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MOTIVIC SERRE INVARIANTS MODULO THE SQUARE OF $\mathbb{L}-1$

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Abstract. Motivic Serre invariants defined by Loeser and Sebag are elements of the Grothendieck ring of varieties modulo $\mathbb{L}-1$. In this paper, we show that we can lift these invariants to modulo the square of $\mathbb{L}-1$ after tensoring the Grothendieck ring with $\mathbb{Q}$ under certain assumptions.

1. Introduction

Let K be a complete discrete valuation field with a perfect residue field k. For a smooth projective irreducible K-variety X, Loeser and Sebag [9] defined the motivic Serre invariant $S(X)$. This invariant belongs to the ring $K_0(\mathrm{Var}_k)/(\mathbb{L}-1)$, where $K_0(\mathrm{Var}_k)$ is the Grothendieck ring of k-varieties and $\mathbb{L} := [\mathbb{A}^1_k]$, the class of an affine line in this ring. Let $K_0(\mathrm{Var}_k)_{\mathbb{Q}} := K_0(\mathrm{Var}_k) \otimes_{\mathbb{Z}} \mathbb{Q}$. In this paper, we construct, under a certain assumption, an invariant

$$\tilde{S}(X) \in K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)^2$$

which coincides with $S(X)$ in $K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)$.

Remark 1.1. Loeser and Sebag defined the motivic Serre invariant more generally for smooth quasi-compact separated rigid K-spaces. For the sake of simplicity, we consider only the case where X is a projective variety.

Let $\mathcal{O}$ be the valuation ring of K. The assumption we will make is that the desingularization theorem and the weak factorization theorem hold; their precise statements are as follows:

Assumption 1.2.

(1) *(Desingularization) There exists a regular projective flat $\mathcal{O}$-scheme $\mathcal{X}$ with the generic fiber $\mathcal{X}_K := \mathcal{X} \otimes_{\mathcal{O}} K = X$ such that the special fiber $\mathcal{X}_k := \mathcal{X} \otimes_{\mathcal{O}} k$ is a simple normal crossing divisor in $\mathcal{X}$. (We call such an $\mathcal{X}$ a* regular snc model *of X.)*

(2) *(Weak factorization) Let $\mathcal{X}$ and $\mathcal{X}'$ be regular snc models of X. Then there exist finitely many regular snc models of X,*

$$\mathcal{X}_0 = \mathcal{X},\ \mathcal{X}_1, \dots, \mathcal{X}_n = \mathcal{X}',$$

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such that for every i, either the birational map $\mathcal{X}_i \dashrightarrow \mathcal{X}_{i+1}$ is the blowup along a regular center $Z \subset \mathcal{X}_{i+1,k}$ which has normal crossings[1] *with $\mathcal{X}_{i+1,k}$ or its inverse $\mathcal{X}_{i+1} \dashrightarrow \mathcal{X}_i$ has the same description with $\mathcal{X}_{i+1,k}$ replaced with $\mathcal{X}_{i,k}$.*

When X has dimension one, this assumption holds, as is well-known. Indeed the above desingularization theorem in this case follows from the desingularization theorem for excellent surfaces by Abhyankar, Hironaka and Lipman (see [8]), while the weak factorization follows from the fact that every proper birational morphism of regular integral noetherian schemes of dimension two factors into a sequence of finitely many blowups at closed points. The last fact is well-known in the case of varieties over an algebraically closed field (for instance, [5, V, Cor. 5.4]) and is valid even in our situation as proved in [7, Th. 4.1] in a more general context. Assumption 1.2 holds also when k has characteristic zero. This follows from the recent generalizations to excellent schemes respectively by Temkin [12, 13] and by Abramovich and Temkin [2] of the Hironaka desingularization theorem and the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1].

