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SIMULTANEOUSLY PREPERIODIC POINTS FOR FAMILIES OF POLYNOMIALS IN NORMAL FORM

DRAGOS GHIOCA, LIANG-CHUNG HSIA, AND KHOA DANG NGUYEN

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ABSTRACT. Let $d > m > 1$ be integers, let $c_1, \dots, c_{m+1}$ be distinct complex numbers, and let $\mathbf{f}(z) := z^d + t_1 z^{m-1} + t_2 z^{m-2} + \cdots + t_{m-1} z + t_m$ be an m-parameter family of polynomials. We prove that the set of m-tuples of parameters $(t_1, \dots, t_m) \in \mathbb{C}^m$ with the property that each c_i (for $i = 1, \dots, m+1$) is preperiodic under the action of the corresponding polynomial $\mathbf{f}(z)$ is contained in finitely many hypersurfaces of the parameter space $\mathbb{A}^m$.

1. INTRODUCTION

The principle of unlikely intersections for 1-parameter families of rational functions $\mathbf{f}_t$ predicts that given two starting points c_1 and c_2 which are not persistently preperiodic for the family $\mathbf{f}$, if there exist infinitely many parameters t such that both c_1 and c_2 are preperiodic for $\mathbf{f}_t$, then the two starting points are dynamically related; for more details, see [BD11, BD13, GH13, GHT13, GHT15, GHT16, GKN16, GKNY17, MZ10, MZ12, MZ14]. For higher dimensional families of rational functions, there are very few definitive results, generally limited to 2-parameter families of dynamical systems; see [GHT15, Theorem 1.4] and [GHT16, Theorem 1.4]. In this paper we prove the following result regarding unlikely intersections for arithmetic dynamics in higher dimensional parameter spaces.

Theorem 1.1. *Let $d > m > 1$ be integers, let $c_1, \dots, c_{m+1} \in \mathbb{C}$, and let*

$$\mathbf{f}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m \tag{1.2}$$

be an m-parameter family of polynomials of degree d. For each point $\mathbf{a} = (a_1, \dots, a_m)$ of $\mathbb{A}^m(\mathbb{C})$ we let $\mathbf{f}_\mathbf{a}$ be the corresponding polynomial defined over $\mathbb{C}$ obtained by specializing each t_i to a_i for $i = 1, \dots, m$. Let $\operatorname{Prep}(c_1, \dots, c_{m+1})$ be the set consisting of parameters $\mathbf{a} \in \mathbb{A}^m(\mathbb{C})$ such that each starting point c_i (for $i = 1, \dots, m+1$) is preperiodic for $\mathbf{f}_\mathbf{a}$. If the points c_i are distinct, then $\operatorname{Prep}(c_1, \dots, c_{m+1})$ is not Zariski dense in $\mathbb{A}^m$.

The polynomials $\mathbf{f}(z)$ as in Theorem 1.1 are in *normal form*; i.e., they are monic of degree d and the coefficient of z^{d-1} is 0. Since each polynomial g is *conjugate* with a polynomial in normal form, i.e., there exists a linear polynomial μ such that $\mu^{-1} \circ g \circ \mu$ is in normal form, one can focus on the dynamics corresponding to

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polynomials as in Theorem 1.1. In [GHT16, Theorem 1.4], the special case $m = 2$ in Theorem 1.1 was proven, while the case of an arbitrary m was conjectured in [GHT16, Question 1.1]. Our Theorem 1.1 answers completely the problem raised in [GHT16].

If one considers m (distinct) starting points c_i, then the set $\text{Prep}(c_1, \dots, c_m)$ is Zariski dense in $\mathbb{A}^m$, as proven in [DeM16, Theorem 1.6] (see also [GNT15] for a discussion regarding all possible preperiodicity portraits simultaneously realized for m starting points by an m-parameter family of polynomials). On the other hand, there are numerous examples when the Zariski closure of $\text{Prep}(c_1, \dots, c_{m+1})$ is positive dimensional, and it may even have codimension 1 in $\mathbb{A}^m$ (see also [GHT16, Introduction]). For example, if $m = 3$, d is even and $c_2 = -c_1$ while $c_4 = -c_3$, then the Zariski closure of $\text{Prep}(c_1, c_2, c_3, c_4)$ contains the plane $\mathcal{P}$ given by the equation $t_2 = 0$ in the parameter space $\mathbb{A}^3$. Indeed, the specialization

