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HARMONIZABLE STABLE FIELDS: REGULARITY AND WOLD-TYPE DECOMPOSITIONS

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Abstract. In this article, we examine the structure of harmonizable stable fields. We start by examining horizontal and vertical regularity. We find equivalent conditions for horizontal and vertical regularity in terms of the harmonizable stable field's spectral measure. We then give a Wold-type decomposition in this setting. After that, we consider strong regularity. Here too, we give equivalent conditions for strong regularity in terms of the field's spectral measure. In addition, we show that strong regularity is equivalent to the field's ability to be represented by a moving average random field. We finish this article with a four-fold Wold-type decomposition.

1. Introduction

In this article, we examine harmonizable stable fields. These fields may be thought of as infinite energy analogues of weakly stationary random fields. Much is known about weakly stationary random fields and the results in this article parallel those known results. For background on weakly stationary random fields, one might consult [7], [8] or [10]. Although the purpose of this paper is to generalize these results to harmonizable stable fields, much of the motivation and inspiration for this article was found in [1], [2], and [9]. In [1], [2], and [9], the authors developed the analogous theory for harmonizable stable sequences. The results in those papers may be thought of as one-variable analogues of the results found in this paper. Although one might hope that the results for harmonizable stable sequences generalize directly to harmonizable stable fields, that is not the case. The results are similar, but there are some differences. Much of the reason for this lies in the fact that function theory in the unit disc of the complex plane does not generalize directly to function theory in the unit bi-disc. In particular, outer functions on the unit bi-disc do not share all the properties of outer functions on the unit disc. We show how one might handle these shortcomings.

We finish this section by giving a brief breakdown of this article. In section 2, we give definitions and background material needed to present and prove our main results. In section 3, we study horizontal regularity. Here, we find equivalent conditions for horizontal regularity in terms of the field's spectral measure. We finish this section with a Wold-type decomposition in the context of horizontal regularity and singularity. Analogous results are presented in the vertical direction without proofs for use in section 4. In section 4, we study strong regularity.

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Here too, we find equivalent conditions for strong regularity in terms of the field's spectral measure. In addition, we find equivalent conditions for strong regularity in terms of the field's ability to be represented by a moving average random field. We end section 4 and this article with a four-fold Wold-type decomposition.

2. Harmonizable stable fields

In this section, we give the fundamental properties of harmonizable stable fields, which we will use throughout this note. One might consult [1], [4] or [5] for the analogous development for harmonizable stable sequences. We follow much of that development in this section.

A real-valued random variable X is called symmetric α-stable, $0 < \alpha \leq 2$, abbreviated $S\alpha S$, if its characteristic function $\varphi_X(t) = E\exp\{itX\}$, $t \in \mathbb{R}$ has the form

$$\varphi_X(t) = \exp\{-c|t|^\alpha\}, \quad t \in \mathbb{R},$$

for some $c \geq 0$. A finite collection of real-valued random variables are called jointly $S\alpha S$ if all linear combinations of these random variables are $S\alpha S$. A complex random variable $Z = X + iY$ is called isotropic α-stable if X and Y are jointly $S\alpha S$ and Z has a radially symmetric distribution; that is, $e^{i\theta}Z$ and Z have the same distribution for all real θ. This is equivalent to the following requirement on $\varphi_Z(t) = E\exp\left\{i\operatorname{Re}(\bar{t}Z)\right\}$, $t \in \mathbb{C}$, the characteristic function of Z,

$$\varphi_Z(t) = \exp\{-c|t|^\alpha\}, \quad t \in \mathbb{C},$$

for some $c \geq 0$. As in [5], we can define a length on Z, by

$$\|Z\| = \begin{cases} c^{1/\alpha}, & \text{for } 1 \leq \alpha \leq 2, \\ c, & \text{for } 0 < \alpha < 1. \end{cases}$$

This length gives a metric on any family of random variables with the property that any linear combination of its members is an isotropic α-stable random variable. We point out that if Z_1 and Z_2 are independent, then

$$\|Z_1 + Z_2\|^\alpha = \|Z_1\|^\alpha + \|Z_2\|^\alpha \quad \text{for } 1 \leq \alpha \leq 2$$

and

$$\|Z_1 + Z_2\| = \|Z_1\| + \|Z_2\| \quad \text{for } 0 < \alpha < 1.$$

Now, let $\mathcal{Z}$ be an independently scattered complex isotropic α-stable variable valued set function defined on $\mathcal{B}(\mathbb{T}^2)$, the Borel subsets of $\mathbb{T}^2$, the distinguished boundary of the unit bi-disc. That is, for all disjoint sets $\Delta_1, \cdots, \Delta_n \in \mathcal{B}(\mathbb{T}^2)$, $\mathcal{Z}(\Delta_1), \cdots, \mathcal{Z}(\Delta_n)$ are independent with

$$\varphi_{\mathcal{Z}(\Delta_k)}(t) = \begin{cases} \exp\{-|t|^\alpha \|\mathcal{Z}(\Delta_k)\|^\alpha\} & \text{for } 1 \leq \alpha \leq 2 \\ \exp\{-|t|^\alpha \|\mathcal{Z}(\Delta_k)\|\} & \text{for } 0 < \alpha < 1 \end{cases}, \quad t \in \mathbb{C}.$$

Using $\mathcal{Z}$, we define

$$\mu(\Delta) = \begin{cases} \|\mathcal{Z}(\Delta)\|^\alpha & \text{for } 1 \leq \alpha \leq 2 \\ \|\mathcal{Z}(\Delta)\| & \text{for } 0 < \alpha < 1 \end{cases}, \quad \text{for } \Delta \in \mathcal{B}(\mathbb{T}^2),$$

and observe that μ is a finite measure defined on $\mathcal{B}(\mathbb{T}^2)$. From this, it follows that if $f \in L^\alpha(\mathbb{T}^2, \mu)$ and $\mathcal{X} = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f(e^{i\lambda}, e^{i\theta})\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta})$, then

$$\varphi_{\mathcal{X}}(t) = \exp\left\{ -|t|^\alpha \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} \left| f(e^{i\lambda}, e^{i\theta}) \right|^\alpha d\mu(e^{i\lambda}, e^{i\theta}) \right\}, \quad t \in \mathbb{C}.$$

