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SMOOTHING OF WEIGHTS IN THE BERNSTEIN APPROXIMATION PROBLEM

ANDREW BAKAN AND JÜRGEN PRESTIN

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ABSTRACT. In 1924 S. Bernstein [Bull. Soc. Math. France **52** (1924), 399-410] asked for conditions on a uniformly bounded $\mathbb{R}$ Borel function (weight) $w : \mathbb{R} \to [0, +\infty)$ which imply the denseness of algebraic polynomials $\mathcal{P}$ in the seminormed space C_w^0 defined as the linear set $\{f \in C(\mathbb{R}) \mid w(x)f(x) \to 0$ as $|x| \to +\infty\}$ equipped with the seminorm $\|f\|_w := \sup_{x\in\mathbb{R}} w(x)|f(x)|$. In 1998 A. Borichev and M. Sodin [J. Anal. Math **76** (1998), 219-264] completely solved this problem for all those weights w for which $\mathcal{P}$ is dense in C_w^0 but for which there exists a positive integer $n = n(w)$ such that $\mathcal{P}$ is not dense in $C^0_{(1+x^2)^n w}$. In the present paper we establish that if $\mathcal{P}$ is dense in $C^0_{(1+x^2)^n w}$ for all $n \geq 0$, then for arbitrary $\varepsilon > 0$ there exists a weight $W_\varepsilon \in C^\infty(\mathbb{R})$ such that $\mathcal{P}$ is dense in $C^0_{(1+x^2)^n W_\varepsilon}$ for every $n \geq 0$ and $W_\varepsilon(x) \geq w(x) + \mathrm{e}^{-\varepsilon|x|}$ for all $x \in \mathbb{R}$.

1. Introduction

Let $C(\mathbb{R})$ be the linear space of all continuous real-valued functions on $\mathbb{R}$ and $\mathcal{W}(\mathbb{R})$ the set of all uniformly bounded on $\mathbb{R}$ Borel functions $w : \mathbb{R} \to \mathbb{R}^+ := [0, +\infty)$ which have an unbounded support $S_w := \{x \in \mathbb{R} \,|\, w(x) > 0\}$ and satisfy $|x|^n w(x) \to 0$ as $|x| \to \infty$ for all $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Denote by $\mathcal{P}$ the set of all algebraic polynomials with real coefficients and by $C^\infty(\mathbb{R})$ the family of all real-valued infinitely continuously differentiable functions on $\mathbb{R}$.

For $w \in \mathcal{W}(\mathbb{R})$ the seminormed space $C_w^0(\mathbb{R})$ consists of the linear set of all $f \in C(\mathbb{R})$ with $\lim_{|x|\to+\infty} w(x)\, f(x) = 0$ and the semi-norm $\|\cdot\|_w$, where $\|f\|_w := \sup_{x\,\in\,\mathbb{R}} w(x)\,|f(x)|$.

We recall the definition of the so-called *upper Baire function* M_F of $F : \mathbb{R} \to \mathbb{R}$ as $M_F(x) := \lim_{\delta\downarrow 0} \sup_{y\in(x-\delta,x+\delta)} F(y)$ (see [15, p. 129]). If F is locally bounded from above, then M_F is an upper semicontinuous function and $F(x) \leq M_F(x)$, $x \in \mathbb{R}$. It is easy to verify that for arbitrary $-\infty < A < B < +\infty$, $w \in \mathcal{W}(\mathbb{R})$ and $f \in C(\mathbb{R})$ we have

$$\sup_{x\in(A,B)} w(x)\,|f(x)| = \sup_{x\in(A,B)} M_w(x)\,|f(x)|\,.$$

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This means that the seminormed spaces $C_w^0(\mathbb{R})$ and $C_{M_w}^0(\mathbb{R})$ coincide identically and, in particular, $\mathcal{P}$ is dense in $C_w^0(\mathbb{R})$ iff it is dense in $C_{M_w}^0(\mathbb{R})$. Thus, it is possible to assume everywhere below that $w \in \mathcal{W}^*(\mathbb{R})$ where $\mathcal{W}^*(\mathbb{R})$ denotes the family of all those $w \in \mathcal{W}(\mathbb{R})$ which are upper semicontinuous on $\mathbb{R}$, i.e., $M_w(x) \equiv w(x)$ for all $x \in \mathbb{R}$.

Introduce

$$\mathcal{W}^{\mathrm{dens}}(\mathbb{R}) := \left\{ w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is dense in } C_w^0(\mathbb{R}) \right\}. \tag{1.1}$$

In 1924 S. Bernstein [4] asked for conditions on $w \in \mathcal{W}^*(\mathbb{R})$ to be in $\mathcal{W}^{\mathrm{dens}}(\mathbb{R})$. This problem is known as *Bernstein's approximation problem*. Various results towards a final solution of Bernstein's approximation problem have been obtained independently by L. Carleson [8], H. Pollard [17], S. N. Mergelyan [14] and L. de Branges [5] (see also the surveys of P. Koosis [11], A. Poltoratski [18] and M. Sodin [19]).

The solution of Bernstein's problem given by L. de Branges [5] in 1959 was slightly improved in 1996 by M. Sodin and P. Yuditskii [20] and attained the following form.

Let f be an entire function, Λ_f be the set of all its zeros, $0 \leq r, \rho < \infty$ and $\sigma_f(\rho) := \overline{\lim}_{r\to\infty} r^{-\rho} \log M_f(r)$, where $M_f(r) := \sup_{|z|=r} |f(z)|$. We say that f is of *minimal exponential type* if $\sigma_f(1) = 0$. Denote by $\mathcal{E}_0(\mathbb{R})$ the family of all entire functions f of minimal exponential type which are real on the real axis (in short, real) and have only real simple zeros.

Theorem A (L. de Branges, 1959 [5]). *Let $w \in \mathcal{W}^*(\mathbb{R})$. Then $\mathcal{P}$ is not dense in $C_w^0(\mathbb{R})$ if and only if there exists an entire function $B \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_B \subset S_w = \{x \in \mathbb{R} \mid w(x) > 0\}$ and*

$$\sum_{\lambda\in\Lambda_B} \frac{1}{w(\lambda)|B'(\lambda)|} < +\infty.$$

In 1958 S. Mergelyan [14] proved that if algebraic polynomials are dense in $C_w^0(\mathbb{R})$ but are not dense in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for some positive integer n, then w has countable support and the number of points in the set $\{x \in \mathbb{R} \mid w(x) > 0,\ |x| < R\}$ is $o(R)$ as $R \to +\infty$. Motivated by this result, A. Borichev and M. Sodin in 1998 [6] divided Bernstein's approximation problem into two parts.

Definition 1. Let $w \in \mathcal{W}^*(\mathbb{R})$. It is said that algebraic polynomials $\mathcal{P}$ are *regularly dense* in $C_w^0(\mathbb{R})$ if they are dense in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for all $n \in \mathbb{N}_0$.

Algebraic polynomials $\mathcal{P}$ are called *singularly dense* in $C_w^0(\mathbb{R})$ if they are dense in $C_w^0(\mathbb{R})$ but not in $C_{(1+x^2)^n w}^0(\mathbb{R})$ for a certain $n \in \mathbb{N} := \{1, 2, \dots\}$.

Similarly to (1.1), we denote

$$\begin{aligned} \mathcal{W}^{\mathrm{reg}}(\mathbb{R}) &:= \left\{ w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is regularly dense in } C_w^0(\mathbb{R}) \right\}, \\ \mathcal{W}^{\mathrm{sing}}(\mathbb{R}) &:= \left\{ w \in \mathcal{W}^*(\mathbb{R}) \mid \mathcal{P} \text{ is singularly dense in } C_w^0(\mathbb{R}) \right\}. \end{aligned}$$

It is obvious that $\mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ and $\mathcal{W}^{\mathrm{sing}}(\mathbb{R})$ are two non-intersecting classes of weights and

$$\mathcal{W}^{\mathrm{dens}}(\mathbb{R}) = \mathcal{W}^{\mathrm{reg}}(\mathbb{R}) \sqcup \mathcal{W}^{\mathrm{sing}}(\mathbb{R}),$$

where the symbol $\sqcup$ denotes the union of two non-intersecting sets.