$$\mathbf{g}(z) := z^d + t_1 z^2 + t_3$$

of $\mathbf{f}(z) = z^d + t_1 z^2 + t_2 z + t_3$ along $\mathcal{P}$ yields a 2-parameter family of even polynomials, and due to the relations between the starting points c_i, we know that all 4 starting points are preperiodic under the action of $\mathbf{g}$ if and only if c_1 and c_3 are preperiodic under the action of $\mathbf{g}$. Another application of [DeM16, Theorem 1.6] yields that there exists a Zariski dense set of points $(t_1, t_3) \in \mathbb{C}^2$ such that both c_1 and c_3 are preperiodic for $\mathbf{g}$, thus proving that $\mathcal{P}$ is contained in the Zariski closure of $\text{Prep}(c_1, c_2, c_3, c_4)$.

We note that if $m = 1$ in Theorem 1.1, then whenever $c_2 = \zeta_d \cdot c_1$, for some d-th root of unity ζ_d, we have that for each parameter t, the point c_1 is preperiodic under the action of $\mathbf{f}(z) = z^d + t$ if and only if c_2 is preperiodic under the action of $\mathbf{f}(z)$. In [BD11, Theorem 1.1], it was shown that the above linear relation is also necessary so that there exist infinitely many parameters t such that both c_1 and c_2 are preperiodic under the action of $z \mapsto z^d + t$. However, when $m > 1$, there exists no linear automorphism of the entire family $\mathbf{f}(z)$ (as opposed to the automorphism $z \mapsto \zeta_d \cdot z$ when $m = 1$), and this allows us to prove Theorem 1.1.

Finally, we observe that if one were to consider a different family of polynomials $\mathbf{h}$ of degree d with m parameters, but this time corresponding to monomials which are *not* of consecutive degrees, then it may very well be that $\text{Prep}(c_1, \dots, c_{m+1})$ is Zariski dense in $\mathbb{A}^m$. Indeed, if $\mathbf{h}(z) := z^d + t_1 z^3 + t_2 z$ is a 2-parameter family of odd polynomials (i.e., d is odd), then $\text{Prep}(c_1, c_2, -c_2)$ is *always* Zariski dense in $\mathbb{A}^2$ since c_2 is preperiodic whenever $-c_2$ is preperiodic, and therefore, essentially, we deal with 2 starting points and 2 parameters. On the other hand, our family of polynomials $\mathbf{f}(z)$ from (1.2) prohibits the possibility of any symmetries between the orbits of the starting points c_i.

We sketch now the plan for our paper. In section 2 we state in Theorem 2.2 a key result proven in [GHT16] for our problem. With the notation as in Theorem 1.1, assuming $\text{Prep}(c_1, \dots, c_{m+1})$ is Zariski dense in $\mathbb{A}^m$, [GHT16, Theorem 5.1] yields that for each point $\mathbf{a} := (a_1, \dots, a_m) \in \mathbb{C}^m$ in the parameter space, if m of the starting points c_i are preperiodic under the action of $\mathbf{f_a}$, then all $m + 1$ starting points c_i are preperiodic under the action of $\mathbf{f_a}$. Our strategy is to consider various lines $L \subset \mathbb{A}^m$ along which each c_i for $i = 1, \dots, m-1$ is preperiodic under the action of $\mathbf{f}$. Letting $\mathbf{g}_t$ be the 1-parameter family of polynomials obtained by specializing $\mathbf{f}$ along L, [GHT16, Theorem 5.1] (coupled with [DeM16, Theorem 1.6]) yields that there exist infinitely many parameters $t \in \mathbb{C}$ such that both c_m and c_{m+1} are preperiodic for $\mathbf{g}_t$. Then [BD13, Theorem 1.3] yields that the points c_m and c_{m+1} are dynamically related with respect to the family $\mathbf{g}_t$. In section 3, using an in-depth analysis of this information for two different lines L, we derive a contradiction, thus proving Theorem 1.1. It is interesting to note that this strategy works as long as $m > 2$. However, we note that the case $m = 2$ was proven in [GHT16, Theorem 1.4] using a similar strategy, but this time extracting slightly different information from using a single line L in the parameter plane $\mathbb{A}^2$ along which c_1 is fixed.