A complex random field $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is called harmonizable $S\alpha S$ with spectral measure μ, if μ is a finite (positive) measure defined on $\mathcal{B}(\mathbb{T}^2)$ with

$$E \exp\left\{ i \operatorname{Re}\left(\bar{t} \sum_{j=1}^{N} z_j X_{t_j, l_j} \right) \right\} = \exp\left\{ -|t|^\alpha \int_{[\pi,\pi)} \int_{[\pi,\pi)} \left| \sum_{j=1}^{N} z_j e^{-it_j\lambda - il_j\theta} \right|^\alpha d\mu(e^{i\lambda}, e^{i\theta}) \right\},$$

where $t, z_j \in \mathbb{C}$, and $t_j, l_j \in \mathbb{Z}$, for $j = 1, \cdots, N$. We see from this that $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is (strictly) stationary. We may define a harmonizable $S\alpha S$ field equivalently through its spectral representation

$$X_{m,n} = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-im\lambda - in\theta}\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta}),$$

where $\mathcal{Z}$ is an independently scattered complex isotropic α-stable variable valued set function defined on $\mathcal{B}(\mathbb{T}^2)$. If $L(X)$ is the closure in probability of the linear span of $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$, then the correspondence between f and

$$\int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f(e^{i\lambda}, e^{i\theta})\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta})$$

gives an isomorphism between $L^\alpha(\mathbb{T}^2, \mu)$ and $L(X)$, that sends $e^{-im\lambda - in\theta}$ to $X_{m,n}$. Hence, every $Y \in L(X)$ has a representation of the form

$$\int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f(e^{i\lambda}, e^{i\theta})\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta})$$

for some $f \in L^\alpha(\mathbb{T}^2, \mu)$ and has a radially symmetric distribution.

When $1 < \alpha \leq 2$ and $Y_1, Y_2 \in L(X)$ with representing functions $f_1, f_2 \in L^\alpha(\mathbb{T}^2, \mu)$, the covariation of Y_1 with Y_2 is defined by

$$[Y_1, Y_2]_\alpha = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f_1(e^{i\lambda}, e^{i\theta})\, |f_2(e^{i\lambda}, e^{i\theta})|^{\alpha-1}\, \overline{\operatorname{sgn}(f_2(e^{i\lambda}, e^{i\theta}))}\, d\mu(e^{i\lambda}, e^{i\theta}),$$

where $\operatorname{sgn}(f(e^{i\lambda}, e^{i\theta}))$ is a complex measurable function of modulus one such that $f(e^{i\lambda}, e^{i\theta}) = |f(e^{i\lambda}, e^{i\theta})| \operatorname{sgn}(f(e^{i\lambda}, e^{i\theta}))$. By Hölder's inequality, we have that $|[Y_1, Y_2]_\alpha| \leq \|f_1\|_{L^\alpha(\mathbb{T}^2,\mu)} \|f_2\|^{\alpha-1}_{L^\alpha(\mathbb{T}^2,\mu)}$ with equality if and only if $Y_1 = zY_2$, where $z \in \mathbb{C}$. The covariation of the harmonizable $S\alpha S$ sequence has the form

$$[X_{m,n}, X_{k,l}]_\alpha = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-i(m-k)\lambda - i(n-l)\theta}\, d\mu(e^{i\lambda}, e^{i\theta}).$$

Note that the covariance of a weakly stationary random field has this form.

For $Y_1, Y_2 \in L(X)$, if $[Y_1, Y_2]_\alpha = 0$, we say that Y_2 is orthogonal to Y_1, and write $Y_2 \perp Y_1$, which is non-symmetric and was introduced by R. C. James in [6]. When $Y_2 \perp Y_1$ and $Y_1 \perp Y_2$, we say that Y_1 and Y_2 are mutually orthogonal. We now make an important distinction between the Gaussian case, $\alpha = 2$, and the non-Gaussian case, $1 < \alpha < 2$, when it comes to the relationship between independence and orthogonality. When $\alpha = 2$, the independence of Y_1 and Y_2 is equivalent to the mutual orthogonality of Y_1 and Y_2 and the mutual orthogonality of Y_1 and $\overline{Y_2}$. However, when $1 < \alpha < 2$, the independence of Y_1 and Y_2 implies the mutual orthogonality of Y_1 and Y_2 and the mutual orthogonality of Y_1 and $\overline{Y_2}$, but it is **not generally true** that the mutual orthogonality of Y_1 and Y_2 and the mutual orthogonality of Y_1 and $\overline{Y_2}$ implies the independence of Y_1 and Y_2. This is because when $0 < \alpha < 2$, the independence of Y_1 and Y_2 is equivalent to their representing functions f_1 and f_2 having disjoint support; that is, $f_1 \cdot f_2 = 0$ $[\mu]$-a.e., see [12], while the mutual orthogonality of Y_1 and Y_2, when $1 < \alpha < 2$ means, by definition, that $\int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f_1(e^{i\lambda}, e^{i\theta}) \, |f_2(e^{i\lambda}, e^{i\theta})|^{\alpha-1} \, \overline{\operatorname{sgn}(f_2(e^{i\lambda}, e^{i\theta}))} \, d\mu(e^{i\lambda}, e^{i\theta}) = 0 = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} f_2(e^{i\lambda}, e^{i\theta}) \, |f_1(e^{i\lambda}, e^{i\theta})|^{\alpha-1} \, \overline{\operatorname{sgn}(f_1(e^{i\lambda}, e^{i\theta}))} \, d\mu(e^{i\lambda}, e^{i\theta})$.

In the sections that follow, most theorems are stated for $1 < \alpha \leq 2$. This is because orthogonality is used in their proofs. For those theorems that do not use orthongonality, the theorems are stated and proved for $0 < \alpha \leq 2$.

3. Horizontal and vertical regularity

For a harmonizable $S\alpha S$ field $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$, we define the concepts of regularity and singularity. Let $L^1(X : m)$ denote the closure in probability of the linear span of $\{X_{k,l} : k \leq m, l \in \mathbb{Z}\}$ and let $L^1(X : -\infty) = \bigcap_m L^1(X : m)$. $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is called horizontally regular if $L^1(X : -\infty) = \{0\}$ and horizontally singular if $L^1(X : -\infty) = L(X)$. Similarly, let $L^2(X : n)$ denote the closure in probability of the linear span of $\{X_{k,l} : k \in \mathbb{Z}, l \leq n\}$ and let $L^2(X : -\infty) = \bigcap_n L^2(X : n)$. $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is called vertically regular if $L^2(X : -\infty) = \{0\}$ and vertically singular if $L^2(X : -\infty) = L(X)$. These definitions are consistent with those given for weakly stationary random fields. It will be advantageous to redefine these concepts in $L^\alpha(\mathbb{T}^2, \mu)$, using the isomorphism that takes $X_{m,n}$ to $e^{im\lambda + in\theta}$. Under this isomorphism, it is straightforward to see that the following definitions are equivalent to those just given. $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is horizontally regular if $M_{-\infty}^{(\alpha,1)} = \{0\}$ and horizontally singular if $M_{-\infty}^{(\alpha,1)} = L^\alpha(\mathbb{T}^2, \mu)$, where $M_{-\infty}^{(\alpha,1)} = \bigcap_m M_m^{(\alpha,1)}$ and $M_m^{(\alpha,1)}$ is equal to the span closure in $L^\alpha(\mathbb{T}^2, \mu)$ of $\{e^{ik\lambda + il\theta} : k \leq m, l \in \mathbb{Z}\}$. Similarly, $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is vertically regular if $M_{-\infty}^{(\alpha,2)} = \{0\}$ and vertically singular if $M_{-\infty}^{(\alpha,2)} = L^\alpha(\mathbb{T}^2, \mu)$, where $M_{-\infty}^{(\alpha,2)} = \bigcap_n M_n^{(\alpha,2)}$ and $M_n^{(\alpha,2)}$ is equal to the span closure in $L^\alpha(\mathbb{T}^2, \mu)$ of $\{e^{ik\lambda + il\theta} : k \in \mathbb{Z}, l \leq n\}$.