Let $\mathcal{X}$ be a regular snc model of X, let $\mathcal{X}_{\mathrm{sm}}$ be its $\mathcal{O}$-smooth locus and let $\mathcal{X}_{\mathrm{sm},k} := \mathcal{X}_{\mathrm{sm}} \otimes_{\mathcal{O}} k$. Then $\mathcal{X}_{\mathrm{sm}}$ is a weak Neron model of X in the sense of [3] and by definition,

$$S(X) = [\mathcal{X}_{\mathrm{sm},k}] \in K_0(\mathrm{Var}_k)/(\mathbb{L}-1).$$

To define our invariant $\tilde{S}(X)$, we also need information on the non-smooth locus of $\mathcal{X}$. Regard $\mathcal{X}_k$ as a divisor and write it as $\mathcal{X}_k = \sum_{i \in I} a_i D_i$, where D_i are the irreducible components of $\mathcal{X}_k$ and a_i are the multiplicities of D_i in $\mathcal{X}$ respectively. For a subset $H \subset I$, we define

$$D_H^\circ := \bigcap_{h \in H} D_h \setminus \bigcup_{i \in I \setminus H} D_i.$$

When $H = \{i\}$, we abbreviate it to D_i°, and when $H = \{i, j\}$, to D_{ij}°. These locally closed subsets give the stratification

$$\mathcal{X}_k = \bigsqcup_{\emptyset \neq H \subset I} D_H^\circ$$

and the stratification

$$\mathcal{X}_{\mathrm{sm},k} = \bigcup_{i \in I:\, a_i = 1} D_i^\circ.$$

From the second stratification, we see that

$$S(X) = \sum_{i \in I:\, a_i = 1} [D_i^\circ] \in K_0(\mathrm{Var}_k)/(\mathbb{L}-1).$$

Loeser and Sebag proved in the paper cited above that this is independent of the model $\mathcal{X}$ and depends only on X.

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[1] That *Z has normal crossings with $\mathcal{X}_{i+1,k}$* means that for every closed point $x \in \mathcal{X}_{i+1,k}$, there exists a regular system of parameters $x_1, \ldots, x_d \in \mathcal{O}_{\mathcal{X}_{i+1},x}$ such that in an open neighborhood of x, the support of the special fiber $\mathcal{X}_{i+1,k}$ is the zero locus of $\prod_{v \in V} x_v$ for some subset $V \subset \{1, \ldots, d\}$ and Z is the common zero locus of x_w, $w \in W$, for some $W \subset \{1, \ldots, d\}$.

Definition 1.3. For a regular snc model $\mathcal{X}$ of X, we define

$$\tilde{S}(\mathcal{X}) := \sum_{i\in I\colon a_i=1}[D_i^\circ] + \sum_{\substack{\{i,j\}\subset I\colon\\ (a_i,a_j)=1}} \frac{1}{a_ia_j}[D_{ij}^\circ](1-\mathbb{L})$$

as an element of $K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)^2$. Here (a,b) denotes the greatest common divisor of a and b.

Obviously, the two invariants $S(X)$ and $\tilde{S}(\mathcal{X})$ coincide when they are sent to $K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)$ by the natural maps.

The following is our main theorem:

Theorem 1.4. *Let X be a smooth projective K-variety. Under Assumption* 1.2*, the invariant $\tilde{S}(\mathcal{X})$ is independent of the chosen regular snc model $\mathcal{X}$ and depends only on X.*

The theorem allows us to think of $\tilde{S}(\mathcal{X})$ as an invariant of X and denote it by $\tilde{S}(X)$, which is what was mentioned at the beginning of this Introduction.

2. Preparatory reductions

We generalize the invariant $\tilde{S}(\mathcal{X})$ as follows. Let $\mathcal{X}$ be a regular flat $\mathcal{O}$-scheme of finite type such that $\mathcal{X}_K$ is smooth and $\mathcal{X}_k = \bigcup_{i\in I} D_i$ is a simple normal crossing divisor in $\mathcal{X}$. (We no longer suppose that $\mathcal{X}$ or $\mathcal{X}_K$ is projective.) For a constructible subset $C \subset \mathcal{X}_k$, we define

$$\tilde{S}(\mathcal{X}, C) := \sum_{\substack{i\in I\colon\\ a_i=1}}[D_i^\circ \cap C] + \sum_{\substack{\{i,j\}\subset I\colon\\ (a_i,a_j)=1}} \frac{1}{a_ia_j}[D_{ij}^\circ \cap C](1-\mathbb{L})$$

as an element of $K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)^2$.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be the blowup along a smooth irreducible center $Z \subset \mathcal{X}_k$ which has normal crossings with $\mathcal{X}_k$. Then, $\mathcal{Y}$ is an $\mathcal{O}$-scheme satisfying the same conditions as $\mathcal{X}$ does, and we can similarly define $\tilde{S}(\mathcal{Y}, C')$ for a constructible subset $C' \subset \mathcal{Y}_k$.