2. Useful results

We start by recalling the traditional assumption from algebraic dynamics that for a polynomial f and a positive integer n, we denote by $f^n = f \circ \cdots \circ f$ its composition with itself n times; furthermore, f^0 always denotes the identity function. A point a is called *preperiodic* under the action of f if its forward orbit under f consists of only finitely many distinct elements; i.e., there exist integers $n > m \geq 0$ such that $f^n(a) = f^m(a)$. Also, as a matter of notation, $\mathbb{N}$ denotes the set of all positive integers, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

It will be useful for our proof of Theorem 1.1 to know all polynomials commuting with an iterate of a given polynomial. Before stating [Ngu15, Theorem 2.3], we recall first the definition of the *d-th Chebyshev polynomial* $T_d(z)$ (for some integer $d \geq 2$), i.e., the unique polynomial satisfying the identity $T_d(z + 1/z) = z^d + 1/z^d$ for all z. We have [Ngu15, Theorem 2.3]:

Theorem 2.1. *Let K be an algebraically closed field of characteristic* 0*, let $d \geq 2$ be an integer, and let $g(z) \in K[z]$ be a polynomial of degree $d > 1$ which is not conjugate to z^d or to $\pm T_d(z)$.*

(a) *If $h(z) \in K[z]$ has degree at least* 2 *such that h commutes with an iterate of g, i.e., $h \circ g^n = g^n \circ h$ for some $n \in \mathbb{N}$, then h and g have a common iterate.*
(b) *Let $M(g^\infty)$ denote the collection of all linear polynomials commuting with an iterate of g. Then $M(g^\infty)$ is a finite cyclic group under composition.*
(c) *Let $\tilde{g}(z) \in K[z]$ be a polynomial of minimum degree $\tilde{d} \geq 2$ such that $\tilde{g}$ commutes with an iterate of g. Then there exists $D = D_g > 0$ relatively prime to the order of $M(g^\infty)$ such that $\tilde{g} \circ L = L^D \circ \tilde{g}$ for every $L \in M(g^\infty)$.*
(d) *$\{\tilde{g}^m \circ L \colon m \in \mathbb{N}_0$ and $L \in M(g^\infty)\} = \{L \circ \tilde{g}^m \colon m \in \mathbb{N}_0$ and $L \in M(g^\infty)\}$, and this set describes exactly all polynomials h commuting with an iterate of g.*

We state now the key result (proven in [GHT16, Theorem 5.1]) which we will use for deriving the conclusion in Theorem 1.1.

Theorem 2.2 ([GHT16])**.** *Let $d > m > 1$ be integers, let $c_1, \dots, c_{m+1}$ be distinct complex numbers, and let $\mathbf{f}(z) := z^d + t_1 z^{m-1} + \cdots + t_{m-1} z + t_m$ be an m-parameter family of polynomials of degree d. For each point $\mathbf{a} = (a_1, \dots, a_m)$ of $\mathbb{A}^m(\mathbb{C})$ we let $\mathbf{f}_\mathbf{a}$ be the corresponding polynomial defined over $\mathbb{C}$ obtained by specializing each t_i to a_i for $i = 1, \dots, m$. Let* $\mathrm{Prep}(c_1, \dots, c_{m+1})$ *be the set consisting of parameters $\mathbf{a} \in \mathbb{A}^m(\mathbb{C})$ such that each starting point c_i (for $i = 1, \dots, m+1$) is preperiodic for $\mathbf{f}_\mathbf{a}$. Assume* $\mathrm{Prep}(c_1, \dots, c_{m+1})$ *is Zariski dense in $\mathbb{A}^m$. Then for each $\mathbf{a} \in \mathbb{C}^m$ such that $c_1, \dots, c_m$ are preperiodic for $\mathbf{f}_\mathbf{a}$, we have that also c_{m+1} is preperiodic for $\mathbf{f}_\mathbf{a}$.*

We let L be a line in the parameter space $\mathbb{A}^m$ parametrized with respect to the coordinates $(t_1,\dots,t_m)$ of $\mathbb{A}^m$ as follows:

$$t_1 := t \text{ and } t_i = \alpha_i t + \beta_i \text{ for } i = 2,\dots,m,$$

for some complex numbers α_i, β_i. Furthermore, we assume that

$$\alpha_2 \neq 0. \tag{2.3}$$

We let $\mathbf{g} := \mathbf{g}_t$ be the specialization of $\mathbf{f}$ along the line L, i.e.,

$$\mathbf{g}_t(z) = z^d + tz^{m-1} + \sum_{i=2}^{m} (\alpha_i t + \beta_i) z^{m-i}. \tag{2.4}$$

The next result is essential for the proof of Theorem 1.1.