In what follows, μ_j $j = 1, 2$, are the marginals of μ. That is, for all $B \in \mathcal{B}(\mathbb{T})$,

$$\mu_1(B) = \mu(B \times \mathbb{T})$$

and

$$\mu_2(B) = \mu(\mathbb{T} \times B).$$

Finally, we introduce the following notation. Let $A \subseteq \mathbb{Z}^2$ and g be in $L^\alpha(\mathbb{T}^2, \mu)$, $1 < \alpha \leq 2$. Then, we define

$$[A]_\mu = \overline{\operatorname{span}} \left\{ e^{im\lambda + in\theta} : (m,n) \in A \right\}$$

and

$$[g]_{A,\mu} = \overline{\text{span}}\left\{e^{im\lambda+in\theta}g(e^{i\lambda}, e^{i\theta}) : (m,n) \in A\right\},$$

where, in both cases, the closure is in $L^\alpha(\mathbb{T}^2, \mu)$.

Now, let $A_\lambda = \{(m,n) : m \geq 0, n \in \mathbb{Z}\}$ and $A_\theta = \{(m,n) : m \in \mathbb{Z}, n \geq 0\}$ and define $H^\alpha_\lambda(\mathbb{T}^2, \mu) = [A_\lambda]_\mu$ and $H^\alpha_\theta(\mathbb{T}^2, \mu) = [A_\theta]_\mu$. Also, let $B_\lambda = \{(m, 0) : m \geq 0\}$ and $B_\theta = \{(0,n) : n \geq 0\}$ and define $H^\alpha(\mathbb{T}_\lambda) = [B_\lambda]_{\sigma^2}$ and $H^\alpha(\mathbb{T}_\theta) = [B_\theta]_{\sigma^2}$, where here and henceforth, σ^2 will denote normalized Lebesgue measure on $\mathbb{T}^2$. Note that $H^\alpha(\mathbb{T}_\lambda)$ and $H^\alpha(\mathbb{T}_\theta)$ are just the usual Hardy spaces, with respect to the appropriate variable, on the unit circle in the complex plane. Using the present notation, we recall that a function g in $H^\alpha(\mathbb{T}_\lambda)$ is called outer if $[g]_{B_\lambda,\sigma^2} = H^\alpha(\mathbb{T}_\lambda)$. The definition is defined analogously for functions in $H^\alpha(\mathbb{T}_\theta)$. Henceforth, σ will denote normalized Lebesgue measure on $\mathbb{T}$. Now, if ξ is a finite measure on $\mathcal{B}(\mathbb{T})$, then for a product measure of the form $\sigma \otimes \xi$ and for f in $H^\alpha_\lambda(\mathbb{T}^2, \sigma\otimes\xi)$ and θ fixed, we define the cut function, $f_\theta(e^{i\lambda}) = f(e^{i\lambda}, e^{i\theta})$. Note that $f_\theta \in H^\alpha(\mathbb{T}_\lambda)$ for $[\mu_2]$-a.e. $e^{i\theta}$. We say f in $H^\alpha_\lambda(\mathbb{T}^2, \sigma\otimes\xi)$ is θ-outer if f_θ is outer for $[\xi]$-a.e. $e^{i\theta}$. In an analogous way, we define λ-outer.

Theorem 1. *Let $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ be a harmonizable $S\alpha S$ field with $1 < \alpha \leq 2$ and spectral measure μ. Then, the following are equivalent:*

(1) *$X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is horizontally regular.*

(2) *μ is absolutely continuous with respect to $\sigma \otimes \mu_2$ with density f satisfying*

$$\int_{[-\pi,\pi)} \log\left(f(e^{i\lambda}, e^{i\theta})\right) d\sigma(e^{i\lambda}) > -\infty \text{ for } [\mu_2]\text{-a.e. } e^{i\theta}. \tag{I}$$

(3) *μ is absolutely continuous with respect to $\sigma\otimes\mu_2$ with density $f(e^{i\lambda}, e^{i\theta}) = \left|\varphi(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where φ is a θ-outer function in $H^\alpha_\lambda(\mathbb{T}^2, \sigma\otimes\mu_2)$.*

Proof. (1) $\Rightarrow$ (2): Suppose $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is horizontally regular. By definition, $M^{(\alpha,1)}_{-\infty} = \{0\}$. Since $1 < \alpha \leq 2$, it follows that $M^{(2,1)}_m \subseteq M^{(\alpha,1)}_m$, for $m \in \mathbb{Z}$, and hence $M^{(2,1)}_{-\infty} \subseteq M^{(\alpha,1)}_{-\infty} = \{0\}$. It then follows from the theory of weakly stationary random fields (see [8] or [10]) that such a μ is absolutely continuous with respect to $\sigma\otimes\mu_2$ with density f satisfying satisfying (I).

(2) $\Rightarrow$ (3): Suppose that μ is absolutely continuous with respect to $\sigma\otimes\mu_2$ with density f satisfying (I). Let E be the subset of $\mathbb{T}$ for which the above inequality holds. Now, define, for $e^{i\theta} \in E$,

$$\varphi(z, e^{i\theta}) = \exp\left[\frac{1}{\alpha}\int_{[-\pi,\pi)} \frac{e^{i\lambda}+z}{e^{i\lambda}-z} \log\left(f(e^{i\lambda}, e^{i\theta})\right) d\sigma(e^{i\lambda})\right] \qquad |z| < 1,$$

and for $e^{i\theta} \notin E$, define $\varphi(z, e^{i\theta}) = 1$. It is well known from the theory of functions (see [11]), that for each $e^{i\theta} \in E$, φ_θ is an outer function and the radial limits of φ_θ exist for $[\sigma]$-a.e. $e^{i\lambda}$ and $|\varphi(e^{i\lambda}, e^{i\theta})|^\alpha = f(e^{i\lambda}, e^{i\theta})$ for $[\sigma]$-a.e. $e^{i\lambda}$, where $\varphi(e^{i\lambda}, e^{i\theta})$ denotes the radial limit function of $\varphi(z, e^{i\theta})$. It follows that $f(e^{i\lambda}, e^{i\theta}) = |\varphi(e^{i\lambda}, e^{i\theta})|^\alpha$ $[\sigma\otimes\mu_2]$-a.e., φ is in $H^\alpha_\lambda(\mathbb{T}^2, \sigma\otimes\mu_2)$, and φ is θ-outer.