Thus, the finding of conditions on a given weight $w \in \mathcal{W}^*(\mathbb{R})$ to be in $\mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ or in $\mathcal{W}^{\mathrm{sing}}(\mathbb{R})$ divides Bernstein's approximation problem into two independent parts: regular and singular, respectively. A complete solution of the singular part was given by A. Borichev and M. Sodin [6] in 1998.

Theorem B. *Let $w \in \mathcal{W}^*(\mathbb{R})$. Algebraic polynomials $\mathcal{P}$ are singularly dense in $C^0_w(\mathbb{R})$ if and only if w is discrete and there exist an entire function $E \in \mathcal{E}_0(\mathbb{R})$ and a non-negative integer n such that*

$$w(x) = \sum_{\lambda\in\Lambda_E} w(\lambda)\,\chi_\lambda(x), \quad x \in \mathbb{R}, \qquad \chi_\lambda(x) := \begin{cases} 0, & \text{if } x \neq \lambda, \\ 1, & \text{if } x = \lambda, \end{cases}$$

$$\sum_{\lambda\in\Lambda_E} \frac{1}{(1+\lambda^2)^k\; w(\lambda)\,|E'(\lambda)|} \quad \begin{cases} < +\infty, & \text{if } k = n+1, \\ = +\infty, & \text{if } k = n, \end{cases}$$

and

$$\sum_{\lambda\in\Lambda_F} \frac{1}{w\,(\lambda)\,|F'(\lambda)|} = +\infty$$

for arbitrary transcendental entire functions F of minimal exponential type such that $\Lambda_F \subset \Lambda_E$ and E/F is transcendental.

The regular part of Bernstein's approximation problem is still open, but the following important result holds.

Theorem C (M. Sodin, 1996 [19])**.** *If $w \in \mathcal{W}^{\rm reg}(\mathbb{R})$, then $w(x) + \mathrm{e}^{-\delta|x|} \in \mathcal{W}^{\rm reg}(\mathbb{R})$ for every $\delta > 0$.*

The following statement about perturbations of zeros of an entire function was proved in [2, Lemma 5, p. 237] (2005).

Lemma A. *For an arbitrary entire function $B \in \mathcal{E}_0(\mathbb{R})$ with zeros $\Lambda_B = \{b_n\}_{n\geq 1}$ there exist a constant $C > 0$ and a sequence of real positive numbers $\{\delta_n\}_{n\geq 1}$ such that for any sequence of real numbers $\{d_n\}_{n\geq 1}$ satisfying*

$$|b_n - d_n| \leq \delta_n, \qquad n \geq 1,$$

one can find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_n\}_{n\geq 1}$ and

$$|B'(b_n)| \leq C \cdot |D'(d_n)|, \quad n \geq 1.$$

If the set of real numbers $\{|B'(b_n)|\}_{n\geq 1}$ in Lemma A is bounded from below, then the result of Lemma A can be improved as follows.

Lemma 1. *Let $B \in \mathcal{E}_0(\mathbb{R})$ and Λ_B denote the set of its zeros. Assume that*

$$\sum_{\lambda\in\Lambda_B} \frac{1}{|B'(\lambda)|} < \infty. \tag{1.2}$$

Then, for arbitrary $\delta > 0$ there exist constants $C_\delta = C_\delta(B), \rho_\delta = \rho_\delta(B) > 0$ such that for any set of real numbers $\{d_\lambda\}_{\lambda\in\Lambda_B}$ satisfying

$$|\lambda - d_\lambda| \leq \rho_\delta\, \mathrm{e}^{-\delta|\lambda|}, \quad \lambda \in \Lambda_B, \tag{1.3}$$

one can find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_\lambda\}_{\lambda\in\Lambda_B}$ and

$$|B'(\lambda)| \leq C_\delta\; |D'(d_\lambda)|\,, \quad \lambda \in \Lambda_B. \tag{1.4}$$

Observe that the proof of Lemma 1 in section 3 gives the explicit expressions for the constants ρ_δ and C_δ in (1.3) and in (1.4). Lemma 1 is instrumental for the proof of the next statement.

Lemma 2. *Let $\varepsilon > 0$ and $w \in \mathcal{W}^{\rm reg}(\mathbb{R})$. Then,*

$$w_\varepsilon(x) := \sup_{|t|\le \mathrm{e}^{-\varepsilon|x|}} \left(w(x+t) + \mathrm{e}^{-\varepsilon|x+t|}\right) \in \mathcal{W}^{\rm reg}(\mathbb{R}). \tag{1.5}$$

Proof. In view of [10, Example 1, p. 8], the function

$$\beta_\varepsilon(x) := w(x) + \mathrm{e}^{-\varepsilon|x|} \tag{1.6}$$

is upper semicontinuous on $\mathbb{R}$, and an application of [10, Theorem 1.2, p. 4] to the supremum in (1.5) yields for each $x \in \mathbb{R}$ the existence of $\theta_\varepsilon(x) \in [-1,1]$ such that

$$w_\varepsilon(x) = \beta_\varepsilon\left(x + \theta_\varepsilon(x)\mathrm{e}^{-\varepsilon|x|}\right), \quad x \in \mathbb{R}. \tag{1.7}$$

To prove $w_\varepsilon \in \mathcal{W}^*(\mathbb{R})$, let $x_0 \in \mathbb{R}$, $\{x_n\}_{n\ge1} \subset \mathbb{R}$, $\lim_{n\to\infty} x_n = x_0$, $\mathbb{N}_1 := \{n \ge 1 \mid |\theta_\varepsilon(x_n)|\mathrm{e}^{-\varepsilon|x_n|} > \mathrm{e}^{-\varepsilon|x_0|}\}$ and let us choose an infinite sequence $\mathbb{N}_2 := \{n_k\}_{k\ge1} \subset \mathbb{N}$ such that $\overline{\lim}_{n\to\infty} w_\varepsilon(x_n) = \lim_{k\to\infty} w_\varepsilon(x_{n_k})$. Since for every $n \in \mathbb{N}\setminus\mathbb{N}_1$ we have $|\theta_\varepsilon(x_n)|\mathrm{e}^{-\varepsilon|x_n|} \le \mathrm{e}^{-\varepsilon|x_0|}$, (1.7) and (1.5) yield $w_\varepsilon(x_n) \le w_\varepsilon(x_0)$, $n \in \mathbb{N}\setminus\mathbb{N}_1$. Thus, if $\mathbb{N}_2 \cap (\mathbb{N}\setminus\mathbb{N}_1)$ is infinite, then $\overline{\lim}_{n\to\infty} w_\varepsilon(x_n) \le w_\varepsilon(x_0)$. Otherwise, it suffices to consider the case $\mathbb{N}_2 \subset \mathbb{N}_1$ in which $\lim_{k\to\infty} |\theta_\varepsilon(x_{n_k})| = 1$ and therefore $\lim_{k\to\infty} w_\varepsilon(x_{n_k}) \le \max\{\beta_\varepsilon(x_0 - \mathrm{e}^{-\varepsilon|x_0|}), \beta_\varepsilon(x_0 + \mathrm{e}^{-\varepsilon|x_0|})\} \le w_\varepsilon(x_0)$, by virtue of (1.7) and the upper semicontinuity of β_ε. This completes the proof of $w_\varepsilon \in \mathcal{W}^*(\mathbb{R})$ (see [16, Theorem 2, p. 150]).

Assume that $w_\varepsilon \notin \mathcal{W}^{\rm reg}(\mathbb{R})$. Then, for some $m \in \mathbb{N}_0$ we have $(1+x^{2m})w_\varepsilon \notin \mathcal{W}^{\rm dens}(\mathbb{R})$, and by Theorem A there exists an entire function $F \in \mathcal{E}_0(\mathbb{R})$ such that

$$\sum_{\lambda\in\Lambda_F} \frac{1}{(1+\lambda^{2m})\, w_\varepsilon(\lambda)\, |F'(\lambda)|} < \infty. \tag{1.8}$$

It follows from $w_\varepsilon \in \mathcal{W}^*(\mathbb{R})$ that $\sum_{\lambda\in\Lambda_F} 1/|F'(\lambda)| < \infty$, and therefore (1.2) holds for $B = F$.