Theorem 1.4 follows from:

Proposition 2.1. *Let $\mathcal{X}$ be as above. For any constructible subset $C \subset \mathcal{X}_k$, we have*

$$\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$

Indeed, Theorem 1.4 is a direct consequence of this proposition with $C = \mathcal{X}_k$ and Assumption 1.2.

In what follows, we will prove this proposition. First we will reduce it to the local situation by using:

Lemma 2.2.

(1) *If C is the disjoint union $\bigsqcup_{s=1}^{l} C_s$ of constructible subsets C_s, then*

$$\tilde{S}(\mathcal{X}, C) = \sum_{s=1}^{l} \tilde{S}(\mathcal{X}, C_s).$$

(2) *Let $\mathcal{X} = \bigcup_{\lambda\in\Lambda} U_\lambda$ be an open covering. Suppose that for every constructible subset $C \subset \mathcal{X}_k$ and for every $\lambda \in \Lambda$,*

$$\tilde{S}(\mathcal{X}, C\cap U_\lambda) = \tilde{S}(\mathcal{Y}, f^{-1}(C\cap U_\lambda)).$$

Then, for every constructible subset $C \subset \mathcal{X}_k$, we have
$$\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$

Proof. The first assertion is obvious. To show the second one, we first claim that there exists a stratification $C = \bigsqcup_{s=0}^{n} C_s$ with C_s constructible such that each C_s is contained in some U_λ. Indeed we can take C_0 as $C \cap U_\lambda$ such that C and C_0 have equal dimension, then construct C_1 applying the same procedure to $C \setminus U_\lambda$ and so on.

By the assumption, for every s, $\tilde{S}(\mathcal{X}, C_s) = \tilde{S}(\mathcal{Y}, f^{-1}(C_s))$. Now, from the first assertion, we get
$$\tilde{S}(\mathcal{X}, C) = \sum_s \tilde{S}(\mathcal{X}, C_s) = \sum_s \tilde{S}(\mathcal{Y}, f^{-1}(C_s)) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).$$
□

Let $x \in \mathcal{X}_k$ be a closed point and take a local coordinate system $x_1, \dots, x_d \in \mathcal{O}_{\mathcal{X},x}$. By shrinking $\mathcal{X}$ if necessary, we may suppose that $x_1, \dots, x_d$ are global sections of $\mathcal{O}_{\mathcal{X}}$ and that the special fiber $\mathcal{X}_k$ is the zero locus of $\prod_{i=1}^{d'} x_i$, $d' \le d$ (thus we identify I with $\{1, \dots, d'\}$) and Z is the common zero locus of x_j, $j \in J$, for some subset $J \subset \{1, \dots, d\}$. From the first assertion of the above lemma, since we obviously have
$$\tilde{S}(\mathcal{X}, C \setminus Z) = \tilde{S}(\mathcal{Y}, f^{-1}(C \setminus Z)),$$
we may also assume that
$$C \subset Z. \tag{2.1}$$
In a few following sections, we will prove Proposition 2.1 in this situation, discussing separately in the cases $(\sharp I =) d' = 1$, $d' = 2$ and $d' \ge 3$. Before that, we prepare some notation and a lemma.

Notation 2.3. For $i \in I$, let D_i be the prime divisor of $\mathcal{X}$ given by $x_i = 0$ and let $E_i \subset \mathcal{Y}_k$ be its strict transform. Let $E_0 \subset \mathcal{Y}_k$ be the exceptional divisor of the blowup $f \colon \mathcal{Y} \to \mathcal{X}$. We denote $f^{-1}(C)$ by $\tilde{C}$.

The multiplicity of E_i in $\mathcal{Y}_k$ is a_i for $i \in I$ and
$$a_0 := \sum_{Z \subset D_i} a_i \tag{2.2}$$
for $i = 0$. We will use the following lemma several times.