Proposition 2.5. *Let $K = \overline{\mathbb{C}(t)}$, and let $\mathbf{h}[z] \in K[z]$. With the above notation (2.4) for $\mathbf{g}$, if $\mathbf{h}$ commutes with an iterate of $\mathbf{g}$, then $\mathbf{h} = \mathbf{g}^\ell$ for some $\ell \in \mathbb{N}_0$.*

Proof. The desired conclusion follows from the next three lemmas coupled with Theorem 2.1 describing all polynomials commuting with an iterate of a given polynomial.

Lemma 2.6. *With the above notation, $\mathbf{g}(z)$ is not conjugate (over K) to z^d or to $\pm T_d(z)$.*

Proof of Lemma 2.6. Since z^d, $T_d(z)$ and also $\mathbf{g}(z)$ are polynomials in normal form, then assuming that for some linear polynomial $\mu \in K(z)$ we have that $\mu^{-1} \circ \mathbf{g} \circ \mu$ is either z^d or $\pm T_d(z)$, we conclude that $\mu(z) = \zeta \cdot z$ for some root of unity ζ. Indeed, letting $\mu(z) = az + b$, we get first that $b = 0$ since $\mathbf{g}(z)$, z^d and $\pm T_d(z)$ have coefficient equal to 0 for their monomial of degree $d-1$. Then equating the leading coefficient in each of the above polynomials yields that a must be a root of unity. Because z^d and $\pm T_d(z)$ have constant coefficients, i.e., there is no dependence on t, we conclude that $\mathbf{g}$ is not conjugate to a monomial or $\pm$Chebyshev polynomial. ☐

Lemma 2.7. *If $\mu(z)$ is a linear polynomial commuting with an iterate of $\mathbf{g}$, then $\mu(z) = z$ for all z.*

Proof of Lemma 2.7. We let $\mu(z) = az + b$ and assume $\mu \circ \mathbf{g}^n = \mathbf{g}^n \circ \mu$ for some $n \in \mathbb{N}$. Again using the fact that $\mathbf{g}$ (and thus also $\mathbf{g}^n$) is in normal form, we conclude that $b = 0$. Then using the fact that $\mathbf{g}^n(z)$ has nonzero terms of degrees $d^n - d + m - 1$ and $d^n - d + m - 2$ (using (2.4) and (2.3) along with an easy induction on n), we conclude that $1 = a^{d^n - d + m - 2} = a^{d^n - d + m - 3}$; hence $a = 1$, as claimed. ☐

Lemma 2.8. *There is no polynomial $\mathbf{h}_1(z) \in K[z]$ and no integer $e > 1$ such that $\mathbf{h}_1^e = \mathbf{g}$.*

Proof of Lemma 2.8. We argue by contradiction and therefore assume that $\mathbf{h}_1^e = \mathbf{g}$ with some integer $e > 1$ and some polynomial $\mathbf{h}_1 \in K[z]$ of degree $s > 1$. Furthermore, we assume $\mathbf{h}_1$ has minimal degree among all such polynomials. According to Theorem 2.1 part (d) along with Lemmas 2.6 and 2.7, we know that all polynomials commuting with $\mathbf{g}$ are of the form $\mathbf{h}_1^n$ for some $n \in \mathbb{N}_0$.

First, we claim that $\mathbf{h}_1(z) \in \mathbb{C}(t)[z]$. Indeed, otherwise there exists some Galois automorphism τ of K fixing $\mathbb{C}(t)$ such that $\mathbf{h}_2 := (\mathbf{h}_1)^\tau \neq \mathbf{h}_1$ (i.e., some coefficient of $\mathbf{h}_1$ is not fixed by τ). But then also $\mathbf{h}_2^e = \mathbf{g}$ (since each coefficient of $\mathbf{g}$ is fixed by τ) and therefore $\mathbf{h}_2 = \mathbf{h}_1$ since they both have the same degree and commute with $\mathbf{g}$. This is a contradiction, and so $\mathbf{h}_1(z) \in \mathbb{C}(t)[z]$.