By Theorem C,

$$\beta_\varepsilon \in \mathcal{W}^{\rm reg}(\mathbb{R}). \tag{1.9}$$

From (1.8) and (1.7) we obtain

$$\sum_{\lambda\in\Lambda_F} \frac{1}{(1+\lambda^{2m})\ \beta_\varepsilon(\lambda+\theta_\varepsilon(\lambda)\mathrm{e}^{-\varepsilon|\lambda|})\ |F'(\lambda)|} < \infty. \tag{1.10}$$

Applying Lemma 1 for $\delta = \varepsilon/2$, we find $T_\varepsilon > 0$ such that $\mathrm{e}^{-\varepsilon x/2} \le \rho_{\varepsilon/2}$, $x \ge T_\varepsilon$, and then we find an entire function $D \in \mathcal{E}_0(\mathbb{R})$ with zeros $\Lambda_D = \{d_\lambda\}_{\lambda\in\Lambda_B}$, where

$$d_\lambda = \lambda, \quad \lambda \in \Lambda_F \cap [-T_\varepsilon, T_\varepsilon], \quad d_\lambda = \lambda + \theta(\lambda)\mathrm{e}^{-\varepsilon|\lambda|}, \quad \lambda \in \Lambda_F \setminus [-T_\varepsilon, T_\varepsilon].$$

Hence, in view of (1.4) and (1.10) we have

$$\begin{aligned}
\infty &> \sum_{\lambda\in\Lambda_F\setminus[-T_\varepsilon,T_\varepsilon]} \frac{1}{(1+\lambda^{2m})\ \beta_\varepsilon\left(\lambda+\theta_\varepsilon(\lambda)\mathrm{e}^{-\varepsilon|\lambda|}\right)\ |F'(\lambda)|} \\
&\ge \frac{1}{C_{\varepsilon/2}} \sum_{\lambda\in\Lambda_F\setminus[-T_\varepsilon,T_\varepsilon]} \frac{1+d_\lambda^{2m}}{1+\lambda^{2m}}\ \frac{1}{(1+d_\lambda^{2m})\ \beta_\varepsilon(d_\lambda)\ |D'(d_\lambda)|} \\
&\ge \frac{1}{2^{2m}C_{\varepsilon/2}} \sum_{\lambda\in\Lambda_F\setminus[-T_\varepsilon,T_\varepsilon]} \frac{1}{(1+d_\lambda^{2m})\ \beta_\varepsilon(d_\lambda)\ |D'(d_\lambda)|},
\end{aligned}$$

from which it follows that

$$\sum_{\lambda\in\Lambda_D} \left(1+\lambda^{2m}\right)^{-1} \beta_\varepsilon(\lambda)^{-1} \ \left|D'(\lambda)\right|^{-1} < \infty.$$

By Theorem A this means that $(1+x^{2m})\cdot\beta_\varepsilon \notin \mathcal{W}^{\mathrm{dens}}(\mathbb{R})$ and therefore $\beta_\varepsilon \notin \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$. This contradicts (1.9) and finishes the proof of Lemma 2. □

We are now ready to prove our main result.

Theorem 1. *For arbitrary $w\in\mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ and $\varepsilon>0$ there exists $W_\varepsilon\in C^\infty(\mathbb{R})$ such that $W_\varepsilon\in\mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ and $W_\varepsilon(x)\geq w(x)+\mathrm{e}^{-\varepsilon|x|}$ for all $x\in\mathbb{R}$.*

Proof. Since the statement of the theorem for $\varepsilon=\varepsilon_0>0$ implies its validity for all $\varepsilon\geq\varepsilon_0$, we can assume without loss of generality that $\varepsilon\in(0,1)$.

Let w_ε be defined as in (1.5), β_ε as in (1.6) and

$$\Omega_\rho(x) := \sup_{|s|\leq\rho\mathrm{e}^{-\varepsilon|x|}} \beta_\varepsilon(x+s), \quad x\in\mathbb{R}, \quad \rho\in(0,1].$$

Since

$$w(x)\leq w(x)+\mathrm{e}^{-\varepsilon|x|}=\beta_\varepsilon(x)\leq\Omega_\rho(x)\leq w_\varepsilon(x), \quad x\in\mathbb{R}, \tag{1.11}$$

by Lemma 2,

$$\Omega_\rho\in\mathcal{W}^{\mathrm{reg}}(\mathbb{R}), \quad \rho\in(0,1].$$

For arbitrary $\omega\in C^\infty(\mathbb{R})$ satisfying

$$0<\omega(x)\leq\mathrm{e}^{-\varepsilon|x|}/4, \quad x\in\mathbb{R}, \tag{1.12}$$

let us introduce

$$K_\omega(x,t) := \left(\int_{-\omega(x)}^{\omega(x)} \exp\left(-\frac{\omega(x)^2}{\omega(x)^2-s^2}\right)\mathrm{d}s\right)^{-1} \exp\left(-\frac{\omega(x)^2}{\omega(x)^2-t^2}\right), \tag{1.13}$$

where $t\in(-\omega(x),\omega(x))$, $x\in\mathbb{R}$ and $K_\omega(x,\pm\omega(x)):=0$. For example, we may take $\omega(x)=(1/4)\exp(-x^2-\varepsilon^2/4)$. Obviously,

$$\int_{-\omega(x)}^{\omega(x)} K_\omega(x,t)\mathrm{d}t=1, \quad x\in\mathbb{R},$$

and therefore the weight

$$W_\varepsilon(x) := \int_{-\omega(x)}^{\omega(x)} K_\omega(x,t)\Omega_{1/2}(x+t)\mathrm{d}t = \int_{x-\omega(x)}^{x+\omega(x)} K_\omega(x,t-x)\Omega_{1/2}(t)\mathrm{d}t \tag{1.14}$$

belongs to $C^\infty(\mathbb{R})$.

Let $x\in\mathbb{R}$ be arbitrary and let $t\in\mathbb{R}$ satisfy $|t|\leq\omega(x)$. Then, by (1.12) we have that $|t|\leq\mathrm{e}^{-\varepsilon|x|}/4$, and the inequalities $\mathrm{e}^{1/4}\leq 4/3$ and $0<\varepsilon<1$ imply that $(3/4)\mathrm{e}^{-\varepsilon|x|}\leq\mathrm{e}^{-\varepsilon|x+t|}\leq(4/3)\mathrm{e}^{-\varepsilon|x|}$. Thus, for every $\rho\in(1/3,1]$,

$$(3\rho-1)\mathrm{e}^{-\varepsilon|x|}/4\leq\rho\mathrm{e}^{-\varepsilon|x+t|}+t\leq(16\rho+3)\mathrm{e}^{-\varepsilon|x|}/12,$$

and therefore

$$\Omega_{(3\rho-1)/4}(x)\leq\Omega_\rho(x+t)\leq\Omega_{(16\rho+3)/12}(x), \quad \rho\in(1/3,1), \quad |t|\leq\mathrm{e}^{-\varepsilon|x|}/4, \quad x\in\mathbb{R},$$

from which we infer for $\rho=1/2$ that

$$\beta_\varepsilon(x)\leq\Omega_{1/8}(x)\leq\Omega_{1/2}(x+t)\leq\Omega_{11/12}(x)\leq\Omega_1(x)=w_\varepsilon(x), \qquad |t|\leq\omega(x),\ x\in\mathbb{R}. \tag{1.15}$$

In view of (1.11) this means that the weight W_ε satisfies

$$w(x) + \mathrm{e}^{-\varepsilon|x|} \le W_\varepsilon(x) \le w_\varepsilon(x), \quad x \in \mathbb{R}. \tag{1.16}$$

It follows from the right-hand side inequality of (1.16) that $W_\varepsilon \in \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$, and therefore the left-hand side inequality of (1.16) completes the proof. □

Since the weight W_ε defined in (1.14) depends on an arbitrary function $\omega \in C^\infty(\mathbb{R})$ satisfying (1.12), we prove in the next corollary that the special choice $\omega = \phi_\varepsilon$ yields a good upper estimate for W_ε'. Here,

$$\phi_\varepsilon(x) := \frac{\mathrm{e}^{-\varepsilon}}{4\kappa} \int_{-1}^{1} \mathrm{e}^{-\frac{1}{1-t^2}} \mathrm{e}^{-\varepsilon|x+t|} \mathrm{d}t, \quad x \in \mathbb{R}, \quad \varepsilon > 0, \tag{1.17}$$

$$\kappa := \int_{-1}^{1} \mathrm{e}^{-\frac{1}{1-t^2}} \mathrm{d}t = \frac{K_1(1/2) - K_0(1/2)}{\sqrt{\mathrm{e}}} \in \left(\frac{1.2}{\mathrm{e}}, \frac{1.21}{\mathrm{e}} \right), \tag{1.18}$$

K_1, K_0 are modified Bessel functions (see [9, (13), p. 5]) and (1.18) is proved in section 4.