Lemma 2.4. *For $i \in I \setminus J$, if $C \subset Z \cap D_i$, then we have $\tilde{C} \subset E_i$.*

Proof. The morphism $\tilde{C} \to C$ is a $\mathbb{P}^{\sharp J - 1}$-bundle. The divisor E_i is the blowup of D_i along $Z \cap D_i$, which has codimension $\sharp J$ in D_i. It follows that $E_i \cap \tilde{C} \to C$ is also a $\mathbb{P}^{\sharp J - 1}$-bundle. Hence $\tilde{C}$ and $E_i \cap \tilde{C}$ coincide and the lemma follows. □

3. The case $d' = 1$

We now begin the proof of Proposition 2.1 in the situation described just before Notation 2.3. In this section, we consider the case $d' = 1$.

Since $Z \subset \mathcal{X}_k$, recalling $I = \{1, \dots, d'\}$, we see that $1 \in J$. Then
$$\tilde{S}(\mathcal{X}, C) = \begin{cases} [C] & (a_1 = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

From (2.2), $a_0 = a_1$, and $(a_0, a_1) = a_1$. Hence, if $a_1 \neq 1$, then

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C).$$

If $a_1 = 1$, then recalling that $C \subset Z$, we see that $\tilde{C} \subset E_0 = f^{-1}(Z)$ and that

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = [\tilde{C} \setminus E_1] + [E_1 \cap \tilde{C}](1 - \mathbb{L}).$$

To compute the right hand side of this equality, we first observe that $\tilde{C}$ is a $\mathbb{P}^{\sharp J-1}$-bundle over C. The divisor E_1 is the blowup of D_1 along Z. Therefore $E_1 \cap \tilde{C}$ is a $\mathbb{P}^{\sharp J-2}$-bundle over C. Hence

$$\begin{aligned}
\tilde{S}(\mathcal{Y}, \tilde{C}) &= [C]([\mathbb{P}^{\sharp J-1}] - [\mathbb{P}^{\sharp J-2}]) + [C][\mathbb{P}^{\sharp J-2}](1 - \mathbb{L}) \\
&= [C]\left(\mathbb{L}^{\sharp J-1} + (1 + \mathbb{L} + \cdots + \mathbb{L}^{\sharp J-2})(1 - \mathbb{L})\right) \\
&= [C](\mathbb{L}^{\sharp J-1} + 1 - \mathbb{L}^{\sharp J-1}) \\
&= [C] \\
&= \tilde{S}(\mathcal{X}, C).
\end{aligned}$$

We conclude that if $d' = 1$, then $\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C})$.

4. The case $d' = 2$

Next we consider the case $d' = 2$. We have

$$C = (C \cap D_1^\circ) \sqcup (C \cap D_2^\circ) \sqcup (C \cap D_{12}^\circ).$$

From the case $\sharp I = 1$ treated in the last section, we have

$$\tilde{S}(\mathcal{X}, C \cap D_i^\circ) = \tilde{S}(\mathcal{Y}, f^{-1}(C \cap D_i^\circ)) \quad (i = 1, 2).$$

Therefore, from Lemma 2.2, replacing C with $C \cap D_{12}^\circ$, we may suppose that

$$C \subset D_{12}^\circ = D_1 \cap D_2. \tag{4.1}$$

Then we have

$$\tilde{S}(\mathcal{X}, C) = \begin{cases} \frac{1}{a_1 a_2}[C](1 - \mathbb{L}) & ((a_1, a_2) = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

We next compute $\tilde{S}(\mathcal{Y}, \tilde{C})$ separately in the case $Z \subset D_1 \cap D_2$ and in the case $Z \not\subset D_1 \cap D_2$.

In the former case, we have $a_0 = a_1 + a_2 \neq 1$ and

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{\substack{i \in \{1,2\}: \\ (a_0, a_i) = 1}} \frac{1}{a_0 a_i}[\tilde{C} \cap E_{0i}^\circ](1 - \mathbb{L}).$$

If $(a_1, a_2) \neq 1$, then $(a_0, a_1) \neq 1$ and $(a_0, a_2) \neq 1$, which show that $\tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C)$. If $(a_1, a_2) = 1$, then we have $(a_0, a_1) = (a_0, a_2) = 1$, and

$$\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{i=1}^{2} \frac{1}{a_0 a_i}[\tilde{C} \cap E_{0i}^\circ](1 - \mathbb{L}).$$