Second, we claim that $\mathbf{h}_1 \in \mathbb{C}[t][z]$. Because $\mathbf{h}_1^e = \mathbf{g}$, we know that $\mathbf{h}_1(z) = \sum_{i=0}^{s} a_i z^i$ for some $a_i \in \mathbb{C}(t)$; since $\mathbf{g}(z)$ is monic, we have that a_s is a root of unity. Now, assuming $i_1 \in \{0, \dots, s-1\}$ is the largest integer such that $a_{i_1} \notin \mathbb{C}[t]$, an induction on e yields that the coefficient of $z^{s^e - s + i_1}$ in $\mathbf{h}_1^e = \mathbf{g}$ is not contained in $\mathbb{C}[t]$, which is a contradiction.

So, we know that $\mathbf{h}_1(z) \in \mathbb{C}[t][z]$. Since $\mathbf{g}$ is in normal form, we conclude that $\mathbf{h}_1$ must have no nonzero term of degree $s-1$. Now, let D be the maximum degree in t of the coefficients of $\mathbf{h}_1$; clearly, $D \geq 1$ since $\mathbf{g}_t$ is not a constant family in t. Then for all but finitely many $c \in \mathbb{C}$, the degree in t of $\mathbf{h}_1(c)$ equals D; let c be one such complex number. An easy computation (using the fact that $\mathbf{h}_1(z)$ has no terms of degree $s-1$) yields that the degree in t of $\mathbf{h}_1^e(c)$ equals Ds^{e-1}. On the other hand, the degree in t of $\mathbf{g}(c)$ is at most 1. So, the assumption that $e > 1$ yields a contradiction, thus concluding the proof of Lemma 2.8. □

Lemma 2.6 allows us to apply Theorem 2.1 in order to determine all polynomials commuting with an iterate of $\mathbf{g}$. Then Lemma 2.7 along with Theorem 2.1 yields that the set of all polynomials commuting with an iterate of $\mathbf{g}$ consists of all compositional powers of some polynomial $\mathbf{g}_0$. On the other hand, Lemma 2.8 yields that $\mathbf{g}$ is not a compositional power of another polynomial; therefore $\mathbf{g}_0 = \mathbf{g}$. This concludes our proof of Proposition 2.5. □

3. Proof of our main result

Proof of Theorem 1.1. Since the case $m = 2$ was proven in [GHT16, Theorem 1.4], we assume from now on that $m > 2$. Also, we proceed by contradiction; i.e., we assume that the set $\mathrm{Prep}(c_1, \dots, c_{m+1})$ is Zariski dense in $\mathbb{A}^m$. This allows us to apply Theorem 2.2.

Now, since the numbers c_i are distinct, clearly we can find $(m-1)$ of them whose sum is nonzero; so, without loss of generality, we assume that

$$\sum_{i=1}^{m-1} c_i \neq 0. \tag{3.1}$$

For each function (not necessarily injective) $\sigma : \{1, \dots, m-1\} \longrightarrow \{1, \dots, m-1\}$, we let $L_\sigma \subset \mathbb{A}^m$ be the line in the parameter space along which the following relations hold:

$$\mathbf{f}(c_i) = c_{\sigma(i)} \text{ for each } i = 1, \dots, m-1. \tag{3.2}$$

Indeed, in order to solve the system of equations (3.2) in the variables t_i, we let $t_1 := t$ and then solve each of the t_i's (for $i = 2, \dots, m$) in terms of the variable t, and in each case we get that t_i is a polynomial $T_{\sigma,i}(t)$ of degree at most 1. The fact that the system (3.2) is solvable follows from Cramer's Rule using the fact that the coefficients matrix is an invertible Vandermonde matrix since $c_i \neq c_j$ if $1 \leq i < j \leq m-1$.

Thus, the points c_i (for $i = 1, \dots, m-1$) are preperiodic along L_σ; we let $\mathbf{g}_\sigma = \mathbf{g}_{\sigma,t}$ be the specialization of $\mathbf{f}$ along the line L_σ. Furthermore, there exist polynomials $A, B_\sigma \in \mathbb{C}[z]$ such that

$$\mathbf{g}_{\sigma,t}(z) = A(z)t + B_\sigma(z). \tag{3.3}$$

A simple computation (using the fact that $A(z)$ is a monic polynomial of degree $m-1$ and that $\mathbf{g}_{\sigma,t}(c_i) = A(c_i)t + B(c_i)$ is a constant polynomial in t for each $i = 1, \ldots, m-1$) yields that

$$A(z) = \prod_{i=1}^{m-1} (z - c_i), \tag{3.4}$$

which confirms the fact that $A(z)$ is independent of the choice of the function σ. So, there exist some complex numbers α_i and $\beta_{\sigma,i}$ (for $i = 2, \ldots, m$) such that

$$\mathbf{g}_{\sigma,t}(z) = z^d + tz^{m-1} + (\alpha_2 t + \beta_{\sigma,2})z^{m-2} + \cdots + (\alpha_{m-1}t + \beta_{\sigma,m-1})z + \alpha_m t + \beta_{\sigma,m}. \tag{3.5}$$