Corollary 1. *Let $\varepsilon \in (0,1)$, $w \in \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ and w_ε be defined as in* (1.5). *Then there exists a weight $W_\varepsilon \in C^\infty(\mathbb{R}) \cap \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ such that $W_\varepsilon(x) \ge w(x) + \mathrm{e}^{-\varepsilon|x|}$ and $|W_\varepsilon'(x)| \le 74\, \mathrm{e}^{\varepsilon|x|} w_\varepsilon(x)$ for all $x \in \mathbb{R}$.*

Theorem 1 allows us to assume without loss of generality that each weight in the regular part of Bernstein's approximation problem is continuous and positive on the whole real axis. It also allows us to apply to this part of the problem the sufficient conditions for the denseness of algebraic polynomials in $C_w^0(\mathbb{R})$ obtained earlier under this assumption (see [17, p. 869], [14, p. 80]). On the other hand, Lemma 2 makes it possible to replace any weight $w \in \mathcal{W}^*(\mathbb{R})$ by the greater step function

$$\begin{aligned} \widehat{w}(x) &= \sum_{n \in \mathbb{Z}} w_n \chi_{[\sigma_n \log(1+|n|),\, \sigma_n \log(1+|n+1|)]}(x), \quad x \in \mathbb{R}, \\ w_n &:= \sup_{x \in [\sigma_n \log(1+|n|),\, \sigma_n \log(1+|n+1|)]} w(x), \quad \sigma_n := \mathrm{sign}(n), \quad n \in \mathbb{Z}, \end{aligned}$$

such that algebraic polynomials are regularly dense in $C_w^0(\mathbb{R})$ if and only if they are regularly dense in $C_{\widehat{w}}^0(\mathbb{R})$. Here, $\mathrm{sign}(n)$ is equal to 1 if $n > 0$, 0 if $n = 0$ and -1 if $n < 0$.

Notice also that Theorem 1 can be efficiently applied to a representation of the so-called p-regular measures for $1 \le p < \infty$. Recall (see [6, p. 250]) that a non-negative Borel measure μ on $\mathbb{R}$ is called *p-regular* if all its moments $\int_{\mathbb{R}} x^n \mathrm{d}\mu(x)$, $n \ge 0$, are finite and algebraic polynomials are dense in $L_p(\mathbb{R}, (1+x^2)^{np} \mathrm{d}\mu(x))$ for every $n \ge 0$. Here, for arbitrary non-negative Borel measures μ, ν on $\mathbb{R}$ and $g \in L_1(\mathbb{R}, \mathrm{d}\mu)$, we write $\mathrm{d}\nu(x) = g(x)\mathrm{d}\mu(x)$ or $\mathrm{d}\nu = g\mathrm{d}\mu$ if $\nu(A) = \int_A g(x)\mathrm{d}\mu(x)$ for arbitrary Borel subset A of $\mathbb{R}$. According to [3, Lemma 4, p. 203], if μ is p-regular, then there exists a finite non-negative Borel measure ν on $\mathbb{R}$ and $w \in \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ such that $\mathrm{d}\mu = w^p\, \mathrm{d}\nu$ (the converse is evident). Taking for this w the weight W_ε from Theorem 1, we obtain $\mathrm{d}\mu = w^p\, \mathrm{d}\nu = W_\varepsilon^p (w/W_\varepsilon)^p\, \mathrm{d}\nu = W_\varepsilon^p\, \mathrm{d}\widetilde{\nu}$ where $\widetilde{\nu}$ is also a non-negative finite Borel measure on $\mathbb{R}$ as follows from $\mathrm{d}\widetilde{\nu} = (w/W_\varepsilon)^p\, \mathrm{d}\nu$ and $w(x) \le W_\varepsilon(x)$ for all $x \in \mathbb{R}$. Thus, the following assertion holds.

Corollary 2. *Let $1 \le p < \infty$ and a measure μ be p-regular. Then, for every $\varepsilon > 0$, there exists a finite non-negative Borel measure ν_ε on $\mathbb{R}$ and a weight $W_\varepsilon \in C^\infty(\mathbb{R}) \cap \mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ such that $W_\varepsilon(x) \ge \mathrm{e}^{-\varepsilon|x|}$ for all $x \in \mathbb{R}$ and $\mathrm{d}\mu = W_\varepsilon^p\, \mathrm{d}\nu_\varepsilon$.*

2. Auxiliary results

Lemma 3. *Let the real numbers a, b, x and $\Delta \in (0,1)$ satisfy*

$$b \in (a - \Delta^2, a + \Delta^2), \quad 0 \notin (a - \Delta, a + \Delta) \ \text{ and } \ x \notin (a - 2\Delta, a + 2\Delta). \tag{2.1}$$

Then,

$$\left| \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{b}\right)^{-1} \right| \le (1 + \Delta)^2 .$$

Proof. The conditions (2.1) imply that $|a| \ge \Delta$, $b \in (a - \Delta, a + \Delta)$ and therefore that $|x - b| \ge \Delta$. Thus, $||b| - |a|| \le |b - a| \le \Delta^2$ and $|b| \le |a| + \Delta^2$, i.e., $|b|/|a| \le 1 + \Delta^2/|a| \le 1 + \Delta$. Finally,

$$\begin{aligned} \left| \frac{1 - x/a}{1 - x/b} \right| &= \frac{|b|}{|a|} \frac{|x - a|}{|x - b|} = \frac{|b|}{|a|} \frac{|b - a + (x - b)|}{|x - b|} \\ &\le \frac{|b|}{|a|} \frac{|b - a| + |x - b|}{|x - b|} \le (1 + \Delta) \cdot \left(1 + \frac{|b - a|}{|x - b|}\right) \le (1 + \Delta)^2 , \end{aligned}$$

which completes the proof. □

Lemma 4. *Let $\varepsilon \in (0, 1/(2\mathrm{e}))$, $C_\varepsilon \in (0, +\infty)$ and f be an entire function satisfying*

$$|f(z)| \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}, \quad z \in \mathbb{C}. \tag{2.2}$$

Then,

$$|f'(z)|, \quad \left| \frac{f(z)}{z - \lambda} \right| \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}, \quad \lambda \in \Lambda_f, \quad z \in \mathbb{C}.$$

Proof. Cauchy's formula [21, (3), p. 81]

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \int_{|z - \zeta| = 1/\varepsilon} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^2}$$

and (2.2) for any $z \in \mathbb{C}$ yield

$$\begin{aligned} |f'(z)| &\le \varepsilon \max_{|\zeta - z| = 1/\varepsilon} |f(\zeta)| \le \varepsilon C_\varepsilon \max_{|\zeta - z| = 1/\varepsilon} \mathrm{e}^{\varepsilon|\zeta|} \\ &\le \varepsilon C_\varepsilon \mathrm{e}^{\varepsilon(|z| + 1/\varepsilon)} = \varepsilon \mathrm{e} C_\varepsilon \mathrm{e}^{\varepsilon|z|} \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}. \end{aligned}$$