(3) $\Rightarrow$ (1): Suppose μ is absolutely continuous with respect to $\sigma\otimes\mu_2$ with density $f(e^{i\lambda}, e^{i\theta}) = \left|\varphi(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where φ is a θ-outer function in $H^\alpha_\lambda(\mathbb{T}^2, \sigma\otimes\mu_2)$. Let $J : L^\alpha(\mathbb{T}^2, \mu) \to L^\alpha(\mathbb{T}^2, \sigma\otimes\mu_2)$ be the linear transformation that takes $e^{im\lambda+in\theta}$ to $e^{im\lambda+in\theta}\overline{\varphi}(e^{i\lambda}, e^{i\theta})$. Note that J is an onto isometry and that

$$J(L^1(X : m)) = \overline{\text{span}}\{e^{ik\lambda+in\theta}\overline{\varphi}(e^{i\lambda}, e^{i\theta}) : k \leq m, n \in \mathbb{Z}\}.$$

It suffices to prove that $\bigcap_m J(L^1(X:m)) = \{0\}$. To see this, note that

$$\int_{[-\pi,\pi)}\int_{[-\pi,\pi)} e^{-ij\lambda+ik\lambda+in\theta}\overline{\varphi}(e^{i\lambda},e^{i\theta})\,d\sigma(e^{i\lambda})d\mu_2(e^{i\theta}) = 0, \text{ for } k<j,$$

since $\varphi(e^{i\lambda},e^{i\theta}) \in H^\alpha(\mathbb{T}_\lambda)$ for $[\mu_2]$-a.e. $e^{i\theta}$. This says that $e^{ij\lambda}$ is orthogonal to $\left\{e^{ik\lambda+in\theta}\overline{\varphi}(e^{i\lambda},e^{i\theta}) : k<j, n\in\mathbb{Z}\right\}$. Therefore, if $\psi \in \bigcap\limits_m J(L^1(X:m))$, then $e^{ij\lambda}$ is orthogonal to ψ for all $j\in\mathbb{Z}$. Since $e^{-in\theta}\psi \in \bigcap_m J(L^1(X:m))$, for all $n\in\mathbb{Z}$, it follows that $e^{ij\lambda+in\theta}$ is orthogonal to ψ for all $(j,n)\in\mathbb{Z}^2$. We can conclude from this that $\psi = 0$ $[\sigma\otimes\mu_2]$-a.e. □

Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ relative to $\sigma\otimes\mu_2$. Here, μ_a is absolutely continuous with respect to $\sigma\otimes\mu_2$ and μ_s is singular with respect to $\sigma\otimes\mu_2$. We will use $\dfrac{d\mu_a}{d(\sigma\otimes\mu_2)}$ to denote the Radon-Nikodym derivative of μ_a with respect to $\sigma\otimes\mu_2$.

Theorem 2. *Let $0<\alpha\le 2$; then $\displaystyle\int_{[-\pi,\pi)} \log\frac{d\mu_a(e^{i\lambda},e^{i\theta})}{d(\sigma\otimes\mu_2)}\,d\sigma(e^{i\lambda}) = -\infty$ for $[\mu_2]$-a.e. $e^{i\theta}$, if and only if $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is horizontally singular.*

Proof. ($\Rightarrow$) From the assumption and the theory of weakly stationary random fields (see [8] or [10]), it follows that $M_n^{(2,1)} = L^2(\mathbb{T}^2,\mu)$ for all n. Now $M_n^{(2,1)} \subseteq M_n^{(\alpha,1)}$ for all n and $M_n^{(\alpha,1)}$ is the closure of $M_n^{(2,1)}$ in $L^\alpha(\mathbb{T}^2,\mu)$. Therefore, $M_n^{(\alpha,1)} = L^\alpha(\mathbb{T}^2,\mu)$ for all n. That is, $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is horizontally singular.

($\Leftarrow$) If $M_n^{(\alpha,1)} = L^\alpha(\mathbb{T}^2,\mu)$ for all n, then it follows that $M_n^{(2,1)} = M_n^{(\alpha,1)} \cap L^2(\mathbb{T}^2,\mu) = L^\alpha(\mathbb{T}^2,\mu)\cap L^2(\mathbb{T}^2,\mu) = L^2(\mathbb{T}^2,\mu)$ for all n. Therefore, by the theory of weakly stationary random fields (see [8] or [10]), $\displaystyle\int_{[-\pi,\pi)} \log\frac{d\mu_a(e^{i\lambda},e^{i\theta})}{d(\sigma\otimes\mu_2)}\,d\sigma(e^{i\lambda}) = -\infty$ for $[\mu_2]$-a.e. $e^{i\theta}$. □

Before we state the next theorem, let U_1 and U_2 be the operators on $L(X)$ with the property that $U_1X_{m,n} = X_{m+1,n}$ and $U_2X_{m,n} = X_{m,n+1}$.

Theorem 3. *Let $1<\alpha\le 2$ and $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ be a harmonizable $S\alpha S$ field. Then, there exists a unique decomposition*

$$X_{m,n} = X_{m,n}^{(1)} + X_{m,n}^{(2)},$$

where

(1) *$X_{m,n}^{(1)}$ and $X_{m,n}^{(2)}$ are in $L(X)$ for all $(m,n)\in\mathbb{Z}^2$.*
(2) *$U_1X_{m,n}^{(j)} = X_{m+1,n}^{(j)}$ and $U_2X_{m,n}^{(j)} = X_{m,n+1}^{(j)}$ for $j=1,2$ and for all $(m,n)\in\mathbb{Z}^2$.*
(3) *$X_{m,n}^{(1)}$ and $X_{m,n}^{(2)}$ are independent.*
(4) *$X_{m,n}^{(1)}$ is horizontally regular and $X_{m,n}^{(2)}$ is horizontally singular.*