Furthermore, according to (3.4), we have that

$$\alpha_2 = -\sum_{i=1}^{m-1} c_i \neq 0. \tag{3.6}$$

Equation (3.4) yields that for any $c \notin \{c_1, \ldots, c_{m-1}\}$, we have that $\deg_t(\mathbf{g}_{\sigma,t}(c)) = 1$ and furthermore (by induction), for any $n \geq 1$, we have that

$$\deg_t \left(\mathbf{g}^n_{\sigma,t}(c)\right) = d^{n-1}. \tag{3.7}$$

Because the points c_i (for $i = 1, \ldots, m-1$) are persistently preperiodic for $\mathbf{g}_{\sigma,t}$, Theorem 2.2 yields that for each parameter $t \in \mathbb{C}$, we have that c_m is preperiodic for $\mathbf{g}_{\sigma,t}$ if and only if c_{m+1} is preperiodic for $\mathbf{g}_{\sigma,t}$. Note that there exist infinitely many parameters $t \in \mathbb{C}$ such that c_m (and therefore also c_{m+1}) is preperiodic for $\mathbf{g}_{\sigma,t}$ since $\deg_t \left(\mathbf{g}^n_{\sigma,t}(c_m)\right) \to \infty$ as shown in (3.7); then the statement follows from [GHT13, Proposition 9.1] (see also [DeM16, Theorem 1.6] for a more general result on dynamically active marked points). Then [BD13, Theorem 1.3] yields that there exists some polynomial $\mathbf{h}(z) = \mathbf{h}_\sigma(x) \in \mathbb{C}[t][z]$ commuting with an iterate of $\mathbf{g}_\sigma$ and there exist positive integers n_m, n_{m+1} such that $\mathbf{g}^{n_m}_\sigma(c_m) = \mathbf{h}\left(\mathbf{g}^{n_{m+1}}_\sigma(c_{m+1})\right)$. Proposition 2.5 (see also (3.6)) allows us to assume that $\mathbf{h}$ is the identity. Furthermore, using (3.7), we conclude that $n_m = n_{m+1} =: n$. Next we prove that we may assume that $n = 2$.

Proposition 3.8. *Let n be an integer larger than 2. If $\mathbf{g}^n_{\sigma,t}(c_m) = \mathbf{g}^n_{\sigma,t}(c_{m+1})$, then $\mathbf{g}^{n-1}_{\sigma,t}(c_m) = \mathbf{g}^{n-1}_{\sigma,t}(c_{m+1})$.*

Proof of Proposition 3.8. First we prove that there exists some d-th root of unity ζ such that $\mathbf{g}^{n-1}_\sigma(c_m) = \zeta \cdot \mathbf{g}^{n-1}_\sigma(c_{m+1})$, and then we will prove that actually $\zeta = 1$.

Using (3.7), we have that, as a polynomial in t,

$$\begin{aligned}\mathbf{g}^n_{\sigma,t}(c_m) &= \left(\mathbf{g}^{n-1}_{\sigma,t}(c_m)\right)^d + t\left(\mathbf{g}^{n-1}_{\sigma,t}(c_m)\right)^{m-1} + \sum_{i=2}^{m}(\alpha_i t + \beta_{\sigma,i}) \cdot \left(\mathbf{g}^{n-1}_{\sigma,t}(c_m)\right)^{m-i} \\ &= \left(\mathbf{g}^{n-1}_{\sigma,t}(c_m)\right)^d + O\left(t^{d^{n-2}(m-1)+1}\right),\end{aligned} \tag{3.9}$$

where the big-O term from (3.9) denotes the fact that the remaining powers of t from the expansion of $\mathbf{g}^n_{\sigma,t}(c_m)$ have degree bounded by $d^{n-2}(m-1)+1$. A similar formula holds for $\mathbf{g}^n_{\sigma,t}(c_{m+1})$. Therefore, the equality $\mathbf{g}^n_{\sigma,t}(c_m) = \mathbf{g}^n_{\sigma,t}(c_{m+1})$ yields that

$$\deg_t \left(\mathbf{g}^{n-1}_{\sigma,t}(c_m)^d - \mathbf{g}^{n-1}_{\sigma,t}(c_{m+1})^d\right) \leq d^{n-2}(m-1) + 1. \tag{3.10}$$

