates but works in the case of infinite branching, will be employed.

Proof of Theorem 2.7. Since the upper estimate has already been established in section 3, to complete the proof it has to be shown that $c_{(D,\infty)}(T_n) \geq c(\infty)\sqrt{n}$ for some constant $c(\infty) > 0$.

If $\alpha_n = 3c_{(D,\infty)}(T_n)$, then there exists a map F_n of $V(T_n)$ into $V(D_{m(n),\infty})$ for some $m(n) \in \mathbb{N}$ satisfying (1) with $C = \alpha_n$ and the scaling factor being an integer power of 2, that is,

$$\forall u, v \in V(T_n) \quad 2^{p(n)} d_{T_n}(u,v) \leq d_{D_{m(n),\infty}}(F_n(u), F_n(v)) \leq \alpha_n 2^{p(n)} d_{T_n}(u,v) \tag{10}$$

for some $p(n) \in \mathbb{Z}$. With the help of the same argument as in Theorem 2.6, it may be assumed that $p(n) \geq 0$.

Since $\{c_{(D,\infty)}(T_n)\}_{n=1}^{\infty}$ as well as $\{\sqrt{n}\}_{n=1}^{\infty}$ are sequences of positive numbers, it suffices to prove that the inequality of the form $c_{(D,\infty)}(T_n) \geq c\sqrt{n}$ holds for some $c > 0$ and sufficiently large n.

Assume that $n > 9$ and denote by $d = d(n)$ the largest integer satisfying

$$2^d < 2^{p(n)} \cdot \left\lfloor \frac{n}{3} \right\rfloor . \tag{11}$$

Consider any vertex $\tau_{\lfloor \frac{n}{3} \rfloor}$ of generation $\lfloor \frac{n}{3} \rfloor$ in T_n. Let $\tau_0, \ldots, \tau_{\lfloor \frac{n}{3} \rfloor}$ be a path joining the root τ_0 and $\tau_{\lfloor \frac{n}{3} \rfloor}$ in T_n. Inequality (10) implies that

$$d_{D_{m(n),\infty}}(F_n(\tau_i), F_n(\tau_{i+1})) \leq \alpha_n 2^{p(n)}$$

and

$$d_{D_{m(n),\infty}}(F_n(\tau_0), F_n(\tau_{\lfloor \frac{n}{3} \rfloor})) \geq 2^{p(n)} \left\lfloor \frac{n}{3} \right\rfloor .$$

By combining these inequalities with condition (11) and Observation 4.1(2), we conclude that there exists $i \in \{0, \ldots, \lfloor \frac{n}{3} \rfloor\}$ such that

$$d_{D_{m(n),\infty}}(F_n(\tau_i), v) \leq \alpha_n 2^{p(n)} \tag{12}$$

for some v of generation d in $D_{m(n),\infty}$.

Inequality (10) together with (12) implies that descendants of τ_i of generation n in T_n will be mapped onto vertices whose distances from v are at least $(n-i)2^{p(n)} - \alpha_n 2^{p(n)}$. One has

$$(n-i)2^{p(n)} - \alpha_n 2^{p(n)} \geq \left(\frac{2}{3} n - \alpha_n \right) 2^{p(n)}. \tag{13}$$

If $\alpha_n \geq \frac{n}{3}$, the conclusion of the theorem holds with $c(\infty) = \frac{1}{9}$. Therefore, assume that $\alpha_n \leq \frac{n}{3}$. In this case, the right-hand side of (13) is not less than

$$\frac{n}{3} 2^{p(n)} \overset{(11)}{>} 2^d > 2^{d-1}.$$

Thence, on each path joining τ_i with one of its descendants of generation n (in T_n) there is a vertex which is mapped by F_n outside the union of two subdiamonds of height 2^{d-1} with the common vertex at v.

Let x_1 and x_2 be the different from v tops/bottoms of the subdiamonds mentioned in the previous paragraph. The statement about the paths mentioned in the previous paragraph implies that on each path joining τ_i with one of its descendants (in T_n) of generation n there is a vertex, the F_n-image of which is at distance at most $\alpha_n 2^{p(n)}$ from either x_1 or x_2. In fact, it is clear that this condition holds for the first vertex on the path whose F_n-image is outside the union of the subdiamonds.