For arbitrary $\lambda \in \Lambda_f$ and $z \in \mathbb{C}$ satisfying $|z - \lambda| \ge 1/(2\varepsilon)$ it follows from (2.2) that

$$\left| \frac{f(z)}{z - \lambda} \right| \le 2\varepsilon C_\varepsilon \mathrm{e}^{\varepsilon|z|} \le C_\varepsilon \mathrm{e}^{\varepsilon|z|},$$

which by the maximum modulus principle [21, p. 165] yields

$$\begin{aligned} \left| \frac{f(z)}{z - \lambda} \right| &\le \max_{|\zeta - \lambda| = 1/(2\varepsilon)} \left| \frac{f(\zeta)}{\zeta - \lambda} \right| = 2\varepsilon \max_{|\zeta - \lambda| = 1/(2\varepsilon)} |f(\zeta)| \\ &\le 2\varepsilon C_\varepsilon \max_{|\zeta - \lambda| = 1/(2\varepsilon)} \mathrm{e}^{\varepsilon|\zeta|} \le 2\varepsilon C_\varepsilon \mathrm{e}^{\varepsilon(|z| + 1/\varepsilon)} \le 2\varepsilon \mathrm{e} C_\varepsilon \mathrm{e}^{\varepsilon|z|} \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}, \end{aligned}$$

provided that $|z - \lambda| \le 1/(2\varepsilon)$. This finishes the proof of Lemma 4. □

Lemma 5. *Let $\varepsilon \in (0, 1/(2\mathrm{e}))$, $C_\varepsilon \in (0, +\infty)$ and B be an entire function from the class $\mathcal{E}_0(\mathbb{R})$ satisfying*

$$\text{(2.3)} \qquad \text{(a)}\ \ |B(z)| \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}, \quad z \in \mathbb{C}, \qquad \text{(b)}\ \ \Theta_B := \sum_{\lambda\in\Lambda_B} \frac{1}{|B'(\lambda)|} < \infty.$$

Then, for arbitrary $\lambda \in \Lambda_B$ the inequality

$$\text{(2.4)} \qquad \left|\frac{B(x)}{x-\lambda}\right| \ge |B'(\lambda)|/2$$

holds for every real x satisfying

$$\text{(2.5)} \qquad |x-\lambda| \le \frac{\mathrm{e}^{-\varepsilon}}{1+2C_\varepsilon\Theta_B}\mathrm{e}^{-\varepsilon|\lambda|}.$$

Thus,

$$\text{(2.6)} \qquad \min_{\mu\in\Lambda_B\setminus\{\lambda\}} |\lambda-\mu| > \frac{\mathrm{e}^{-\varepsilon}}{1+2C_\varepsilon\Theta_B}\mathrm{e}^{-\varepsilon|\lambda|}, \quad \lambda \in \Lambda_B.$$

Proof. Let $\lambda \in \Lambda_B$ and

$$B_\lambda(x) := \frac{B(x)}{x-\lambda}.$$

Obviously, $B_\lambda(\lambda) = B'(\lambda)$, and it follows from Lemma 4 that

$$|B_\lambda'(z)| \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}, \quad z \in \mathbb{C}.$$

Furthermore, (2.3)(b) yields

$$|B'(\lambda)| \ge \Theta_B^{-1}.$$

Assume that $x \in [-1, 1]$ and

$$|B_\lambda(x+\lambda) - B_\lambda(\lambda)| > |B_\lambda(\lambda)|/2.$$

Then,

$$\begin{aligned} \Theta_B^{-1}/2 &\le |B'(\lambda)|/2 = |B_\lambda(\lambda)|/2 < |B_\lambda(x+\lambda) - B_\lambda(\lambda)| \\ &= \left|\int_0^{|x|} B_\lambda'(\lambda+\sigma t)\mathrm{d}t\right| \le C_\varepsilon \int_0^{|x|} \mathrm{e}^{\varepsilon|\lambda+\sigma t|}\mathrm{d}t \le C_\varepsilon \mathrm{e}^{\varepsilon(1+|\lambda|)}|x| \\ &< \left(C_\varepsilon + \Theta_B^{-1}/2\right)\mathrm{e}^{\varepsilon(1+|\lambda|)}|x|, \end{aligned}$$

where $\sigma = 1$ if $x > 0$ and $\sigma = -1$ if $x < 0$. This means that if

$$|x| \le \frac{\mathrm{e}^{-\varepsilon(1+|\lambda|)}}{1+2C_\varepsilon\Theta_B},$$

then

$$|B_\lambda(x+\lambda) - B'(\lambda)| \le |B'(\lambda)|/2,$$

and therefore

$$\begin{aligned} |B_\lambda(\lambda+x)| &= |B'(\lambda) + B_\lambda(\lambda+x) - B'(\lambda)| \\ &\ge |B'(\lambda)| - |B_\lambda(\lambda+x) - B'(\lambda)| \ge |B'(\lambda)|/2, \end{aligned}$$

which was to be proved. □

Lemma 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be integrable on every compact segment $[a, b]$ of the real line, $\omega \in C^\infty(\mathbb{R})$ be strictly positive on $\mathbb{R}$, κ be defined in* (1.18) *and*

$$f_\omega(x) := \frac{1}{\kappa}\int_{-1}^{1} f(x + t\omega(x)) \exp\left(-\frac{1}{1-t^2}\right) \mathrm{d}t, \ x \in \mathbb{R}. \tag{2.7}$$

Then, $f_\omega \in C^\infty(\mathbb{R})$ and for every $x \in \mathbb{R}$ we have

$$f'_\omega(x) = \frac{1}{\kappa\,\omega(x)}\int_{-1}^{1} f(x + t\omega(x))\frac{2t}{(t^2-1)^2} \exp\left(-\frac{1}{1-t^2}\right) \mathrm{d}t$$

$$-\frac{\omega'(x)}{\omega(x)} f_\omega(x) + \frac{2\,\omega'(x)}{\kappa\,\omega(x)}\int_{-1}^{1} f(x + t\omega(x))\frac{t^2}{(t^2-1)^2} \exp\left(-\frac{1}{1-t^2}\right) \mathrm{d}t. \tag{2.8}$$

Proof. Since

$$\kappa f_\omega(x) = \frac{1}{\omega(x)}\int_{x-\omega(x)}^{x+\omega(x)} f(t) \exp\left(-\frac{\omega(x)^2}{\omega(x)^2 - (t-x)^2}\right) \mathrm{d}t, \quad x \in \mathbb{R},$$

then $f_\omega \in C^\infty(\mathbb{R})$ and for arbitrary $x \in \mathbb{R}$ we obtain

$$\kappa f'_\omega(x) = -\kappa f_\omega(x)\frac{\omega'(x)}{\omega(x)} + \frac{1}{\omega(x)}\int_{x-\omega(x)}^{x+\omega(x)} f(t) T_\omega(x, t)\mathrm{d}t,$$

where

$$T_\omega(x, t) := \frac{\mathrm{d}}{\mathrm{d}x}\exp\left(-\frac{\omega(x)^2}{\omega(x)^2 - (t-x)^2}\right)$$

$$= \left[\frac{2\,\omega(x)\,\omega'(x)(t-x)^2}{\left((t-x)^2 - \omega(x)^2\right)^2} + \frac{2\,\omega(x)^2(t-x)}{\left((t-x)^2 - \omega(x)^2\right)^2}\right] \exp\left(-\frac{\omega(x)^2}{\omega(x)^2 - (t-x)^2}\right),$$

from which (2.8) follows easily by the change of variables. Lemma 6 is proved. □

3. Proof of Lemma 1

3.1. If Lemma 1 is proved for $\delta = \delta_0 > 0$, then for arbitrary $\delta_1 > \delta_0$ it follows from $|\lambda - d_\lambda| \le \rho_{\delta_0} \mathrm{e}^{-\delta_1|\lambda|} \le \rho_{\delta_0} \mathrm{e}^{-\delta_0|\lambda|}$, $\lambda \in \Lambda_B$ that Lemma 1 also holds for $\delta = \delta_1$ with $C_{\delta_1} = C_{\delta_0}$ and $\rho_{\delta_1} = \rho_{\delta_0}$. Therefore, it is sufficient to prove Lemma 1 only for those numbers δ which satisfy

$$0 < \delta < 1/\mathrm{e}.$$

3.2. Let B be an entire function satisfying the conditions of Lemma 1. Then, these conditions are met by any translation of B of the form $B_{T_a}(z) := B(z + a)$, $a \in \mathbb{R} \setminus \{0\}$ because $\Lambda_{B_{T_a}} = \Lambda_B - a$, $\Theta_{B_{T_a}} = \Theta_B$ and $B_{T_a} \in \mathcal{E}_0(\mathbb{R})$, where Θ_B denotes the value of the series in (1.2).