Now, let ζ_d be a primitive d-th root of unity. If there is no $i \in \{0, \dots, d-1\}$ such that $\mathbf{g}_{\sigma,t}^{n-1}(c_m) = \zeta_d^i \cdot \mathbf{g}_{\sigma,t}^{n-1}(c_{m+1})$, then

$$\left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^d - \left(\mathbf{g}_{\sigma,t}^{n-1}(c_{m+1})\right)^d = \prod_{i=0}^{d-1} \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m) - \zeta_d^i \cdot \mathbf{g}_{\sigma,t}(c_{m+1})^{n-1}\right) \tag{3.11}$$

is a polynomial in t of degree at least $\deg_t \left(\mathbf{g}_{\sigma,t}(c_m)\right)^{d-1} = d^{(n-2)} \cdot (d-1)$ since at most one of the terms from the product appearing in (3.11) may have degree less than $\deg_t \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)$. This contradicts (3.10) (note that $n > 2$), thus proving that one of the terms in the product appearing in (3.11) must be 0, and so there exists a root of unity $\zeta = \zeta_d^{i_0}$ (for some $i_0 = 0, \dots, d-1$) such that

$$\mathbf{g}_{\sigma,t}^{n-1}(c_m) = \zeta \cdot \mathbf{g}_{\sigma,t}^{n-1}(c_{m+1}). \tag{3.12}$$

Next we prove that $\zeta = 1$ (i.e., $i_0 = 0$). For this we need to refine the expansion from (3.9), as follows:

$$\begin{aligned} \mathbf{g}_{\sigma,t}^{n}(c_m) = {} & \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^d + t \cdot \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^{m-1} \\ & + (\alpha_2 t + \beta_{\sigma,2}) \cdot \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^{m-2} + O\left(t^{d^{n-2}(m-3)+1}\right), \end{aligned} \tag{3.13}$$

and similarly, using (3.12), we get

$$\begin{aligned} \mathbf{g}_{\sigma,t}^{n}(c_{m+1}) = {} & \left(\mathbf{g}_{\sigma,t}^{n-1}(c_{m+1})\right)^d + t \cdot \left(\mathbf{g}_{\sigma,t}^{n-1}(c_{m+1})\right)^{m-1} \\ & + (\alpha_2 t + \beta_{\sigma,2}) \cdot \left(\mathbf{g}_{\sigma,t}^{n-1}(c_{m+1})\right)^{m-2} + O\left(t^{d^{n-2}(m-3)+1}\right) \\ = {} & \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^d + t \cdot \zeta^{m-1} \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^{m-1} \\ & + (\alpha_2 t + \beta_2) \cdot \zeta^{m-2} \left(\mathbf{g}_{\sigma,t}^{n-1}(c_m)\right)^{m-2} + O\left(t^{d^{n-2}(m-3)+1}\right). \end{aligned} \tag{3.14}$$

The equality $\mathbf{g}_{\sigma,t}^{n}(c_m) = \mathbf{g}_{\sigma,t}^{n}(c_{m+1})$ coupled with expansions (3.13) and (3.14) yields first that $\zeta^{m-1} = 1$, and then re-using (3.13) and (3.14) yields that $\zeta^{m-2} = 1$. So, $\zeta = 1$, as desired. ☐

So, we know that $\mathbf{g}_{\sigma,t}^{2}(c_m) = \mathbf{g}_{\sigma,t}^{2}(c_{m+1})$. Using (3.3), we get that

$$\begin{aligned} 0 & = \mathbf{g}_{\sigma,t}^{2}(c_m) - \mathbf{g}_{\sigma,t}^{2}(c_{m+1}) \\ & = \left(A(c_m)t + B_\sigma(c_m)\right)^d - \left(A(c_{m+1})t + B_\sigma(c_{m+1})\right)^d \\ & + t\left(A(c_m)t + B_\sigma(c_m)\right)^{m-1} - t\left(A(c_m)t + B_\sigma(c_m)\right)^{m-1} + O\left(t^{m-1}\right). \end{aligned} \tag{3.15}$$

Comparing the terms of degree d we get

$$A(c_{m+1}) = \xi \cdot A(c_m), \tag{3.16}$$

for some $\xi \in \mathbb{C}$ such that $\xi^d = 1$. Note that ξ is independent of σ, since $A(z)$ is independent of σ.