Now, let us fix such a path and estimate from below the generation r of the first vertex on this path, whose F_n-image is outside the union of the subdiamonds. It can be seen by using inequality (10) that the earliest generation r for which it is possible for such image to be outside the union of the subdiamonds has to satisfy

$$(r-i)2^{p(n)}\alpha_n + 2^{p(n)}\alpha_n > 2^{d-1},$$

whence

$$(r-i) \geq \frac{2^{d-1}}{2^{p(n)}\alpha_n} - 1.$$

The choice of d (see the line preceding (11)) implies that

$$2^{d+1} \geq 2^{p(n)} \cdot \left\lfloor \frac{n}{3} \right\rfloor , \tag{14}$$

and hence

$$(r-i) \geq \frac{1}{4\alpha_n}\left\lfloor\frac{n}{3}\right\rfloor - 1.$$

Next, consider four different descending paths in T_n starting at different descendants of τ_i of generation $(i+2)$. Along each of these paths we pick the first vertex whose F_n-image is outside the union of the two subdiamonds. Let v_1, v_2, v_3, and v_4 be the picked vertices and suppose that these vertices belong to generations r_1, r_2, r_3, and r_4, respectively, in T_n.

First, assume that $r_j > i+2$ for $j = 1, 2, 3, 4$, while the case where $r_j = i+2$ for some j will be considered at the very end of the proof. By the argument above, each r_j satisfies

$$(r_j - i) \geq \frac{1}{4\alpha_n}\left\lfloor\frac{n}{3}\right\rfloor - 1.$$

Therefore the pairwise distances between vertices v_1, v_2, v_3, and v_4 are at least:

$$r_{j_1} + r_{j_2} - 2i - 2 \geq \frac{1}{2\alpha_n}\left\lfloor\frac{n}{3}\right\rfloor - 4, \quad j_1, j_2 \in \{1,2,3,4\},\ j_1 \neq j_2.$$

The argument above implies that under the assumption $r_j > i+2$ the image $F_n(v_j)$ is at distance at most $2^{p(n)}\alpha_n$ to either x_1 or x_2. As a result, at least two of these images are at distance at most $2 \cdot 2^{p(n)}\alpha_n$ from each other. Using (10) one concludes that

$$2^{p(n)}\left(\frac{1}{2\alpha_n}\left\lfloor\frac{n}{3}\right\rfloor - 4\right) \leq 2 \cdot 2^{p(n)}\alpha_n.$$

It is easy to see that this inequality implies that $\alpha_n \geq c\sqrt{n}$ for some constant $c > 0$ and sufficiently large n.

Now comes the case where $r_j = i+2$ for some $j \in \{1,2,3,4\}$. In this case, the distance between $F_n(v_j)$ and v, on one hand, is $> 2^{d-1}$, and, on the other hand, by (12) and (10), it is $\leq 3\alpha_n 2^{p(n)}$. This leads to $3\alpha_n 2^{p(n)} > 2^{d-1}$. Combining with (14), one obtains

$$3\alpha_n 2^{p(n)} > 2^{p(n)-2} \cdot \left\lfloor\frac{n}{3}\right\rfloor.$$

Thus, $\alpha_n \geq \frac{1}{12} \cdot \left\lfloor\frac{n}{3}\right\rfloor$, yielding that in this case $c_{(D,\infty)}(T_n) \geq \frac{1}{36} \cdot \left\lfloor\frac{n}{3}\right\rfloor$. This inequality is sufficient for our purposes. This completes the proof of Theorem 2.7. □

5. Acknowledgments

The work on this paper was inspired by a question on the order of distortion of embeddings of binary trees into diamonds, asked by Assaf Naor during the National AMS meeting in Baltimore, Maryland, 2014. The question about analogues of the main result for diamonds with high branching was asked by Thomas Schlumprecht during the Workshop in Analysis and Probability, Texas A&M University, 2016.

The last-named author gratefully acknowledges the support by National Science Foundation DMS-1201269 and by the Summer Support of Research program of St. John's University during different stages of work on this paper. Part of the work on this paper was done when the last-named author was a participant in the NSF supported Workshop in Analysis and Probability, Texas A&M University, 2016.

The authors would like to thank the referee for a careful reading of the paper and for suggesting important improvements to the presentation.