We show that if Lemma 1 is proved for the function B, then it also holds for any B_{T_a}, $a \in \mathbb{R} \setminus \{0\}$, with constants $\rho_\delta(B_{T_a}) = \mathrm{e}^{-\delta|a|}\rho_\delta(B)$ and $C_\delta(B_{T_a}) = C_\delta(B)$.

Let $\delta > 0$, a be an arbitrary non-zero real number and $E := B_{T_a}$. If $\{e_\lambda\}_{\lambda\in\Lambda_E}$ is any collection of real numbers satisfying $|\lambda - e_\lambda| \le \rho_\delta(E) \exp(-\delta|\lambda|)$, $\lambda \in \Lambda_E$, then in view of $\Lambda_E = \Lambda_B - a$ we have

$$|\lambda - a - e_{\lambda-a}| \le \rho_\delta(E)\mathrm{e}^{-\delta|\lambda-a|} \le \mathrm{e}^{\delta|a|}\rho_\delta(E)\mathrm{e}^{-\delta|\lambda|} = \rho_\delta(B)\mathrm{e}^{-\delta|\lambda|}, \quad \lambda \in \Lambda_B,$$

and therefore the numbers $d_\lambda := e_{\lambda-a} + a$, $\lambda \in \Lambda_B$, satisfy condition (1.3). Thus, there exists an entire function $D \in \mathcal{E}_0(\mathbb{R})$ such that $\Lambda_D = \{d_\lambda\}_{\lambda\in\Lambda_B}$ and $|B'(\lambda)| \le C_\delta(B)\,|D'(d_\lambda)|$, $\lambda \in \Lambda_B$. Then, for the function $G(z) := D(z+a)$ we have $G \in \mathcal{E}_0(\mathbb{R})$, $\Lambda_G = \Lambda_D - a = \{d_\lambda - a\}_{\lambda\in\Lambda_B} = \{d_{\lambda+a} - a\}_{\lambda\in\Lambda_E} = \{e_\lambda\}_{\lambda\in\Lambda_E}$ and $|E'(\lambda)| = |B'(a+\lambda)| \le C_\delta(B)\,|D'(d_{a+\lambda})| = C_\delta(B)\,|G'(d_{a+\lambda} - a)| = C_\delta(B)\,|G'(e_\lambda)|$, $\lambda \in \Lambda_E$. This implies the validity of Lemma 1 for B_{T_a}, as claimed.

We conclude that to prove Lemma 1 for all translations B_{T_a}, $a \in \mathbb{R}$, of the entire function B it is sufficient to prove it for at least one of them. We specify the translation of B by choosing an $a \in \mathbb{R}\setminus\Lambda_B$ such that $\min_{\lambda\in\Lambda_B,\lambda>a}(\lambda - a) = \min_{\lambda\in\Lambda_B,\lambda<a}(a-\lambda)$ if Λ_B is unbounded in both directions, $a > 1 + \max\Lambda_B$ if Λ_B is bounded from above and $a < -1 + \min\Lambda_B$ if Λ_B is bounded from below. Considering such B_{T_a} as the initial function B in Lemma 1, we can therefore assume that the set Λ_B of all zeros of B in Lemma 1 obeys the following additional properties:

$$
\begin{array}{llll}
 & \text{(a)} & 0 \notin \Lambda_B; & \\
(3.1) & \text{(b)} & \min\limits_{\lambda\in\Lambda_B,\lambda>0}|\lambda| = \min\limits_{\lambda\in\Lambda_B,\lambda<0}|\lambda| & \text{if } \sup\Lambda_B = +\infty \text{ and } \inf\Lambda_B = -\infty; \\
 & \text{(c)} & \min\Lambda_B > 1 & \text{if } \inf\Lambda_B > -\infty; \\
 & \text{(d)} & \max\Lambda_B < -1 & \text{if } \sup\Lambda_B < +\infty.
\end{array}
$$

Observe that (3.1)(b) means the existence of two neighboring zeros $\lambda_1, \lambda_2 \in \Lambda_B$ of B (i.e., $\lambda_1 < \lambda_2$, $(\lambda_1,\lambda_2)\cap\Lambda_B = \emptyset$) such that $\lambda_1 = -\lambda_2$.

3.3. Denote by Θ_B the value of the series in (1.2) and let

$$\varepsilon := \delta/2 \in (0,\ 1/(2\mathrm{e})\,),\quad \rho_\delta := \left(\frac{\mathrm{e}^{-\varepsilon}}{4 + 8C_\varepsilon\Theta_B}\right)^2 \in (0, 1/16), \tag{3.2}$$

where

$$C_\varepsilon := \sup_{z\in\mathbb{C}} \mathrm{e}^{-\varepsilon|z|}|B(z)| < \infty.$$

Then, for the function B the conditions of Lemma 5 are fulfilled and (2.5) implies that

$$[\lambda_1 - 2\Delta_{\lambda_1},\ \lambda_1 + 2\Delta_{\lambda_1}] \cap [\lambda_2 - 2\Delta_{\lambda_2},\ \lambda_2 + 2\Delta_{\lambda_2}] = \emptyset,\ \ \lambda_1, \lambda_2 \in \Lambda_B,\ \ \lambda_1 \ne \lambda_2, \tag{3.3}$$

where

$$\Delta_\lambda := \sqrt{\rho_\delta}\ \mathrm{e}^{-\varepsilon|\lambda|} \in (0, 1/4),\quad \lambda \in \Lambda_B. \tag{3.4}$$

Actually, assume that there exist $\lambda_1, \lambda_2 \in \Lambda_B$ such that $\lambda_1 < \lambda_2$, $(\lambda_1,\lambda_2)\cap\Lambda_B = \emptyset$ and

$$\lambda_1 + 2\Delta_{\lambda_1} \ge \lambda_2 - 2\Delta_{\lambda_2}. \tag{3.5}$$

By virtue of (2.6),

$$\lambda_1 < \lambda_2 - 4\Delta_{\lambda_2},\quad \lambda_1 + 4\Delta_{\lambda_1} < \lambda_2, \tag{3.6}$$

and therefore

$$\lambda_2 - \lambda_1 > 2\Delta_{\lambda_1} + 2\Delta_{\lambda_2},$$

which contradicts (3.5) and proves (3.3).

Introduce the following neighborhood of Λ_B:

$$\Lambda_B^\Delta := \bigsqcup\nolimits_{\lambda\in\Lambda_B} [\,\lambda - 2\Delta_\lambda,\ \lambda + 2\Delta_\lambda\,]\,. \tag{3.7}$$

We now prove that for any two neighboring zeros $\lambda_1 < \lambda_2$ of B the midpoint of the interval $[\lambda_1, \lambda_2]$ does not belong to Λ_B^{Δ}. In fact, it follows from $\lambda_1, \lambda_2 \in \Lambda_B$, $\lambda_1 < \lambda_2$, $(\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset$ and (3.6) that

$$\frac{\lambda_1 + \lambda_2}{2} < \lambda_2 - 2\Delta_{\lambda_2}\,, \quad \lambda_1 + 2\Delta_{\lambda_1} < \frac{\lambda_1 + \lambda_2}{2},$$

which proves

$$\lambda_1 < \lambda_2,\ \lambda_1, \lambda_2 \in \Lambda_B,\ \ (\lambda_1, \lambda_2) \cap \Lambda_B = \emptyset \ \Rightarrow\ \frac{\lambda_1 + \lambda_2}{2} \notin \Lambda_B^{\Delta}. \tag{3.8}$$

Together with (3.1) this property means that

$$0 \notin \Lambda_B^{\Delta}. \tag{3.9}$$

Actually, if Λ_B is unbounded in both directions, then according to (3.1)(b) the origin is the midpoint of a segment joining two neighboring zeros of B which have opposite signs. It follows from (3.8) that (3.9) holds. In the case when Λ_B is bounded from one side the distance $\min_{\lambda \in \Lambda_B} |\lambda|$ between 0 and Λ_B is greater than 1, by virtue of (3.1)(c),(d). But in view of (3.4), $2\Delta_\lambda < 1/2$, and therefore (3.9) follows readily from (3.7).

