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DIVISION RINGS WITH RANKS

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Abstract. Any superrosy division ring is shown to be centrally finite. Furthermore, division rings satisfying a generalized chain condition on definable subgroups are studied. In particular, a division ring of burden n has dimension at most n over its center, and any definable group of definable automorphisms of a field of burden n has size at most n. Additionally, an alternative proof that division rings interpretable in o-minimal structures are algebraically closed, real closed or the quaternions over a real closed field is given.

1. Introduction

An important aspect in model theory is to analyze algebraic properties of mathematical objects, such as groups and rings, which satisfy certain combinatorial behavior on their definable sets. Some of these combinatorial patterns (e.g. ω-stability, stability, simplicity) yield the existence of suitable chain conditions among definable groups as well as well-behaved rank functions among definable sets. These are important tools to study algebraic properties of groups and rings.

A milestone in classifying fields from a model-theoretic point of view is a result of Macintyre [15] which states that any infinite ω-stable field is algebraically closed. This was generalized to the superstable context by Cherlin and Shelah [3] and therefore, by previous work of Cherlin [2], in fact any infinite superstable division ring is an algebraically closed field. Moreover, Pillay, Scanlon and Wagner [22] showed that a wider class of division rings, namely the supersimple ones, are commutative and have indeed trivial Brauer group. In all these cases the existence of a suitable ordinal-valued rank function plays an essential role. A more general context in which a similar rank function is present is the class of superrosy division rings which includes, besides all before mentioned fields, division rings interpretable in o-minimal structures. In the definable context, such division rings were first analyzed by Otero, Peterzil and Pillay [19] and later by Peterzil and Steinhorn [20]. They were characterized to be either algebraically closed fields, real closed fields or the quaternions over a real closed field. In particular, there exist non-commutative superrosy division rings.

The first part of this paper is dedicated to the study of superrosy division rings using the aforementioned rank function from a purely axiomatic point of view.

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We show that any infinite superrosy division ring is centrally finite (Theorem 2.9). This gives a uniform treatment to all previously mentioned cases.

One consequence of the presence of a well-behaved rank function is the non-existence of an infinite descending chain of definable subgroups, each having infinite index in its predecessor. Weakenings of such a forbidden pattern appear naturally in wider model theoretic classes, such as simple or NTP_2 theories. In these frameworks, a more general notion of rank, namely *weight* in simple theories and *burden* in NTP_2 theories, can be defined. Our aim is to analyze division rings in these contexts. For instance, using machinery from simplicity theory and classical results on division rings such as the Cartan-Brauer-Hua Theorem, we show that a division ring with a simple theory of weight one must be commutative (Corollary 3.5). Furthermore, we study division rings of finite burden, using only a generalized chain condition on definable subgroups. These include division rings interpretable in o-minimal structures and in simple structures of finite rank (e.g. any pseudofinite field or more generally any perfect PAC field with small Galois group). Other examples, which are neither o-minimal nor simple, are pseudo real closed fields with small Galois group [17]. Moreover, a recent result of Chernikov and Simon [5] gives rise to numerous examples: any ultra-product of the p-adics, $\mathbb{R}((t))$, $\mathbb{C}((t))$, $\mathbb{R}((t^{\mathbb{Q}}))$ and $F((t^{\mathbb{Q}}))$ where F is any perfect PAC field with small Galois group, among others. Indeed, all these examples have burden 1. We show that any division ring of burden n has dimension at most n over its center, and in particular a division ring of burden 1 is commutative. Moreover, we prove that a field of burden n and characteristic zero has at most n many definable automorphisms (Proposition 4.10).

The last section includes a remark on division rings interpretable in o-minimal structures. We apply the above results to show that any such division ring is either an algebraically closed field, a real closed field or the quaternions over a real closed field (Theorem 5.1). This gives an alternative approach to the aforementioned result of Peterzil and Steinhorn.

2. Ranks à la Lascar

In this section we aim to study superrosy division rings. For an exposition on superrosy groups and fields, we refer the interested reader to [6].

As being centrally finite and superrosy are both properties of the theory of a division ring, we may work (if necessary) in a saturated elementary extension. In fact, all results presented here extend to arbitrary $\aleph_0$-saturated division rings carrying an ordinal-valued rank function among definable factors (in the pure language of rings), i.e.

$$\mathrm{rk} : \{\text{Definable factors}\} \to \mathrm{Ord},$$

which satisfies the following properties:

(1) A definable factor has rank zero if and only if it is finite.
(2) The rank is preserved under definable bijections.
(3) The Lascar inequalities: For a definable subgroup H of a definable group G we have that

$$\mathrm{rk}(H) + \mathrm{rk}(G/H) \leq \mathrm{rk}(G) \leq \mathrm{rk}(H) \oplus \mathrm{rk}(G/H),$$

where the function $\oplus$ is the smallest symmetric strictly increasing function f among pairs of ordinals such that $f(\alpha, \beta + 1) = f(\alpha, \beta) + 1$. More precisely, every ordinal α can be written in the Cantor normal form as a finite sum $\omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$

for ordinals $\alpha_1 > \ldots > \alpha_k$ and natural numbers $n_1, \ldots, n_k$. If additionally $\beta = \omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_k} \cdot m_k$, then $\alpha \oplus \beta$ is defined to be $\omega^{\alpha_1} \cdot (n_1 + m_1) + \ldots + \omega^{\alpha_k} \cdot (n_k + m_k)$.

Observe that in superrosy division rings, the thorn U-rank satisfies the above properties; see [6, 18].

The existence of such a rank yields some immediate consequences on definable groups of a superrosy division ring, which we state below.