References

[1] F. Baudier, R. Causey, S. J. Dilworth, D. Kutzarova, N. L. Randrianarivony, Th. Schlumprecht, and S. Zhang, On the geometry of the countably branching diamond graphs, J. Funct. Anal., to appear, `https://doi.org/10.1016/j.jfa.2017.05.013`

[2] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. MR2368647

[3] J. Bourgain, *The metrical interpretation of superreflexivity in Banach spaces*, Israel J. Math. **56** (1986), no. 2, 222–230, DOI 10.1007/BF02766125. MR880292

[4] Bo Brinkman and Moses Charikar, *On the impossibility of dimension reduction in l_1*, J. ACM **52** (2005), no. 5, 766–788, DOI 10.1145/1089023.1089026. MR2176562

[5] Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair, *Cuts, trees and l_1-embeddings of graphs*, Combinatorica **24** (2004), no. 2, 233–269, DOI 10.1007/s00493-004-0015-x. MR2071334

[6] William B. Johnson and Gideon Schechtman, *Diamond graphs and super-reflexivity*, J. Topol. Anal. **1** (2009), no. 2, 177–189, DOI 10.1142/S1793525309000114. MR2541760

[7] Benoît R. Kloeckner, *Yet another short proof of Bourgain's distortion estimate for embedding of trees into uniformly convex Banach spaces*, Israel J. Math. **200** (2014), no. 1, 419–422, DOI 10.1007/s11856-014-0024-4. MR3219585

[8] J. R. Lee and A. Naor, *Embedding the diamond graph in L_p and dimension reduction in L_1*, Geom. Funct. Anal. **14** (2004), no. 4, 745–747, DOI 10.1007/s00039-004-0473-8. MR2084978

[9] James R. Lee and Prasad Raghavendra, *Coarse differentiation and multi-flows in planar graphs*, Discrete Comput. Geom. **43** (2010), no. 2, 346–362, DOI 10.1007/s00454-009-9172-4. MR2579701

[10] Ilan Newman and Yuri Rabinovich, *A lower bound on the distortion of embedding planar metrics into Euclidean space*, Discrete Comput. Geom. **29** (2003), no. 1, 77–81, DOI 10.1007/s00454-002-2813-5. MR1946795

[11] Mikhail I. Ostrovskii, *On metric characterizations of some classes of Banach spaces*, C. R. Acad. Bulgare Sci. **64** (2011), no. 6, 775–784. MR2884975

[12] Mikhail I. Ostrovskii, *Metric embeddings*, Bilipschitz and coarse embeddings into Banach spaces, De Gruyter Studies in Mathematics, vol. 49, De Gruyter, Berlin, 2013. MR3114782

[13] Mikhail Ostrovskii, *Metric characterizations of superreflexivity in terms of word hyperbolic groups and finite graphs*, Anal. Geom. Metr. Spaces **2** (2014), 154–168, DOI 10.2478/agms-2014-0005. MR3210894

[14] Mikhail Ostrovskii and Beata Randrianantoanina, *Metric spaces admitting low-distortion embeddings into all n-dimensional Banach spaces*, Canad. J. Math. **68** (2016), no. 4, 876–907, DOI 10.4153/CJM-2015-041-7. MR3518997

[15] Mikhail I. Ostrovskii and Beata Randrianantoanina, *A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces*, J. Funct. Anal. **273** (2017), no. 2, 598–651, DOI 10.1016/j.jfa.2017.03.017. MR3648862

[16] Gilles Pisier, *Martingales in Banach spaces*, Cambridge Studies in Advanced Mathematics, vol. 155, Cambridge University Press, Cambridge, 2016. MR3617459

[17] Y. Rabinovich and R. Raz, *Lower bounds on the distortion of embedding finite metric spaces in graphs*, Discrete Comput. Geom. **19** (1998), no. 1, 79–94, DOI 10.1007/PL00009336. MR1486638

[18] Oded Regev, *Entropy-based bounds on dimension reduction in L_1*, Israel J. Math. **195** (2013), no. 2, 825–832, DOI 10.1007/s11856-012-0137-6. MR3096576

Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242
E-mail address: sleung1@kent.edu

Department of Mathematics and Statistics, Hunter College, CUNY, New York, New York 10065
E-mail address: sarah.nelson07@yahoo.com

Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey
E-mail address: sofia.ostrovska@atilim.edu.tr

Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, New York 11439
E-mail address: ostrovsm@stjohns.edu