Proof. If the conditions of Theorem 2 are satisfied, it follows that $X_{m,n}$ is horizontally singular and so we may set $X_{m,n}^{(1)} = 0$ and $X_{m,n}^{(2)} = X_{m,n}$ and the theorem holds. If, on the other hand, the conditions of Theorem 2 are not satisfied, let E_a

and E_s be the support of μ_a and μ_s respectively, and let

$$R = \left\{ e^{i\theta} : \int_{[-\pi,\pi)} \log \frac{d\mu_a(e^{i\lambda}, e^{i\theta})}{d(\sigma \otimes \mu_2)}\, d\sigma(e^{i\lambda}) > -\infty \right\}.$$

Now, define $N = E_a \cap (\mathbb{T} \times R)$. Since $X_{m,n}$ is harmonizable, there exists an $L(X)$-valued Borel measure $\mathcal{Z}$ on $\mathbb{T}^2$ such that for every (m, n),

$$X_{m,n} = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-(im\lambda + in\theta)}\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta}).$$

Now, define $\mathcal{Z}^{(1)}(A) = \mathcal{Z}(A \cap N)$, $\mathcal{Z}^{(2)}(A) = \mathcal{Z}(A \cap N^c)$,

$$X^{(1)}_{m,n} = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-(im\lambda + in\theta)}\, d\mathcal{Z}^{(1)}(e^{i\lambda}, e^{i\theta}),$$

and

$$X^{(2)}_{m,n} = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-(im\lambda + in\theta)}\, d\mathcal{Z}^{(2)}(e^{i\lambda}, e^{i\theta}).$$

By their construction, it is straightforward to see that $X^{(1)}_{m,n}$ and $X^{(2)}_{m,n}$ satisfy conditions 1, 2 and 3 of the theorem. It remains to show that $X^{(1)}_{m,n}$ is horizontally regular and $X^{(2)}_{m,n}$ is horizontally singular.

First, we will show that $X^{(1)}_{m,n}$ is horizontally regular. By definition, $\mu^{(1)}(B) = \mu(B \cap N)$ for every B in $\mathcal{B}(\mathbb{T}^2)$. By Theorem 1, we need to show that $\mu^{(1)}$ is absolutely continuous with respect to $\sigma \otimes \mu_2^{(1)}$ and

$$\int_{[-\pi,\pi)} \log \frac{d\mu^{(1)}(e^{i\lambda}, e^{i\theta})}{d(\sigma \otimes \mu_2^{(1)})}\, d\sigma(e^{i\lambda}) > -\infty \text{ for } [\mu_2^{(1)}]\text{-a.e. } e^{i\theta}.$$

Let $A \in \mathcal{B}(\mathbb{T}^2)$; then

$$\begin{aligned} \mu^{(1)}(A) &= \int\int_A d\mu^{(1)}(\xi, \eta) = \int\int_{A \cap N} d\mu(\xi, \eta) \\ &= \int\int_{A \cap E_a \cap (\mathbb{T} \times R)} \frac{d\mu_a(\xi, \eta)}{d(\sigma \otimes \mu_2)}\, d(\sigma \otimes \mu_2)(\xi, \eta) \\ &= \int\int_{A \cap (\mathbb{T} \times R)} \frac{d\mu_a(\xi, \eta)}{d(\sigma \otimes \mu_2)}\, d(\sigma \otimes \mu_2)(\xi, \eta) \\ &= \int\int_A \mathbf{1}_{\mathbb{T} \times R} \frac{d\mu_a(\xi, \eta)}{d(\sigma \otimes \mu_2)}\, d(\sigma \otimes \mu_2)(\xi, \eta). \qquad (1) \end{aligned}$$

Let $B \in \mathcal{B}(\mathbb{T})$; then

$$\begin{aligned} \mu_2^{(1)}(B) &= \mu^{(1)}(\mathbb{T} \times B) = \int\int_{\mathbb{T} \times B} \mathbf{1}_{\mathbb{T} \times R} \frac{d\mu_a(\xi, \eta)}{d(\sigma \otimes \mu_2)}\, d(\sigma \otimes \mu_2)(\xi, \eta) \\ &= \int_B \left[\int_{[-\pi,\pi)} \mathbf{1}_{\mathbb{T} \times R}(e^{i\lambda}, \eta) \frac{d\mu_a(e^{i\lambda}, \eta)}{d(\sigma \otimes \mu_2)}\, d\sigma(e^{i\lambda}) \right] d\mu_2(\eta) \\ &= \int_B g(\eta)\, d\mu_2(\eta), \qquad (2) \end{aligned}$$

where

$$g(\eta) = \begin{cases} \int_{[-\pi,\pi)} \frac{d\mu_a(e^{i\lambda}, \eta)}{d(\sigma \otimes \mu_2)}\, d\sigma(e^{i\lambda}) & \text{if } \eta \in R, \\ 0 & \text{if } \eta \notin R. \end{cases}$$

Now, if we define

$$\ell(\xi,\eta)=\begin{cases} 1 & \text{if } g(\eta)=0,\\ \frac{1}{g(\eta)}\mathbf{1}_{\mathbb{T}\times R}(\xi,\eta)\frac{d\mu_a(\xi,\eta)}{d(\sigma\otimes\mu_2)} & \text{if } g(\eta)>0,\end{cases}$$

then

$$\mu^{(1)}(A)=\int\int_A \ell(\xi,\eta)g(\eta)\,d(\sigma\otimes\mu_2)(\xi,\eta)=\int\int_A \ell(\xi,\eta)\,d(\sigma\otimes\mu_2^{(1)})(\xi,\eta).$$

From this we see that $\mu^{(1)}$ is absolutely continuous with respect to $\sigma\otimes\mu_2^{(1)}$ with $\dfrac{d\mu^{(1)}(\xi,\eta)}{d(\sigma\otimes\mu_2^{(1)})}=\ell(\xi,\eta)$, and it is straightforward to see that

$$\int_{[-\pi,\pi)}\log\frac{d\mu^{(1)}(e^{i\lambda},e^{i\theta})}{d(\sigma\otimes\mu_2^{(1)})}\,d\sigma(e^{i\lambda})>-\infty \text{ for } [\mu_2^{(1)}]\text{-a.e. } e^{i\theta},$$

as desired.