Remark 2.1. Let G and H be two definable groups and let $f : H \to G$ be a definable group morphism. Then

$$\mathrm{rk}(\mathrm{Ker} f) + \mathrm{rk}(\mathrm{Im} f) \leq \mathrm{rk}(H) \leq \mathrm{rk}(\mathrm{Ker} f) \oplus \mathrm{rk}(\mathrm{Im} f).$$

Thus, if f is injective, then H and G have the same rank if and only if $\mathrm{Im} f$ has finite index in G. In particular, if H is a subgroup of G, then H and G have the same rank if and only if H has finite index in G.

Remark 2.2. As there is no infinite strictly descending chain of ordinals, any infinite descending chain of definable groups, each of them having infinite index in its predecessor, stabilizes after finitely many steps. In addition, as in a division ring the centralizer of any set is a subdivision ring, any descending chain of centralizers of finite (even definable) sets stabilizes after finitely many steps. In fact, notice that an easy compactness argument yields the existence of a natural number n such that the centralizer of any set equals the centralizers of a subset of size n. In particular, they are definable. Therefore, since in any group G for any sets X and Y we have that $C_G(X) \lneq C_G(Y)$ if and only if $C_G(C_G(X)) \gneq C_G(C_G(Y))$, we obtain the *chain condition on centralizers*, i.e. all (ascending or descending) chains of centralizers of arbitrary subsets are finite.

Definition 2.3. Let X be a definable set of rank $\omega^\alpha \cdot n + \beta$ with $\beta < \omega^\alpha$ and n a positive natural number. A definable subset Y of X is *wide* in X if it has rank at least $\omega^\alpha \cdot n$ and *negligible* with respect to X if its rank is strictly smaller than ω^α. If there is no confusion we simply say that Y is wide or respectively negligible.

Lemma 2.4. *Any superrosy division ring has finite dimension (as a vector space) over any definable non-negligible subdivision ring.*

Proof. Let D be a superrosy division ring of rank $\omega^\alpha \cdot n + \beta$ with $\beta < \omega^\alpha$ and n a positive natural number. Suppose, towards a contradiction, that there is a definable subdivision ring D_0 of rank greater than or equal to ω^α such that D has infinite dimension over D_0. Thus for $1 = \lambda_0, \lambda_1, \ldots, \lambda_n$ linearly independent over D_0 we obtain

$$\begin{aligned}
\mathrm{rk}(D) &\geq \mathrm{rk}\Big(\bigoplus_{i \leq n} D_0\lambda_i\Big) \geq \mathrm{rk}(D_0\lambda_n) + \mathrm{rk}\Big(\bigoplus_{i \leq n} D_0\lambda_i \Big/ D_0\lambda_n\Big) \\
&\geq \omega^\alpha + \mathrm{rk}\Big(\bigoplus_{i < n} D_0\lambda_i\Big) \geq \cdots \geq \omega^\alpha \cdot (n+1) \\
&> \mathrm{rk}(D),
\end{aligned}$$

which yields a contradiction. □

Recall that two groups H and N are said to be *commensurable* if their intersection $H \cap N$ has finite index in each of them. For the following lemma we use Schlichting's Theorem [23], generalized by Bergmann and Lenstra [1] to obtain an invariant subgroup, up to commensurability. See also [24, Theorem 4.2.4].

Fact 2.5 (Schlichting's Theorem). Let $\mathfrak{F}$ be a family of uniformly commensurable subgroups of a group G, i.e. there is a natural number k such that for any H and H^* in $\mathfrak{F}$ the index $[H : H \cap H^*]$ is at most k. Then there is a subgroup N commensurable with any H in $\mathfrak{F}$, which is invariant under any automorphism of G stabilizing $\mathfrak{F}$ setwise.

Lemma 2.6. *Any wide definable additive subgroup of a superrosy division ring has finite index.*

Proof. Let D be a superrosy division ring, which we may assume to be $\aleph_0$-saturated, of rank $\omega^\alpha \cdot n + \beta$ with $\beta < \omega^\alpha$ and n a positive natural number. Suppose towards a contradiction that there is a definable additive subgroup A of D of infinite index and of rank greater than or equal to $\omega^\alpha \cdot n$. Consider the family of $D^\times$-translates of A. As there is no infinite descending chain of definable subgroups, each of them having infinite index in its predecessor, there is a finite intersection H of $D^\times$-translates of A such that for any λ in $D^\times$ the index $[H : H \cap \lambda H]$ is finite. As left translation is an automorphism, any two translates of H are commensurable and hence, a compactness argument yields that the family $\{\lambda H : \lambda \in D^\times\}$ is uniformly commensurable. Thus, by Schlichting's Theorem, we can find a $D^\times$-invariant additive subgroup I, i.e. an ideal of D, which is commensurable with H. As H has infinite index in D, the ideal I has to be trivial and hence H is finite. Now, as there is no infinite decreasing sequence of ordinal numbers above $\omega^\alpha \cdot n$, there exists a finite intersection N of $D^\times$-translates of A such that $\mathrm{rk}(N) \geq \omega^\alpha \cdot n$ and for any λ in $D^\times$ either N and λN are commensurable or $\mathrm{rk}(N \cap \lambda N) < \omega^\alpha \cdot n$. As H is finite, there is some λ in $D^\times$ such that N and λN are not commensurable and hence $\mathrm{rk}(N \cap \lambda N) < \omega^\alpha \cdot n$. Together with the following inequality:

$$\omega^\alpha \cdot n \leq \mathrm{rk}(N) \leq \mathrm{rk}(N \cap \lambda N) \oplus \mathrm{rk}(N/N \cap \lambda N),$$

we get that $\mathrm{rk}(N/N \cap \lambda N) \geq \omega^\alpha$. Hence,

$$\begin{aligned} \mathrm{rk}(N + \lambda N) &\geq \mathrm{rk}(\lambda N) + \mathrm{rk}(N + \lambda N/\lambda N) \\ &= \mathrm{rk}(N) + \mathrm{rk}(N/N \cap \lambda N) \\ &\geq \omega^\alpha \cdot n + \omega^\alpha \\ &> \mathrm{rk}(D), \end{aligned}$$

which yields a contradiction and finishes the proof. □

Corollary 2.7. *Let D be a superrosy division ring. If a definable group morphism from D^+ or $D^\times$ to D^+ has a negligible kernel, its image has finite index in D^+.*