Since $E_1 \cap \tilde{C} = E_0 \cap E_1 \cap \tilde{C} \to C$ is a trivial $\mathbb{P}^{\sharp J-2}$-bundle and $E_1 \cap E_2 \cap \tilde{C} \to C$ is a hyperplane in it, $E_{01}^\circ \cap \tilde{C} \to C$ is a trivial $\mathbb{A}^{\sharp J-2}$-bundle. (Note that if $\sharp J = 2$, then $E_1 \cap E_2 = \emptyset$ and $E_1 \cap \tilde{C} = E^\circ_{01} \cap \tilde{C} \to C$ is an isomorphism and still a trivial $\mathbb{A}^{\sharp J-2}$-bundle.) Similarly for $E^\circ_{02} \cap \tilde{C} \to C$. Hence

$$\begin{aligned}\tilde{S}(\mathcal{Y}, \tilde{C}) &= \left(\frac{1}{(a_1+a_2)a_1} + \frac{1}{(a_1+a_2)a_2}\right)[C]\mathbb{L}^{\sharp J-2}(1-\mathbb{L}) \\ &= \frac{1}{a_1a_2}[C]\mathbb{L}^{\sharp J-2}(1-\mathbb{L}) \\ &\overset{\bigstar}{=} \frac{1}{a_1a_2}[C](1-\mathbb{L}) \\ &= \tilde{S}(\mathcal{X}, C).\end{aligned}$$

Here the equality marked with $\bigstar$ follows from

$$\mathbb{L}(1-\mathbb{L}) = (\mathbb{L}-1)(1-\mathbb{L}) + 1 - \mathbb{L} = 1 - \mathbb{L} \mod (\mathbb{L}-1)^2.$$

In the case $Z \not\subset D_1 \cap D_2$, we have either $Z \subset D_1$ or $Z \subset D_2$. Since the two cases are similar, we only discuss the former case. Since $2 \in I \setminus J$, from assumptions (2.1) and (4.1) and Lemma 2.4, we have $\tilde{C} \subset E_0 \cap E_2$. Since $a_0 = a_1$, $\tilde{C} \to C$ is a $\mathbb{P}^{\sharp J-1}$-bundle and $\tilde{C} \cap E_1 \to C$ is a $\mathbb{P}^{\sharp J-2}$-bundle, we have

$$\begin{aligned}\tilde{S}(\mathcal{Y}, \tilde{C}) &= \frac{1}{a_0a_2}[\tilde{C} \cap E^\circ_{0,2}](1-\mathbb{L}) \\ &= \frac{1}{a_1a_2}[\tilde{C} \setminus E_1](1-\mathbb{L}) \\ &= \frac{1}{a_1a_2}[C][\mathbb{P}^{\sharp J-1} \setminus \mathbb{P}^{\sharp J-2}](1-\mathbb{L}) \\ &= \frac{1}{a_1a_2}[C]\mathbb{L}^{\sharp J-1}(1-\mathbb{L}) \\ &= \frac{1}{a_1a_2}[C](1-\mathbb{L}) \\ &= \tilde{S}(\mathcal{X}, C).\end{aligned}$$

We have completed the proof that $\tilde{S}(\mathcal{Y}, \tilde{C}) = \tilde{S}(\mathcal{X}, C)$, when $d' = 2$.

5. The case $d' \geq 3$

As in the last section, by induction on $\sharp I$, we may suppose that

$$C \subset \bigcap_{i \in I} D_i. \tag{5.1}$$

Then $\tilde{S}(\mathcal{X}, C) = 0$. On the other hand, $\tilde{S}(\mathcal{Y}, \tilde{C})$ is a $\mathbb{Q}$-linear combination of

$$A_i := \left[\tilde{C} \cap E^\circ_{0i}\right](1-\mathbb{L}), \quad i \in I,$$

and

$$B := \delta_{1,a_0}\left[\tilde{C} \cap E^\circ_0\right],$$

with δ_{1,a_0} being the Kronecker delta. Thus it suffices to show that $A_i = 0$, $i \in I$, and that $B = 0$.