It remains to show that $X^{(2)}_{m,n}$ is horizontally singular. We start by pointing out that N^c can be written as the disjoint union $N^c=E_a^c\cup(E_a\cap(\mathbb{T}\times R^c))$. From this observation, we can further decompose $\mathcal{Z}^{(2)}$ as follows. For $A\in\mathcal{B}(\mathbb{T}^2)$, we define

$$\mathcal{Z}_s^{(2)}(A)=\mathcal{Z}(A\cap E_a^c)=\mathcal{Z}(A\cap E_s),$$

since the support of the measure on E_a^c is contained entirely in E_s and

$$\mathcal{Z}_a^{(2)}(A)=\mathcal{Z}(A\cap(E_a\cap(\mathbb{T}\times R^c))).$$

From these definitions, we observe that

$$\mathcal{Z}^{(2)}(A)=\mathcal{Z}_a^{(2)}(A)+\mathcal{Z}_s^{(2)}(A).$$

From this observation, we define

$$X^{(2)}_{m,n}(a)=\int_{[-\pi,\pi)}\int_{[-\pi,\pi)}e^{im\lambda+in\theta}\,d\mathcal{Z}_a^{(2)}(e^{i\lambda},e^{i\theta})$$

and

$$X^{(2)}_{m,n}(s)=\int_{[-\pi,\pi)}\int_{[-\pi,\pi)}e^{im\lambda+in\theta}\,d\mathcal{Z}_s^{(2)}(e^{i\lambda},e^{i\theta})$$

and observe that

$$X^{(2)}_{m,n}=X^{(2)}_{m,n}(a)+X^{(2)}_{m,n}(s).$$

It follows from Theorem 2 that both $X^{(2)}_{m,n}(a)$ and $X^{(2)}_{m,n}(s)$ are horizontally singular. Therefore, it follows that $X^{(2)}_{m,n}$ is also horizontally singular. □

To end this section, we will state, without proofs, the analogous results in the vertical direction. We do this for completeness and for the sake of reference.

Theorem 4. *Let $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ be a harmonizable $S\alpha S$ field with $1<\alpha\le 2$ and spectral measure μ. Then, the following are equivalent:*

(1) *$X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is vertically regular.*
(2) *μ is absolutely continuous with respect to $\mu_1\otimes\sigma$ with density f satisfying*

$$\int_{[-\pi,\pi)}\log\left(f(e^{i\lambda},e^{i\theta})\right)\,d\sigma(e^{i\theta})>-\infty \text{ for } [\mu_1]\text{-a.e. } e^{i\lambda}.$$

(3) *μ is absolutely continuous with respect to $\mu_1\otimes\sigma$ with density $f(e^{i\lambda},e^{i\theta})=\left|\varphi(e^{i\lambda},e^{i\theta})\right|^\alpha$, where φ is a λ-outer function in $H^\alpha_\theta(\mathbb{T}^2,\mu_1\otimes\sigma)$.*

Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ relative to $\mu_1 \otimes \sigma$. Here, μ_a is absolutely continuous with respect to $\mu_1 \otimes \sigma$ and μ_s is singular with respect to $\mu_1 \otimes \sigma$. We will use $\dfrac{d\mu_a}{d(\mu_1 \otimes \sigma)}$ to denote the Radon-Nikodym derivative of μ_a with respect to $\mu_1 \otimes \sigma$.

Theorem 5. *Let* $0 < \alpha \le 2$; *then* $\displaystyle\int_{[-\pi,\pi)} \log \frac{d\mu_a(e^{i\lambda}, e^{i\theta})}{d(\mu_1 \otimes \sigma)}\, d\sigma(e^{i\theta}) = -\infty$ *for* $[\mu_1]$-*a.e.* $e^{i\lambda}$, *if and only if* $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *is vertically singular.*

Theorem 6. *Let* $1 < \alpha \le 2$ *and* $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *be a harmonizable* $S\alpha S$ *field. Then, there exists a unique decomposition*

$$X_{m,n} = \tilde{X}^{(1)}_{m,n} + \tilde{X}^{(2)}_{m,n},$$

where

(1) $\tilde{X}^{(1)}_{m,n}$ *and* $\tilde{X}^{(2)}_{m,n}$ *are in* $L(X)$ *for all* $(m,n) \in \mathbb{Z}^2$.
(2) $U_1 \tilde{X}^{(j)}_{m,n} = \tilde{X}^{(j)}_{m+1,n}$ *and* $U_2 \tilde{X}^{(j)}_{m,n} = \tilde{X}^{(j)}_{m,n+1}$ *for* $j = 1, 2$ *and for all* $(m,n) \in \mathbb{Z}^2$.
(3) $\tilde{X}^{(1)}_{m,n}$ *and* $\tilde{X}^{(2)}_{m,n}$ *are independent.*
(4) $\tilde{X}^{(1)}_{m,n}$ *is vertically regular and* $\tilde{X}^{(2)}_{m,n}$ *is vertically singular.*

4. Strong regularity

We say that $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is strongly regular if it is both horizontally and vertically regular. We note that if we let $L(X : (m,n))$ denote the closure in probability of the linear span of $\{X_{k,l} : k \le m, l \le n\}$ and let $L(X : -\infty) = \bigcap_{m,n} L(X : (m,n))$, then $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is strongly regular if and only if $L(X : -\infty) = \{0\}$. It will be advantageous to redefine this concept in $L^\alpha(\mathbb{T}^2, \mu)$, using the isomorphism that takes $X_{m,n}$ to $e^{im\lambda + in\theta}$. Under this isomorphism, it is straightforward to see that $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is strongly regular if and only if $M^{(\alpha)}_{-\infty} = \{0\}$, where $M^{(\alpha)}_{-\infty} = \bigcap_{m,n} M^{(\alpha)}_{m,n}$ and $M^{(\alpha)}_{m,n}$ is equal to the span closure in $L^\alpha(\mathbb{T}^2, \mu)$ of $\{e^{ik\lambda + il\theta} : k \le m, l \le n\}$.

In the theorems and proofs that follow, we will use the notation introduced in section 3. In addition, we will write either $H^\alpha(\mathbb{T}^2)$ or $H^\alpha(\mathbb{T}^2, \sigma^2)$ for $[Q]_{\sigma^2}$, where $Q = \{(m,n) : m, n \ge 0\}$. Finally, a function f in $H^\alpha(\mathbb{T}^2)$ is called strongly outer if $[f]_{Q,\sigma^2} = H^\alpha(\mathbb{T}^2)$.

Theorem 7. *Let* $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *be a harmonizable* $S\alpha S$ *field with* $1 < \alpha \le 2$ *and spectral measure* μ. *Then, the following are equivalent:*

(1) $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *is strongly regular with* $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda \cap A_\theta]_\mu$.
(2) μ *is absolutely continuous with respect to* σ^2 *with density* $f(e^{i\lambda}, e^{i\theta}) = \left|\varphi(e^{i\lambda}, e^{i\theta})\right|^\alpha$, *where* φ *is a strongly outer function in* $H^\alpha(\mathbb{T}^2)$.
(3) $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *has a moving average representation*

$$X_{m,n} = \sum_{(k,l) \in \mathbb{N}^2} a_{k,l} V_{m-k,n-l},$$

where the random field $V_{m,n}$, $(m,n) \in \mathbb{Z}^2$ *is jointly stationary with* $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$, *satisfies* $L(X : (m,n)) = L(V : (m,n))$, *and is a harmonizable* $S\alpha S$ *field with spectral measure* σ^2 *and thus consists of mutually orthogonal random variables with norm one.*

One will observe in the proof that the $a_{k,l}$'s are the Fourier coefficients of the strongly outer function φ. We also point out that the condition $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda \cap A_\theta]_\mu$ given in (1) is an extra condition used to handle the shortcomings of outer functions in the bi-disc. This additional condition is a natural analog of the strong commuting condition used in the theory of weakly stationary random fields (see [7] or [10]).