Proof. As the kernel is negligible, by Remark 2.1 the image is wide and thus the previous lemma yields the statement. □

Before stating and proving the main result of this section, we show the following lemma which holds under the mere assumption that $D^\times$ satisfies a weak chain condition on centralizers.

Lemma 2.8. *Let D be a non-commutative division ring without an infinite ascending chain of centralizers of an element, and suppose that for any non-central element a in D the center $Z(D)$ is contained in $a^D - a$. Then, for any natural number m, any element of $Z(D)$ has an mth root in D.*

Proof. By Kaplansky's Theorem [14, Theorem 15.15], we can find an element a in D for which none of its powers belong to $Z(D)$. Moreover, after replacing a by one of its powers, we may assume that $C_D(a) = C_D(a^n)$ for any natural number n.

Now, let m be a natural number and let c be a central element. As a^m is non-central, by assumption there is some x in D such that $(a^m)^x - a^m = c$. Observe that a and a^x commute since

$$C_D(a^x) = C_D(a)^x = C_D(a^m)^x = C_D((a^m)^x) = C_D(a^m) = C_D(a)$$

and so

$$(a^x a^{-1})^m - ca^{-m} = ((a^m)^x - c)a^{-m} = 1.$$

Furthermore, as ca^{-m} is also non-central, one can find an element y in D with $(ca^{-m})^y - ca^{-m} = 1$. Thus, similarly as above we get that $C_D(a^y) = C_D(a)$ and so a, a^x and a^y commute. Finally, since the mth power $(a^x a^{-1})^m$ equals $(ca^{-m})^y$, we obtain that c equals $(a^x a^{-1} a^y)^m$ and so it has an mth root. □

Theorem 2.9. *A division ring with a superrosy theory has finite dimension over its center.*

Proof. Suppose, towards a contradiction, that there is a superrosy division ring which has infinite dimension over its center and let D be such a division ring of minimal rank. As D is clearly infinite its rank must be non-zero.

The proof consists of a series of steps.

Step 1. Any proper centralizer has finite dimension over its center:

Since any proper centralizer is a subdivision ring and thus as an additive group has infinite index in D, by Lascar inequalities its rank is strictly smaller than the rank of D. Hence, we conclude by the choice of D.

Step 2. D has infinite dimension over any proper centralizer. In particular, the centralizer of a non-central element is negligible:

Otherwise, it has finite dimension over some subfield by Step 1 and thus it would be finite dimensional over its center by [14, Theorem 15.8]. The second part is an immediate consequence by Lemma 2.4.

Now, for an arbitrary element a, let $\delta_a : D \to D$ be defined as $\delta_a(u) = au - ua$. To ease notation, we also write $[a, u]$ for $au - ua$.

Step 3. For a non-central element a, the map δ_a is a surjective derivation whose kernel is the centralizer of a:

Easy computations yield that it is a derivation and that its kernel is $C_D(a)$. As the $C_D(a)$ is negligible by Step 2, the image of δ_a has finite index in D by Corollary 2.7. Moreover, since $\operatorname{Im} \delta_a$ is a vector space over $C_D(a)$, which is an infinite division ring by [14, Theorem 13.10], it is indeed equal to D.

Step 4. The characteristic is zero:

Suppose that D has characteristic $p > 0$. By Kaplansky's Theorem [14, Theorem 15.15], we can find an element a in D for which none of its powers belong to $Z(D)$. Up to replacing it by one of its powers, we may assume by the chain condition on centralizers that for any natural number n we have that $C_D(a)$ is equal to $C_D(a^n)$.

By Step 3, we can find an element x such that $\delta_a(x) = ax - xa = -1$. Thus, for any $i \geq 1$ conjugation by a^i yields $x^{a^i}a - x^{a^{i-1}}a = 1$. Hence

$$(ax)^{a^p} - ax = x^{a^{p-1}}a - ax = \sum_{i=1}^{p-1} \left(x^{a^i}a - x^{a^{i-1}}a\right) - (ax - xa) = (p-1) + 1 = 0.$$

Whence a^p commutes with ax and so with x. Therefore x belongs to $C_D(a^p)$ which equals to $C_D(a)$, yielding a contradiction.

Step 5. Any non-central element is transcendental over $Z(D)$:

Given a non-central element a, by Step 3 we can find a non-central element u such that $\delta_a(u) = -1$, i.e. $[a, u] = -1$. Additionally, as the characteristic is zero and the map δ_u is a derivation whose kernel clearly contains the center, the element a must be transcendental over the center since $\delta_u(a) = -\delta_a(u) = 1$.

Step 6. For any non-central element a and any x such that $\delta_a(x) = 1$, the intersection $Z(C_D(x)) \cap C_D(a)$ is equal to $Z(D)$:

Let y be an element of $Z(C_D(x)) \cap C_D(a)$. Thus $Z(C_D(y))$ is contained in $C_D(a)$ and thus in the kernel of δ_a. As additionally $\delta_a(x) = 1$, the element x must be transcendental over $Z(C_D(y))$. However, the choice of y yields that $x \in C_D(y)$ and so $C_D(y)$ has infinite dimension over its center. Therefore, we obtain that y belongs to $Z(D)$ by Step 1.