We first show that $B = 0$. If $\sharp(I \cap J) \geq 2$, then

$$a_0 = \sum_{i \in I \cap J} a_i > 1.$$

Hence $B = 0$. If $\sharp(I \cap J) < 2$, then $I \setminus J$ is non-empty. Assumptions (2.1) and (5.1) and Lemma 2.4 show that $\tilde{C} \cap E_0^\circ$ is empty, hence $B = 0$.

Next we show that $A_i = 0$. If $\sharp(I \setminus J) \geq 2$, then from Lemma 2.4, for every $i \in I$, there exists $i' \in I \setminus \{i\}$ such that $\tilde{C} \subset E_{i'}$. Hence $\tilde{C} \cap E_{0i}^\circ = \emptyset$ and $A_i = 0$.

If $\sharp(I \setminus J) = 1$, then by the same reasoning as above, $A_i = 0$ for $i \in I \cap J$. For $i \in I \setminus J$,

$$\tilde{C} \cap E_{0i}^\circ = \mathbb{P}_C^{\sharp J-1} \setminus \bigcup_{j \in I \cap J} H_j,$$

where $\mathbb{P}_C^{\sharp J-1}$ denotes the trivial $\mathbb{P}^{\sharp J-1}$-bundle $\mathbb{P}^{\sharp J-1} \times C$ over C and H_j are coordinate hyperplanes of $\mathbb{P}_C^{\sharp J-1}$. Since $\sharp(I \cap J) \geq 2$,

$$\begin{aligned} A_i &= [C][\mathbb{G}_m^{\sharp(I\cap J)-1} \times \mathbb{A}^{\sharp J - \sharp(I \cap J)}](1-\mathbb{L}) \\ &= -[C]\mathbb{L}^{\sharp J - \sharp(I\cap J)}(\mathbb{L}-1)^{\sharp(I \cap J)} = 0 \mod (\mathbb{L}-1)^2. \end{aligned}$$

If $\sharp(I \setminus J) = 0$, equivalently if $Z \subset D_i$ for every $i \in I$, then for every $i \in I$,

$$\tilde{C} \cap E_{0i}^\circ = \mathbb{P}_C^{\sharp J-2} \setminus \bigcup_{j \in I \setminus \{i\}} H_j,$$

where H_j are coordinate hyperplanes of $\mathbb{P}_C^{\sharp J-2}$. We have

$$A_i = [C][\mathbb{G}_m^{\sharp I-2} \times \mathbb{A}^{\sharp J - \sharp I}](1-\mathbb{L}) = -[C]\mathbb{L}^{\sharp J - \sharp I}(\mathbb{L}-1)^{\sharp I - 1} = 0 \mod (\mathbb{L}-1)^2.$$

We thus have proved that $\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C}) = 0$ also when $d' \geq 3$, which completes the proofs of Proposition 2.1 and Theorem 1.4.

6. Closing comments

It is natural to try to refine $\tilde{S}(X)$ further by lifting it to $K_0(\mathrm{Var}_k)_{\mathbb{Q}}/(\mathbb{L}-1)^n$ for $n > 2$ and by adding extra terms of the form

$$c[D_H^\circ](1-\mathbb{L})^{\sharp H - 1}$$

with $c \in \mathbb{Q}$, $H \subset I$, $\sharp H \geq 3$. However the author did not manage to find such a refinement.

The original invariant considered by Serre [11] and denoted by $i(X)$ was defined for a K-analytic manifold when the residue field k is finite and lives in $\mathbb{Z}/(\sharp k - 1)$. There seems to be no counterpart of $\tilde{S}(X)$ in this context, at least in a naive way, because $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ is a field and the ideal generated by $(\sharp k - 1)^2$ in it is the entire field.

The author has no convincing explanation of the meaning of fractional coefficients appearing in the definition of $\tilde{S}(X)$. However, as a possibly related work, we note that Denef and Loeser [4] previously also considered motivic invariants with coefficients in $\mathbb{Q}$.

Nicaise and Sebag [10, Th. 5.4] gave a nice interpretation of the Euler characteristic representation of $S(X)$ in terms of cohomology of the generic fiber (see also [6] for another proof). It would be interesting to look for a similar interpretation of representations of $\tilde{S}(X)$ or $\tilde{S}(X)$ itself.

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