Proof. (1) $\Rightarrow$ (2): Suppose that $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is strongly regular with $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda \cap A_\theta]_\mu$. By definition, $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ is both horizontally and vertically regular. Hence, we may employ Theorem 1 and Theorem 4. It follows from these theorems that μ is absolutely continuous with respect to $\sigma \otimes \mu_2$ with density $\dfrac{d\mu}{d(\sigma \otimes \mu_2)} = \left|\varphi_1(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where φ_1 is a θ-outer function in $H^\alpha_\lambda(\mathbb{T}^2, \sigma \otimes \mu_2)$ and that μ is absolutely continuous with respect to $\mu_1 \otimes \sigma$ with density $\dfrac{d\mu}{d(\mu_1 \otimes \sigma)} = \left|\varphi_2(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where φ_2 is a λ-outer function in $H^\alpha_\theta(\mathbb{T}^2, \mu_1 \otimes \sigma)$. It follows from these observations that μ_1 and μ_2 are absolutely continuous with respect to σ. This in turn tells us that μ is absolutely continuous with respect to σ^2 and $\dfrac{d\mu}{d\sigma^2}$ may be written in either of the two following forms:

$$\frac{d\mu}{d\sigma^2} = \frac{d\mu}{d(\sigma \otimes \mu_2)} \cdot \frac{d(\sigma \otimes \mu_2)}{d\sigma^2} = \frac{d\mu}{d(\sigma \otimes \mu_2)} \cdot \frac{d\mu_2}{d\sigma}$$

or

$$\frac{d\mu}{d\sigma^2} = \frac{d\mu}{d(\mu_1 \otimes \sigma)} \cdot \frac{d(\mu_1 \otimes \sigma)}{d\sigma^2} = \frac{d\mu}{d(\mu_1 \otimes \sigma)} \cdot \frac{d\mu_1}{d\sigma}.$$

It follows from Theorem 1 of [1] that $\dfrac{d\mu_2}{d\sigma} = \left|g_2(e^{i\theta})\right|^\alpha$, where g_2 is an outer function in $H^\alpha(\mathbb{T}_\theta)$ and that $\dfrac{d\mu_1}{d\sigma} = \left|g_1(e^{i\lambda})\right|^\alpha$, where g_1 is an outer function in $H^\alpha(\mathbb{T}_\lambda)$. It now follows from these additional observations that $\dfrac{d\mu}{d\sigma^2} = \left|\psi_1(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where ψ_1 is a θ-outer function in $H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$ and $\dfrac{d\mu}{d\sigma^2} = \left|\psi_2(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where ψ_2 is a λ-outer function in $H^\alpha_\theta(\mathbb{T}^2, \sigma^2)$. Now, define $q = \dfrac{\psi_1}{\psi_2}$. Then, we have that q is unimodular; that is, $|q| = 1$ $[\sigma^2]$-a.e. So, we may write $q\psi_2 = \psi_1$. We will start by showing that ψ_2 is in $e^{-im\lambda} H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$ for some $m \geq 0$. To this end, suppose that $\hat{\psi_2}(m,n) \neq 0$ for some $m < 0$ and $n \in \mathbb{Z}$. Now, since ψ_1 is in $H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$ and $q\psi_2 = \psi_1$, it follows that $\hat{q}(k,l) = 0$ for all $k \leq -m$ and for all $n \in \mathbb{Z}$. Hence, if for every $m \in \mathbb{Z}$, there exists a $k < m$ such that $\hat{\psi_2}(k,n) \neq 0$ for some $n \in \mathbb{Z}$, then, we would have $q = 0$, thus giving a contradiction. Therefore, there exists an $m \geq 0$ such that $\hat{\psi_2}(k,n) = 0$ for all $k < -m$ and for all $n \in \mathbb{Z}$. It follows that ψ_2 is in $e^{-im\lambda} H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$ and q is in $e^{im\lambda} H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$ for some $m \geq 0$. Now, since ψ_2 is also in $H^\alpha_\theta(\mathbb{T}^2, \sigma^2)$, we have that ψ_2 is in $e^{-im\lambda} H^\alpha_\lambda(\mathbb{T}^2, \sigma^2) \cap H^\alpha_\theta(\mathbb{T}^2, \sigma^2)$. It follows that $\psi_2 = e^{-im\lambda}\varphi$ for some φ in $H^\alpha(\mathbb{T}^2, \sigma^2)$. Since ψ_2 is λ-outer, it follows that φ is also λ-outer. Also, since q is in $e^{im\lambda} H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$, $q = e^{im\lambda} q^*$, where q^* is a unimodular function in $H^\alpha_\lambda(\mathbb{T}^2, \sigma^2)$. Therefore, the equation $q\psi_2 = \psi_1$ can now be written as $q^*\varphi = \psi_1$. Since ψ_1 is θ-outer, it follows that q^*_θ is a constant of modulus one for $[\sigma]$-a.e. $e^{i\theta}$. These observations then imply that φ must also be θ-outer.

Now, since $|\varphi| = |\psi_1|$, we have that $\frac{d\mu}{d\sigma^2} = \left|\varphi(e^{i\lambda}, e^{i\theta})\right|^\alpha$. Finally, the fact that φ is both λ-outer and θ-outer and that $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda \cap A_\theta]_\mu$, it follows from Theorem 3.8 of [3] that φ is a strongly outer function in $H^\alpha(\mathbb{T}^2, \sigma^2)$.