Step 7. For any non-central element a and any x such that $\delta_a(x) = 1$, the map δ_a restricted to $Z(C_D(x))$ is a surjective derivation onto $Z(C_D(x))$ whose kernel is $Z(D)$:

Note first that the kernel is precisely $C_D(a) \cap Z(C_D(x))$ which is equal to the center by Step 6. To prove that δ_a restricted to $Z(C_D(x))$ induces a map to $Z(C_D(x))$, we first see that the image of $C_D(x)$ via δ_a is contained in $C_D(x)$. To do so, let u be an element of $C_D(x)$. By the Jacobi identity, we have

$$\big[a, [u, x]\big] + \big[u, [x, a]\big] + \big[x, [a, u]\big] = 0.$$

As u commutes with x, the first summand is equal to 0 and since $[x, a] = -1$, we obtain that $[u, [x, a]]$ is 0 as well. Therefore $[x, [a, u]]$ must be 0 and hence $\delta_a(u) = [a, u]$ belongs to $C_D(x)$. Second, let v be an arbitrary element of $Z(C_D(x))$. Note first that $\delta_a(v)$ belongs to $C_D(x)$. Furthermore, for u again in $C_D(x)$, the identity

$$\big[u, [a, v]\big] + \big[a, [v, u]\big] + \big[v, [u, a]\big] = 0$$

yields, similarly as above, that $\delta_a(v) = [a, v]$ commutes with u and thus, as u was taken to be arbitrary in $C_D(x)$, we obtain that $\delta_a(v)$ belongs to $C_D(C_D(x))$ and thus to $Z(C_D(x))$.

To see that the derivation δ_a is surjective, notice first that $Z(C_D(x))$ is infinite dimensional over $Z(D)$ by Step 5 and hence, the center is negligible in $Z(C_D(x))$ by Lemma 2.4. In particular, the kernel of the derivation δ_a is negligible. Thus, the image of δ_a has finite index in $Z(C_D(x))^+$ by Corollary 2.7. Since $Z(C_D(x))^+$ has characteristic zero by Step 4, it is divisible and hence the image of δ_a and $Z(C_D(x))^+$ must coincide.

Step 8. For any non-central element a and any x such that $\delta_a(x) = 1$, we obtain $Z(C_D(x)) = a^{Z(C_D(x))} - a$:

Let $\sigma_a : Z(C_D(x))^\times \to D$ be the function that maps an element y to $a^y - a$. We need to show that σ_a is a surjective group morphism from $Z(C_D(x))^\times$ onto $Z(C_D(x))^+$. To show that the image of σ_a is contained in $Z(C_D(x))^+$, note that for any y in $Z(C_D(x))^\times$ we have that

$$\sigma_a(y) = a^y - a = y^{-1}(ay - ya) = y^{-1}\delta_a(y)$$

which belongs to $Z(C_D(x))$ by Step 7. Moreover, as $\sigma_a(1) = 0$ and for any u and v in $Z(C_D(x))$

$$\begin{aligned} \sigma_a(uv) &= a^{uv} - a \\ &= a^{uv} - a^v + a^v - a \\ &= (a^u - a)^v + (a^v - a) \\ &= (a^u - a) + (a^v - a) \\ &= \sigma_a(u) + \sigma_a(v), \end{aligned}$$

the map σ_a is a group morphism. To conclude this step, it remains to show that σ_a is surjective. For this, note first that the kernel of σ_a is clearly $Z(C_D(x))^\times \cap C_D(a)^\times$ which is equal to $Z(D)^\times$ by Step 6. As x is transcendental over $Z(D)$ by Step 5, we get that $Z(C_D(x))$ is infinite dimensional over $Z(D)$ and thus the kernel of σ_a is negligible with respect to $Z(C_D(x))$ by Lemma 2.4. Thus, by Corollary 2.7, the image of σ_a has finite index in $Z(C_D(x))^+$. Since $Z(C_D(x))^+$ is divisible as the characteristic is zero by Step 4, both are indeed equal.

Step 9. The multiplicative group $Z(D)^\times$ is divisible:

By Step 8, we deduce that for any non-central element a in D, the center $Z(D)$ is contained in $a^D - a$ and hence, by Lemma 2.8, any central element has an mth root in D for any natural number m. Therefore, as any non-central element is transcendental over $Z(D)$ by Step 5, any root of a central element must belong to the center.

Step 10. There exists a non-central element b such that for any non-central element y in $Z(C_D(b))$ we have that $Z(C_D(b)) = Z(C_D(y))$:

Otherwise we can find an infinite descending chain of fields

$$Z(C_D(a_0)) \gneq Z(C_D(a_1)) \gneq \ldots \gneq Z(C_D(a_i)) \gneq \ldots$$

with a_{i+1} being a non-central element belonging to $Z(C_D(a_i))$, which yields a contradiction with Remark 2.2.