3.4. If $\{d_\lambda\}_{\lambda \in \Lambda_B}$ are arbitrary numbers satisfying (1.3), it follows from (1.3), (3.2) and (3.4) that

$$d_\lambda \in \left[\,\lambda - \Delta_\lambda^2,\ \lambda + \Delta_\lambda^2\,\right] \subset [\,\lambda - \Delta_\lambda,\ \lambda + \Delta_\lambda\,], \quad \lambda \in \Lambda_B, \tag{3.10}$$

and in view of (3.3) that

$$d_{\lambda_0} \notin [\,\lambda - 2\Delta_\lambda,\ \lambda + 2\Delta_\lambda\,], \quad \lambda_0, \lambda \in \Lambda_B, \quad \lambda_0 \neq \lambda. \tag{3.11}$$

It is worth remembering that according to the Lindelöf theorem [13, Theorem 15, p. 28] a set $\Lambda \subset \mathbb{R} \setminus \{0\}$ is the set of all zeros of some entire function from the class $\mathcal{E}_0(\mathbb{R})$ if and only if there exists a finite limit of $\delta_\Lambda(R)$ and $n_\Lambda(R)/R \to 0$ as $R \to +\infty$. Here,

$$\delta_\Lambda(R) := \sum\nolimits_{\lambda \in \Lambda \cap (-R,R)} 1/\lambda, \quad n_\Lambda(R) := \operatorname{card}\left\{\lambda \in \Lambda \ \middle|\ |\lambda| < R\right\}, \quad R > 0,$$

and $\operatorname{card} A \in \mathbb{N}_0 \cup \{+\infty\}$ denotes the number of elements in a set A. Then, all functions $f \in \mathcal{E}_0(\mathbb{R})$ satisfying $\Lambda_f = \Lambda$ are given by the following formula:

$$f(z) = A \lim_{R \to \infty} \prod\nolimits_{\lambda \in \Lambda \cap (-R,R)} (1 - z/\lambda)\,, \quad A \in \mathbb{R} \setminus \{0\}, \quad z \in \mathbb{C},$$

where $f(0) = A \neq 0$. Thus,

$$B(z) = B(0) \lim_{R \to \infty} \prod\nolimits_{\lambda \in \Lambda_B \cap (-R,R)} (1 - z/\lambda)\,, \quad z \in \mathbb{C}, \tag{3.12}$$

and it follows from $\lim_{R \to +\infty} n_B(R)/R = 0$ that $\sum_{\lambda \in \Lambda_B} 1/\lambda^2 < \infty$.

Denote $\Lambda_D := \{d_\lambda\}_{\lambda \in \Lambda_B}$. Since $\Lambda_B = \{\lambda\}_{\lambda \in \Lambda_B}$ satisfies the conditions of Lindelöf's theorem, they are also met by the set Λ_D because

$$|d_\lambda - \lambda| \leq \frac{\rho_\delta}{\delta^2 \lambda^2}, \quad \lambda \in \Lambda_B,$$

by virtue of (1.3) and the inequality

$$\exp(-x) \leq 1/x^2, \quad x > 0. \tag{3.13}$$

Therefore, Λ_D is the set of all zeros of the entire function

$$D(z) := \lim_{R\to\infty} \prod_{\substack{\lambda\in\Lambda_B \\ d_\lambda\in(-R,R)}} (1 - z/d_\lambda), \quad z \in \mathbb{C}, \tag{3.14}$$

which belongs to the class $\mathcal{E}_0(\mathbb{R})$.

Let m denote the Lebesgue measure on $\mathbb{R}$. Then it follows from (3.4), (3.7) and (3.13) that

$$m\left(\Lambda_B^\Delta\right) \le 4\sqrt{\rho_\delta} \sum_{\lambda\in\Lambda_B} \mathrm{e}^{-\varepsilon|\lambda|} \le 4\varepsilon^{-2}\sqrt{\rho_\delta} \sum_{\lambda\in\Lambda_B} 1/\lambda^2 < \infty.$$

Hence, the set

$$\mathbb{R}_B^+ := [0, +\infty) \setminus \left(\Lambda_B^\Delta \cup -\Lambda_B^\Delta\right)$$

is unbounded, and in view of (3.10), (3.3) and (3.7) we have

$$\{\lambda \mid \lambda \in \Lambda_B \cap (-R, R)\} = \{\lambda \mid \lambda \in \Lambda_B,\ d_\lambda \in (-R, R)\}, \quad R \in \mathbb{R}_B^+. \tag{3.15}$$

3.5. Let us estimate $|B'(\lambda_0)|/|D'(d_{\lambda_0})|$ for arbitrary $\lambda_0 \in \Lambda_B$. It follows from (2.4), (2.5), (3.2) and (3.4) that

$$\left|\frac{B(x)}{x-\lambda}\right| \ge |B'(\lambda)|/2, \quad x \in [\lambda - 4\Delta_\lambda, \lambda + 4\Delta_\lambda], \quad \lambda \in \Lambda_B,$$

and therefore, by (3.10), we have

$$|B'(\lambda_0)| \le \frac{2}{|\lambda_0|} \frac{|B(d_{\lambda_0})|}{\left|1 - \dfrac{d_{\lambda_0}}{\lambda_0}\right|}.$$

Then, by (3.12), (3.14) and (3.15),

$$\begin{aligned} \frac{|B'(\lambda_0)|}{|D'(d_{\lambda_0})|} &\le \frac{2}{|\lambda_0|} \frac{|B(d_{\lambda_0})|}{\left|1 - \dfrac{d_{\lambda_0}}{\lambda_0}\right| |D'(d_{\lambda_0})|} \\ &= \frac{2|d_{\lambda_0}||B(0)|}{|\lambda_0|} \lim_{\substack{R\to+\infty \\ R\in\mathbb{R}_B^+}} \prod_{\substack{\lambda\in\Lambda_B\cap(-R,R) \\ \lambda\ne\lambda_0}} \left|\frac{1 - d_{\lambda_0}/\lambda}{1 - d_{\lambda_0}/d_\lambda}\right|. \end{aligned} \tag{3.16}$$

The relations (3.7) and (3.9) imply that $0 \notin [\lambda - 2\Delta_\lambda,\ \lambda + 2\Delta_\lambda]$ and therefore $|\lambda| \le 2\Delta_\lambda$, which together with the consequence $|d_\lambda| \le |\lambda| + \Delta_\lambda^2$ of (3.10) yields in view of (3.4) $|d_\lambda/\lambda| \le 1 + \Delta_\lambda^2/|\lambda| \le 1 + \Delta_\lambda/2 \le 2$ for every $\lambda \in \Lambda_B$. Thus, in (3.16) we have $|d_{\lambda_0}|/|\lambda_0| \le 2$.