Proposition 3.17. *The quantity $B_\sigma(c_{m+1}) - \xi B_\sigma(c_m)$ is independent of the function σ.*

Proof of Proposition 3.17. Our analysis splits into two cases: either $m < d-1$ or $m = d-1$.

If $m < d-1$, then comparing the coefficient of t^{d-1} in (3.15), we get

$$A(c_m)^{d-1} B_\sigma(c_m) = A(c_{m+1})^{d-1} B_\sigma(c_{m+1}),$$

and so (3.16) yields that $B_\sigma(c_{m+1}) = \xi \cdot B_\sigma(c_m)$ (note that $A(c_m) \neq 0$, according to (3.4)), thus providing the desired conclusion.

If $m = d-1$, then again comparing the coefficient of t^{d-1} in (3.15) yields that

$$0 = dA(c_m)^{d-1}B_\sigma(c_m) - dA(c_{m+1})^{d-1}B_\sigma(c_{m+1}) + A(c_m)^{d-2} - A(c_{m+1})^{d-2}. \tag{3.18}$$

Using (3.16) and (3.18), we obtain that

$$B_\sigma(c_{m+1}) - \xi B_\sigma(c_m) = \frac{\xi - \xi^{-1}}{dA(c_m)}.$$

This concludes the proof of Proposition 3.17. □

Using Lagrange interpolation for the polynomial $B_\sigma(z) - z^d$ which has degree at most $m-2$, one computes that

$$B_\sigma(z) = z^d + \sum_{i=1}^{m-1} \left(c_{\sigma(i)} - c_i^d\right) \cdot \frac{A(z)}{(z - c_i) \cdot A'(c_i)}, \tag{3.19}$$

where $A'(z)$ is the derivative of the polynomial $A(z)$. Next we will consider two special functions σ: one of them is the identity function σ_1 which maps c_i to c_i for each $i = 1, \dots, m-1$, while the second function σ_2 differs from σ_1 only when evaluated at c_1, i.e.,

$$\sigma_2(c_1) = c_2 \text{ and } \sigma_2(c_i) = c_i \text{ for } i = 2, \dots, m-1.$$

Proposition 3.17 yields that

$$B_{\sigma_2}(c_{m+1}) - B_{\sigma_1}(c_{m+1}) = \xi \cdot \left(B_{\sigma_2}(c_m) - B_{\sigma_1}(c_m)\right). \tag{3.20}$$

Using (3.19) along with (3.20) yields that

$$\begin{aligned} 0 &= \frac{(c_2 - c_1)A(c_{m+1})}{A'(c_1)(c_{m+1} - c_1)} - \frac{(c_2 - c_1) \cdot \xi A(c_m)}{A'(c_1)(c_m - c_1)} \\ &= \frac{(c_2 - c_1)A(c_{m+1})}{A'(c_1)} \left(\frac{1}{(c_{m+1} - c_1)} - \frac{1}{(c_m - c_1)}\right) \quad \text{since } A(c_{m+1}) = \xi A(c_m) \\ &= \frac{(c_2 - c_1)A(c_{m+1})}{A'(c_1)} \cdot \frac{c_m - c_{m+1}}{(c_{m+1} - c_1)(c_m - c_1)}. \end{aligned}$$

Therefore, either $c_{m+1} = c_m$ or $A(c_{m+1}) = 0$; i.e., $c_{m+1} = c_i$ for some $i = 1, \dots, m-1$. This contradicts the fact that the starting points c_i are all distinct. In conclusion, $\mathrm{Prep}(c_1, \dots, c_{m+1})$ is contained in finitely many hypersurfaces of $\mathbb{A}^m$. □

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Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: `dghioca@math.ubc.ca`

Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan, ROC

E-mail address: `hsia@math.ntnu.edu.tw`

Department of Mathematics and Statistics, University of Calgary, Calgary, AB T2N 4T4, Canada

E-mail address: `dangkhoa.nguyen@ucalgary.ca`