Step 11. Final contradiction:

For this final step, let b be an element given by Step 10. By Step 3 the derivation δ_b on D is surjective and so we can find an element y such that $\delta_b(y) = -1$. In particular $\delta_y(b) = -\delta_b(y) = 1$. Hence, by Step 7 the restriction of δ_y to $Z(C_D(b))$ is surjective onto $Z(C_D(b))$. As $-b^{-1}$ belongs trivially to $Z(C_D(b))$, there is some c in $Z(C_D(b))$ such that $\delta_y(c) = -b^{-1}$, i.e. $\delta_c(y) = b^{-1}$. Hence, since c commutes with b we obtain that

$$1 = cyb - ycb = c(yb) - (yb)c = \delta_c(yb).$$

Thus, Step 8 yields:

$$Z(C_D(yb)) = c^{Z(C_D(yb))} - c$$

and in particular, for any z in $Z(D)$ we can find an element t in $Z(C_D(yb))$ such that $z = c^t - c$. Now, we define the map $\tau : Z(D)^+ \to D^\times$ as follows:

$$\tau(z) = b^{-1}b^t, \quad \text{where } t \in Z(C_D(yb)) \text{ such that } z = c^t - c.$$

Next, we see that τ is a group isomorphism from $Z(D)^+$ onto $Z(D)^\times$. First, we check that it is well defined. To do so, let z be in $Z(D)$ and consider t and s in $Z(C_D(yb))$ such that $z = c^t - c$ and $z = c^s - c$. Thus $t^{-1}s$ commutes with c. As c is non-central, the choice of b yields that $Z(C_D(c)) = Z(C_D(b))$ and so the product $t^{-1}s$ is an element of $C_D(b)$. Therefore, the value of $\tau(z)$ does not depend on the choice of t. Now, we show that $\operatorname{Im}\tau$ is contained in $Z(D)^\times$. Hence, consider an arbitrary element z of $Z(D)$ and let t be in $Z(C_D(yb))$ such that $c^t - c = z$. Thus t normalizes $Z(C_D(c))$ and so $Z(C_D(b))$. Therefore $\tau(z) = b^{-1}b^t$ belongs to $Z(C_D(b))$. Moreover, as t commutes with yb and so with by since $yb - by = 1$, we have that

$$\begin{aligned} [yb, \tau(z)] &= yb^t - b^{-1}b^t yb \\ &= b^{-1}byb^t - b^{-1}b^t yb \\ &= b^{-1}(byb)^t - b^{-1}(byb)^t \\ &= 0. \end{aligned}$$

Thus $\tau(z)$ commutes with yb and hence with y since $\tau(z)$ belongs to $Z(C_D(b))$. Therefore, we have shown that $\tau(z)$ is in $Z(C_D(b)) \cap C_D(y)$ and so it is central by Step 6.

Now, to prove that τ is a group morphism, note first that $\tau(0) = 1$. Moreover, consider z and z' in $Z(D)$ and let t and s be the corresponding elements from $Z(C_D(yb))$ such that $\tau(z) = b^{-1}b^t$ and $\tau(z') = b^{-1}b^s$. The choice of t and s yields

$$c^{ts} - c = (c^t - c)^s + (c^s - c) = z + z'.$$

Thus

$$\begin{aligned} \tau(z + z') &= b^{-1}(ts)^{-1}bts \\ &= (b^{-1}s^{-1}b)(b^{-1}t^{-1}bt)s \\ &= (b^{-1}t^{-1}bt)(b^{-1}s^{-1}b)s \\ &= \tau(z)\tau(z'). \end{aligned}$$

Hence, it remains to show that τ is an isomorphism. To check that the kernel of τ is trivial, consider an arbitrary element z of the kernel and let t be in $Z(C_D(yb))$ such that $z = c^t - c$. The choice of z yields that $b^{-1}b^t = 1$ and so t commutes with b. As $Z(C_D(b)) = Z(C_D(c))$, we have that t commutes with c and thus $z = 0$. Now, by Remark 2.1 the image of τ has finite index in $Z(D)^\times$ and so it is equal to $Z(D)^\times$ since the latter is divisible by Step 9.

Therefore τ is an isomorphism between $Z(D)^+$ and $Z(D)^\times$ and consequently, we can find a central element z such that $\tau(z) = -1$. Hence, for some t in $Z(C_D(yb))$ we have that $b^t = -b$, whence

$$b^2 = (-b)^2 = (b^t)^2 = (b^2)^t$$

and so t and b^2 commute. On the other hand, the choice of b implies that $Z(C_D(b))$ is equal to $Z(C_D(b^2))$ since b^2 is non-central by Step 5, thus t commutes with b and so

$$-1 = \tau(z) = b^{-1}b^t = b^{-1}b = 1,$$

a blatant contradiction since the characteristic of D is zero by Step 4. This final contradiction finishes the proof. □

3. Chain conditions on uniformly definable subgroups

In this section we study rosy division rings. Its unique property which we use throughout the section is the following chain condition, which is folklore:

Fact 3.1. Let $\mathfrak{F}$ be a family of uniformly definable subgroups of a rosy group; then there are natural numbers n and d such that any descending chain of intersections

$$F_0 \geq F_0 \cap F_1 \geq \cdots \geq F_0 \cap \ldots \cap F_i \geq \cdots$$

with F_i in $\mathfrak{F}$ for all i, each having index at least d in its predecessor, has length at most n.

Proof. By replacing the stratified ranks in simple theories by the ones in rosy theories [6, Definition 1.13], one obtains the statement following the proofs yielding [24, Theorem 4.2.12]. To do so, observe that in a rosy theory two relatively definable subgroups are commensurable if and only if they have the same stratified ranks, and that having the same rank is a type-definable condition. This can be directly seen from the definition of the rank. □

Remark 3.2. Similarly as in Remark 2.2, as the collection of centralizers of one element is a family of uniformly definable additive subgroups and moreover division rings, the above chain condition yields the chain condition on centralizers. Namely, any chain of centralizers of arbitrary sets is finite.

The following result is an easy adaptation of [16, Theorem 3.5].

Proposition 3.3. *A rosy division ring of positive characteristic has finite dimension over its center.*

Proof. Let D be a rosy division ring of positive characteristic, say p. Assume, as we may, that D is infinite and non-commutative. By the ordinary chain condition on centralizers, we may inductively suppose that any centralizer of a non-central element has finite dimension over its center.