Setting in Lemma 3, $x = d_{\lambda_0}$, $a = \lambda$, $b = d_\lambda$ and $\Delta = \Delta_\lambda$ with λ_0 and λ taken from (3.16), we obtain the validity of the conditions (2.1),

$$d_\lambda \in (\lambda - \Delta_\lambda^2, \lambda + \Delta_\lambda^2), \quad 0 \notin (\lambda - \Delta_\lambda, \lambda + \Delta_\lambda), \quad d_{\lambda_0} \notin (\lambda - 2\Delta_\lambda, \lambda + 2\Delta_\lambda),$$

as a consequence of (3.10), (3.7), (3.9), and (3.11). Hence, the factors in (3.16) satisfy

$$\left|(1 - d_{\lambda_0}/\lambda)(1 - d_{\lambda_0}/d_\lambda)^{-1}\right| \le (1 + \Delta_\lambda)^2 = \left(1 + \sqrt{\rho_\delta}\, \mathrm{e}^{-\varepsilon|\lambda|}\right)^2,$$

by virtue of (3.4). It follows therefore from (3.16) that

$$\frac{|B'(\lambda_0)|}{|D'(d_{\lambda_0})|} \le C_\delta := 4|B(0)| \prod_{\lambda\in\Lambda_B} \left(1 + \sqrt{\rho_\delta}\, \mathrm{e}^{-\varepsilon|\lambda|}\right)^2 < \infty, \quad \lambda_0 \in \Lambda_B, \tag{3.17}$$

where the product above is finite in view of (3.13), (3.1)(a) and $\sum_{\lambda\in\Lambda_B} 1/\lambda^2<\infty$. Lemma 1 is proved, and the formulas (3.17), (3.2) together with the reasoning of subsection 3.2 establish the explicit expressions for the constants ρ_δ and C_δ in (1.3) and in (1.4).

4. Proof of Corollary 1

We first prove (1.18). It follows from

$$\int_{-1}^{1} \mathrm{e}^{-\frac{1}{1-t^2}}\,\mathrm{d}t=\frac{1}{\mathrm{e}}\int_0^\infty \frac{\mathrm{e}^{-t}\mathrm{d}t}{\sqrt{t}(t+1)^{3/2}}=\frac{2}{\mathrm{e}}\int_0^\infty \mathrm{e}^{-t}\mathrm{d}\sqrt{\frac{t}{t+1}}=\frac{2}{\mathrm{e}}\int_0^\infty \mathrm{e}^{-t}\sqrt{\frac{t}{t+1}}\mathrm{d}t$$
$$=-\frac{2}{\mathrm{e}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \frac{\mathrm{e}^{-xt}\mathrm{d}t}{\sqrt{t(t+1)}}\Big|_{x=1}=-\frac{2}{\mathrm{e}}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x/2}\int_1^\infty \frac{\mathrm{e}^{-(x/2)t}\mathrm{d}t}{\sqrt{t^2-1)}}\Big|_{x=1}$$

and [9, (19), p. 82] that

$$\kappa=-\frac{2}{\mathrm{e}}\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{x/2}K_0(x/2)\Big|_{x=1}=-\frac{K_0(1/2)+K_0'(1/2)}{\sqrt{\mathrm{e}}}=\frac{K_1(1/2)-K_0(1/2)}{\sqrt{\mathrm{e}}},$$

by virtue of [9, (21), p. 79]. The values in [1, p. 417], $\mathrm{e}^{0.5}K_0(0.5)=1.52410\ldots$ and $\mathrm{e}^{0.5}K_1(0.5)=2.73100\ldots$ finish the proof of (1.18). Similarly, we obtain

$$\int_{-1}^{1}\frac{2|t|}{(t^2-1)^2}\exp\left(-\frac{1}{1-t^2}\right)\mathrm{d}t=\frac{2}{\mathrm{e}},\quad \int_{-1}^{1}\frac{t^2}{(t^2-1)^2}\exp\left(-\frac{1}{1-t^2}\right)\mathrm{d}t=\frac{\kappa}{2}. \tag{4.1}$$

Let the function ϕ_ε be defined in (1.17). In order to prove Corollary 1, we observe that all constant functions belong to the set $C^\infty(\mathbb{R})$, and therefore we can apply Lemma 6 to the function $4\mathrm{e}^\varepsilon\phi_\varepsilon$ which coincides with the function f_ω in (2.7) for $\omega\equiv 1$ and $f(x)=\exp(-\varepsilon|x|)$. Thus, (2.8) and (1.17) yield for every $x\in\mathbb{R}$ that

$$\phi_\varepsilon'(x)=\frac{\mathrm{e}^{-\varepsilon}}{4\kappa}\int_{-1}^{1}\mathrm{e}^{-\varepsilon|x+t|}\frac{2t\mathrm{e}^{-\frac{1}{1-t^2}}}{(1-t^2)^2}\,\mathrm{d}t,\quad \phi_\varepsilon(x)=\frac{\mathrm{e}^{-\varepsilon}}{4\kappa}\int_{-1}^{1}\mathrm{e}^{-\varepsilon|x+t|}\mathrm{e}^{-\frac{1}{1-t^2}}\,\mathrm{d}t. \tag{4.2}$$

It follows from (4.2), (4.1), (1.18) and

$$-|x|-1\le -|x+t|\le 1-|x|,\quad |t|\le 1,\ t,x\in\mathbb{R},$$

that

$$|\phi_\varepsilon'(x)|\le (5/12)\,\mathrm{e}^{-\varepsilon|x|},\qquad (\mathrm{e}^{-2\varepsilon}/4)\,\mathrm{e}^{-\varepsilon|x|}\le\phi_\varepsilon(x)\le(1/4)\,\mathrm{e}^{-\varepsilon|x|},\ x\in\mathbb{R}. \tag{4.3}$$

Let W_ε be defined as in (1.14) with $\omega=\phi_\varepsilon$. Then by Theorem 1, $W_\varepsilon\in C^\infty(\mathbb{R})\cap\mathcal{W}^{\mathrm{reg}}(\mathbb{R})$ and $W_\varepsilon(x)\ge w(x)+\mathrm{e}^{-\varepsilon|x|}$ for all $x\in\mathbb{R}$. Furthermore, W_ε is equal to the function f_ω in (2.7) for $\omega=\phi_\varepsilon$ and $f=\Omega_{1/2}$. Thus, by (2.8), we obtain

$$W_\varepsilon'(x)=\frac{1}{\kappa\,\phi_\varepsilon(x)}\int_{-1}^{1}\Omega_{1/2}(x+t\phi_\varepsilon(x))\frac{2t}{(t^2-1)^2}\exp\left(-\frac{1}{1-t^2}\right)\mathrm{d}t-\frac{\phi_\varepsilon'(x)}{\phi_\varepsilon(x)}W_\varepsilon(x)$$
$$+\frac{2\,\phi_\varepsilon'(x)}{\kappa\,\phi_\varepsilon(x)}\int_{-1}^{1}\Omega_{1/2}(x+t\phi_\varepsilon(x))\frac{t^2}{(t^2-1)^2}\exp\left(-\frac{1}{1-t^2}\right)\mathrm{d}t. \tag{4.4}$$

According to (1.15) and (1.16), $\Omega_{1/2}(x + t\phi_\varepsilon(x)) \le w_\varepsilon(x)$ and $W_\varepsilon(x) \le w_\varepsilon(x)$ for all $|t| \le 1$ and $x \in \mathbb{R}$. Therefore, it follows from (4.1), (4.3) and (4.4) that

$$|W_\varepsilon'(x)| \le w_\varepsilon(x)\left[\frac{4\mathrm{e}^{2\varepsilon}\mathrm{e}^{\varepsilon|x|}}{\kappa}\cdot\frac{2}{\mathrm{e}} + \frac{10\mathrm{e}^{2\varepsilon}}{6} + \frac{2}{\kappa}\cdot\frac{10\mathrm{e}^{2\varepsilon}}{6}\cdot\frac{\kappa}{2}\right]$$
$$\le 10\mathrm{e}^{2\varepsilon}\mathrm{e}^{\varepsilon|x|}w_\varepsilon(x) \le 74\mathrm{e}^{\varepsilon|x|}w_\varepsilon(x),$$

which completes the proof of Corollary 1.

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Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv 01601, Ukraine
E-mail address: `andrew@bakan.kiev.ua`

Institut für Mathematik, Universität zu Lübeck, D-23562 Lübeck, Germany
E-mail address: `prestin@math.uni-luebeck.de`