By Kaplansky's Theorem [14, Theorem 15.15], we can find a non-central element a of D for which no power belongs to the center. Note that the centralizer of a is infinite. Additionally, by Fact 3.1, after replacing a by one of its powers, we may assume that $C_D(a) = C_D(a^p)$. Let δ_a be the derivation of D given by $\delta_a(x) = ax - xa$, which is clearly a definable $C_D(a)$-linear morphism. Since D has characteristic p one can easily see that

$$\delta_{a^{p^n}}(D) = \underbrace{\delta_a \circ \cdots \circ \delta_a}_{p^n}(D).$$

As any descending chain of uniformly definable additive subgroups stabilizes up to finite index after finitely many steps, there exists a natural number n for which the group $\delta_{a^{p^{n+1}}}(D)$ has finite index in $\delta_{a^{p^n}}(D)$. Thus, they are equal as both are vector spaces over the infinite division ring $C_D(a)$. Replacing a by a^{p^n} if necessary, we may assume that $\delta_a(D)$ and $\delta_{a^p}(D)$ coincide, and so do $\delta_a(D)$ and $(\delta_a \circ \delta_a)(D)$. Thus, given an element x of D there is some y such that $\delta_a(x) = \delta_a(\delta_a(y))$ and so $\delta_a(y) - x$ belongs to $C_D(a)$. Hence D is the sum of the two $C_D(a)$-vector spaces $\delta_a(D)$ and $C_D(a)$. Whence, as $C_D(a) = C_D(a^p)$ and $\delta_{a^p} = \delta_a \circ \overset{p}{\ldots} \circ \delta_a$, the latter sum is a direct sum, i.e.

$$D = C_D(a) \oplus \delta_a(D).$$

Now, set H to be the $C_D(a)$-vector space $\delta_a(D)$, and consider the family $\mathfrak{F}$ of $D^\times$-translates of H. As a finite intersection of $D^\times$-translates of H is also a vector space over the infinite division ring $C_D(a)$, by Fact 3.1 applied to $\mathfrak{F}$, there is some finite intersection $I = \lambda_1 H \cap \ldots \cap \lambda_n H$ such that $I = I \cap \lambda H$ for any λ in $D^\times$. Hence, we deduce that I is a (left) proper ideal and so it is trivial. Moreover, the subvector spaces $\lambda_i H$ have codimension one in D and thus D has finite dimension over $C_D(a)$. As by assumption, the centralizer of a has finite dimension over its center, we obtain that D has finite dimension over an infinite subfield and thus over its own center by [14, Theorem 15.8]. □

Proposition 3.4. *A rosy division ring which has only finitely many non-central conjugacy classes is commutative.*

Proof. Let D be a non-commutative rosy division ring, and suppose that D has only finitely many conjugacy classes.

The first step is to show that $Z(D)$ is contained in $b^D - b$ for any non-central element b in D. To do so, set H to be $Z(D) \cap (b^D - b)$ and observe that it is an additive subgroup of $Z(D)$. Now, following the lines of the proof of [24, Theorem 5.6.12] we show that H has only finitely many $Z(D)$-translates. For any z in $Z(D)$ we have that

$$zH = z[(b^D) - b] \cap Z(D) = [(zb^D) - zb] \cap Z(D).$$

Hence, if z and z' are two elements in $Z(D)$ such that $z'b = (zb)^d$ for some d in D, we have that

$$zH = (zH)^d = [(zb)^{Dd} - (zb)^d] \cap Z(D) = [(z'b)^D - (z'b)] \cap Z(D) = z'H.$$

As $Z(D)b$ intersects only finitely many conjugacy classes, the group H has finitely many multiplicative $Z(D)$-translates.

Now, observe that for any two central elements z and z', their difference $z - z'$ belongs to $b^D - b$ if and only if there is some element x from $D^\times$ such that

$$b + z = b^x + z' = (b + z')^x.$$

As there are only finitely many conjugacy classes in the coset $b + Z(D)$, the index of $Z(D) \cap (b^D - b)$ in $Z(D)$ has to be finite. Thus, the finite intersection of all its $Z(D)^\times$-translates, which forms an ideal of $Z(D)$, has finite index in $Z(D)$ as well. If $Z(D)$ is finite, the characteristic of D is positive and thus by Proposition 3.3 and Wedderburn's little Theorem [14, Theorem 13.1] D must be commutative. So we may assume that $Z(D)$ is an infinite field and hence equal to $Z(D) \cap (b^D - b)$. Thus $Z(D)$ is contained in $b^D - b$ for any non-central element b of D.

Now, by Kaplansky's Theorem [14, Theorem 15.15], we can find an element a in D for which none of its powers belong to $Z(D)$. Moreover, by the chain condition on centralizers, after replacing a by one of its powers, we may assume that $C_D(a) = C_D(a^n)$ for any natural number n.

Now, for any natural number m any element of the center has an mth root in D by Lemma 2.8. In particular, there is an infinite sequence $-1 = \xi_0, \xi_1, \xi_2, \dots$ of elements in D with $\xi_k^{2^k} = -1$ for all $k < \omega$. It is clear that all these roots of unity have different conjugacy classes and hence all but finitely many must belong to the center since there are only finitely many non-central conjugacy classes. Furthermore, one can find two different natural numbers i and j such that ξ_i and ξ_j belong to the center and $\xi_i a$ and $\xi_j a$ are conjugates. Thus, there is some x in $D \setminus C_D(a)$ and some non-trivial ζ in the center with $\zeta^m = 1$ for some $m < \omega$ such that $a = \zeta a^x$. Hence $a^m = (\zeta a^x)^m = (a^m)^x$ and so x belongs to the centralizer of a^m which, by the choice of a, coincides with the centralizer of a. This yields the final contradiction. □

The above result yields the following consequence for division rings definable in a simple theory. Recall that in a group G definable in such a theory, an element g of G is *generic* over A if for any h in G with g independent from h over A, the product $g \cdot h$ is independent from h, A. These elements play an essential role in the study of definable groups in this context. The next result focuses on division rings with generic elements of weight one, i.e. given any two independent elements b and c and any generic element a either a is independent from b or from c. A key property, which is unknown for rosy division rings, is that an element is generic for the multiplicative group if and only if it is generic for the additive group; see the proof of [22, Proposition 3.1]. For a detailed exposition, we refer the reader to [24, Chapter 4].

Corollary 3.5. *A division ring with a simple theory and a generic of weight one is a field.*

Proof. Let D be such a division ring and let g be any non-central element. We denote by $\ulcorner g^D \urcorner$ the canonical parameter of the conjugacy class of g in D. Now, let X be the set of non-generic elements of D over $\ulcorner g^D \urcorner$. By [13, Remark 1.1], or more precisely, its proof, the set of non-generic elements over any given small subset forms a subdivision ring. As conjugation by an element of $D^\times$ is an automorphism of D which fixes $\ulcorner g^D \urcorner$, the subdivision ring X is invariant under conjugation and hence it is contained in $Z(D)$ by the Cartan-Brauer-Hua Theorem [14, Theorem 13.17]. In fact, the division ring of non-generics over $\ulcorner g^D \urcorner$ and $Z(D)$ coincide. So g itself is a generic element of D independent from $\ulcorner g^D \urcorner$. Thus for any non-central element g in D, we have that the canonical parameter $\ulcorner g^D \urcorner$ is algebraic over the empty set. Hence D has only finitely many non-central conjugacy classes and whence it is commutative by Proposition 3.4. □

4. Division rings of finite burden

In this section we study division rings in which definable subgroups satisfy a generalized chain condition. More precisely, given a division ring D and a natural number n, we introduce the following property:

$(\dagger)_n$ For any definable subgroups $H_0, \dots, H_n$ of D^+, there exists some $j \le n$ such that $\bigcap_{i \le n} H_i$ has finite index in $\bigcap_{i \ne j} H_i$.

The motivation to analyze division rings fulfilling this property for some natural number n originates in the study of division rings of finite burden, and it is as well satisfied by any superrosy division rings of finite rank. In fact, all examples of fields mentioned in the introduction satisfy $(\dagger)_1$. Below we give the precise definition of burden.

Definition 4.1. Let $\pi(\bar{x})$ be a partial type. An *inp-pattern of depth* κ in $\pi(\bar{x})$ is a family of formulas $\{\psi_\alpha(\bar{x};\bar{y}_\alpha)\}_{\alpha<\kappa}$, an array of parameters $(\bar{a}_{\alpha,j})_{\alpha<\kappa,j<\omega}$ with $|\bar{a}_{\alpha,j}| = |\bar{y}_\alpha|$, and a sequence of natural numbers $(k_\alpha)_{\alpha<\kappa}$ such that:

- the set $\{\psi_\alpha(\bar{x},\bar{a}_{\alpha,j})\}_{j<\omega}$ is k_α-inconsistent for every $\alpha<\kappa$;
- the set $\pi(\bar{x}) \cup \{\psi_\alpha(\bar{x},\bar{a}_{\alpha,f(\alpha)})\}_{\alpha<\kappa}$ is consistent for every $f:\kappa\to\omega$.

We say that a theory has *burden* less than n for some natural number n, if there is no inp-pattern of depth n in the partial type $x = x$. Moreover, a definable division ring has burden n if the formula defining its domain has burden n. A division ring of burden 1 is called *inp-minimal*.

The following result is an immediate consequence of the proof of [4, Proposition 4.5] in the definable context.

Fact 4.2. A definable division ring of burden n satisfies $(\dagger)_n$.

Lemma 4.3. *A division ring satisfying* $(\dagger)_n$ *has dimension at most* n *over any infinite definable subfield.*

Proof. Let D be a division ring satisfying $(\dagger)_n$ with an infinite definable subfield K, and assume that the dimension of D over K is at least $n+1$. Choose K-linearly independent elements $e_0,\dots,e_n$ in D. For $j\le n$, consider the definable K-vector spaces $V_j = \bigoplus_{i\neq j} Ke_i$. Therefore, the condition $(\dagger)_n$ yields the existence of some $k\le n$ such that the index

$$\left[\bigcap_{j\neq k} V_j : \bigcap_j V_j\right] = [Ke_k : \{0\}]$$

is finite, a contradiction. □

Corollary 4.4. *Any infinite division ring satisfying* $(\dagger)_n$ *has dimension at most* n *over its center.*

Proof. Let D be a division ring satisfying $(\dagger)_n$. By Kaplansky's Theorem [14, Theorem 15.15], we may assume that D has an element d of infinite order. Hence $Z(C(d))$ is an infinite definable subfield of D, so Lemma 4.3 implies that D has finite dimension over $Z(C(d))$ and whence it has finite dimension over its center by [14, Theorem 15.8]. Therefore $Z(D)$ must be infinite. Now, we can apply Lemma 4.3 to the center of D and obtain the desired result. □

Immediately we obtain:

Corollary 4.5. *A definable division ring of burden* n *has dimension at most* n *over its center. In particular, an inp-minimal division ring is commutative.*

Moreover, as the quaternions are a finite extension of the inp-minimal field $\mathbb{R}$, they have finite burden. As they are non-commutative, one cannot expect to improve the above results to obtain commutativity.

Another consequence of these results is the descending chain condition among definable subfields.

Corollary 4.6. *Let D be an infinite division ring satisfying $(\dagger)_n$. Then any descending chain of definable infinite subfields has finite length. Therefore, if $\mathfrak{F}$ is a family of definable subfields of D, the intersection of all subfields in $\mathfrak{F}$ is equal to a finite subintersection, and so it is definable.*

Now, we aim to study definable groups of automorphisms of fields satisfying $(\dagger)_n$ for some n. In particular, this applies to fields of finite burden as well as superrosy fields of finite rank. We obtain results of the same nature to the following one of Hrushovski in the superstable case [10, Proposition 3]:

Fact 4.7. Any definable group of automorphisms acting definably on a definable superstable field is trivial.

Proposition 4.8. *If F is a field satisfying $(\dagger)_n$ and the algebraic closure of the prime field of F in F is infinite, then any definable group of automorphisms acting definably on F has size at most n.*

Proof. Assume, as we may, that our structure is sufficiently saturated. Let G be a definable group of automorphisms of F acting definably, let k be the prime field of F and let G_x denote the stabilizer of any element $x \in k^{\mathrm{alg}} \cap F$ in G. As k is fixed by the action of every element in G and G/G_x is in bijection with the orbit of x, the stabilizer G_x has finite index in G. Now, we work with the subgroup

$$H = \bigcap_{x \in k^{\mathrm{alg}} \cap F} G_x$$

of G. Note that it is a type-definable subgroup of G of bounded index. We consider the intersection $\mathrm{Fix}(H) = \bigcap_{\sigma \in H} \mathrm{Fix}(\sigma)$ of definable subfields of F. By Corollary 4.6 it is equal to a finite subintersection. Hence, as additionally $\mathrm{Fix}(H)$ contains the infinite field $k^{\mathrm{alg}} \cap F$, it is a definable infinite subfield of F. Thus, Lemma 4.3 yields that F has at most dimension n over $\mathrm{Fix}(H)$, so H is finite. Hence the group G is a bounded definable group and whence finite by compactness. Now, consider the definable field $\mathrm{Fix}(G)$. By Galois theory we know that F is a finite field extension of $\mathrm{Fix}(G)$ of degree $|G|$, and so $\mathrm{Fix}(G)$ is an infinite definable subfield of F. Hence F has dimension at most n over $\mathrm{Fix}(G)$ by Lemma 4.3 and whence G has size at most n. □

Corollary 4.9. *If F is a field satisfying $(\dagger)_1$ and the algebraic closure of the prime field of F in F is infinite, then any definable group acting definably on F as automorphisms is trivial.*

Observe that if F is Artin-Schreier closed, then the algebraic closure of the prime field of F is infinite in F. Thus, the above result holds for any infinite field of positive characteristic with finite burden and which in addition is NIP [11] or merely n-dependent [8].

We conclude the section with the following result in characteristic zero.

Proposition 4.10. *There are at most n many definable automorphisms of a definable field of characteristic zero satisfying $(\dagger)_n$.*

Proof. Let K be a field satisfying $(\dagger)_n$ and let $\mathcal{H}$ be the family of all definable automorphisms. For σ in $\mathcal{H}$, let F_σ denote the fixed field of σ, which is definable. By Corollary 4.6, the intersection F of all these fixed fields is again definable. Thus, it is infinite since the characteristic is zero, and hence K has dimension at most n

over F by Lemma 4.3. Therefore $\mathrm{Aut}(K/F)$ has size at most n and so does $\mathcal{H}$ since any of its automorphisms fixes F. □

5. Interpretable division rings in o-minimal structures

As pointed out in the introduction, Peterzil and Steinhorn [20, Theorem 4.1] showed that an infinite definable ring without zero divisors in an o-minimal structure is an algebraically closed field, a real closed field or the division ring of the quaternions over a definable real closed field. Next, we generalize the above result to interpretable division rings in o-minimal structures. Different from the proof of Peterzil and Steinhorn, which has a topological flavor, our approach uses the previous sections together with more recent results on groups in o-minimal structures.

Theorem 5.1. *An infinite interpretable division ring in an o-minimal structure is an algebraically closed field, a real closed field or the division ring of the quaternions over a definable real closed field.*

Proof. Let D be a division ring interpretable in an o-minimal structure and let K be its center, which is an interpretable field. As an o-minimal structure has burden 1, the structure $(D, +, \times)$ has finite burden and so D is finite dimensional over K by Corollary 4.5. Consequently, the field K is infinite and interpretable in an o-minimal structure. Alternatively, any o-minimal structure is superrosy [18, Section 5.2], a class of structures which is preserved under interpretation [12, Remark 1.3], and thus we obtain the same conclusion by Theorem 2.9.

By Frobenius' Theorem [14, Theorem 13.12] it suffices to show that K is either algebraically closed or real closed. To do so, we show that K is isomorphic to some definable field in an o-minimal structure. Indeed, by a result of Pillay [21], namely that any infinite definable field in an o-minimal structure is real closed or algebraically closed, the former is enough to conclude.

Now, consider the semidirect product $K^+ \rtimes K^\times$. As this group is clearly interpretable in the given o-minimal structure, it is definably isomorphic to a definable group G by [7, Theorem 8.23]. This isomorphism yields the existence of a definable normal subgroup N of G isomorphic to K^+ and a subgroup H of G isomorphic to $K^\times$ such that G is the (inner) semidirect product of N and H where H acts definably on N by conjugation. In particular, the definable isomorphism induces a definable field structure on N isomorphic to K. This finishes the proof. □

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