(2) $\Rightarrow$ (3): Suppose μ is absolutely continuous with respect to σ^2 with density $f(e^{i\lambda}, e^{i\theta}) = \left|\varphi(e^{i\lambda}, e^{i\theta})\right|^\alpha$, where φ is a strongly outer function in $H^\alpha(\mathbb{T}^2)$. Now, let $U_1 : L^\alpha(\mathbb{T}^2, f\, d\sigma^2) \to L(X)$ be defined by

$$U_1(g) = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} g(e^{i\lambda}, e^{i\theta})\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta}).$$

This is a linear isometry that is onto. Also, let $U_2 : L^\alpha(\mathbb{T}^2, f\, d\sigma^2) \to L^\alpha(\mathbb{T}^2, \sigma^2)$ be defined by $U_2(g) = g\varphi$. This is a linear isometry and is also onto since φ is strongly outer. Then, $U = U_2 U_1^{-1} : L(X) \to L^\alpha(\mathbb{T}^2, \sigma^2)$ and $U(X_{m,n}) = U_2 U_1^{-1}(X_{m,n}) = U_2(e^{-i(m\lambda+n\theta)}) = e^{-i(m\lambda+n\theta)}\varphi(e^{i\lambda}, e^{i\theta})$. Let $V_{m,n} = U^{-1}(e^{-i(m\lambda+n\theta)})$ and the fact that φ is strongly outer gives us that $L(X : (m,n)) = L(V : (m,n))$, $(m,n) \in \mathbb{Z}^2$. Now, since $\varphi \in H^\alpha(\mathbb{T}^2)$, it has a Fourier series

$$\varphi(e^{i\lambda}, e^{i\theta}) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} e^{ik\lambda+il\theta},$$

which converges in $L^\alpha(\mathbb{T}^2, \sigma^2)$. Therefore,

$$X_{m,n} = U^{-1}(e^{-i(m\lambda+n\theta)}\varphi(e^{i\lambda}, e^{i\theta})) = U^{-1}\left(e^{-i(m\lambda+n\theta)} \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} e^{ik\lambda+il\theta}\right)$$

$$= U^{-1}\left(\sum_{(k,l)\in\mathbb{N}^2} a_{k,l} e^{-i((m-k)\lambda+(n-l)\theta)}\right) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} V_{m-k,n-l} \text{ in } L(X).$$

In view of our isomorphism U, we have that

$$E\exp\left\{i\,\mathrm{Re}\left(\bar{t}\sum_{j=1}^N z_j V_{t_j,l_j}\right)\right\}$$
$$= \exp\left\{-|t|^\alpha \int_{[\pi,\pi)} \int_{[\pi,\pi)} \left|\sum_{j=1}^N z_j e^{-i(t_j\lambda+l_j\theta)}\right|^\alpha d\sigma^2(e^{i\lambda}, e^{i\theta})\right\},$$

where $t, z_j \in \mathbb{C}$, and $t_j, l_j \in \mathbb{Z}$, for $j = 1, \cdots, N$. Thus, V_n, $n \in \mathbb{Z}$ is harmonizable $S\alpha S$ with spectral measure σ^2 and thus

$$[V_{m,n}, V_{k,l}]_\alpha = \int_{[-\pi,\pi)} \int_{[-\pi,\pi)} e^{-i(m\lambda+n\theta)} e^{i(k\lambda+l\theta)}\, d\sigma^2(e^{i\lambda}) = \delta_{(m,n),(k,l)},$$

and so the $V_{m,n}$'s are mutually orthogonal with $\|V_{m,n}\|_\alpha = [V_{m,n}, V_{m,n}]_\alpha^{1/\alpha} = 1$, for all $(m,n) \in \mathbb{Z}^2$. The joint stationarity of $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ and $V_{m,n}$, $(m,n) \in \mathbb{Z}^2$ can be seen from the fact that $X_{m,n} = U^{-1}(e^{-i(m\lambda+n\theta)}\varphi(e^{i\lambda}, e^{i\theta}))$ and $V_{m,n} = U^{-1}(e^{-i(m\lambda+n\theta)})$.

(3) $\Rightarrow$ (1): Suppose that $X_{m,n} = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} V_{m-k,n-l}$, with all the conditions of (3) holding. Since $L(X : (m,n)) = L(V : (m,n))$, $(m,n) \in \mathbb{Z}^2$, it follows that $L(X : -\infty) = L(V : -\infty)$. We will show that $L(V : -\infty) = \{0\}$. Let $Y \in L(V : -\infty)$. Then, let $U : L(V) \to L^\alpha(\mathbb{T}^2, \sigma^2)$ be the isomorphism that sends $V_{m,n}$

to $e^{-i(m\lambda+n\theta)}$. Therefore, Y can be represented by some f in $L^\alpha(\mathbb{T}^2,\sigma^2)$; that is, $Y = \int_{[-\pi,\pi)}\int_{[-\pi,\pi)} f(e^{i\lambda},e^{i\theta})\, d\mathcal{Z}_V(e^{i\lambda},e^{i\theta})$. Since $Y \in L(V:-\infty)$, by definition, $Y \in L(V:(m,n))$ for all $(m,n)\in\mathbb{Z}^2$. It follows that

$$f \in \overline{\mathrm{span}}\{e^{-i(k\lambda+l\theta)} : k \le m,\, l \le n\}$$

for all $(m,n)\in\mathbb{Z}^2$. Since Fourier coefficients are unique, it follows that $f=0$. Therefore, $Y=0$. So, $L(V:-\infty)=\{0\}$ and hence $L(X:-\infty)=\{0\}$. So, $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is strongly regular, as desired. Next, note that $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda \cap A_\theta]_\mu$ is equivalent to $L^1(X:0)\cap L^2(X:0) = L(X:(0,0))$. Now, since $L(X:(m,n)) = L(V:(m,n))$ for all $(m,n)\in\mathbb{Z}^2$, it follows that $L^1(X:m) = L^1(V:m)$ for all $m\in\mathbb{Z}$ and $L^2(X:n)=L^2(V:n)$ for all $n\in\mathbb{Z}$. Since, the spectral measure for the random field $V_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is σ^2, it follows that $[A_\lambda]_{\sigma^2}\cap[A_\theta]_{\sigma^2} = [A_\lambda\cap A_\theta]_{\sigma^2}$ is equivalent to $L^1(V:0)\cap L^2(V:0) = L(V:(0,0))$. It is well known and straightforward to see that $[A_\lambda]_{\sigma^2}\cap[A_\theta]_{\sigma^2} = [A_\lambda\cap A_\theta]_{\sigma^2}$. Therefore, $[A_\lambda]_\mu \cap [A_\theta]_\mu = [A_\lambda\cap A_\theta]_\mu$, as desired. □

In sharp contrast with the Gaussian case, $\alpha=2$, where the $V_{m,n}$'s in Theorem 7 are independent, for the non-Gaussian case, $0<\alpha<2$, the $V_{m,n}$'s are not independent random variables. We state this observation as a proposition. The proof follows exactly as in [1], and is included for completeness.

Proposition 1. *No (non-trivial) harmonizable non-Gaussian SαS field $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ with $0<\alpha<2$ is the moving average of an independent SαS field $V_{m,n}$, $(m,n)\in\mathbb{Z}^2$ with $L(X)=L(V)$.*

Proof. Suppose on the contrary that the $V_{m,n}$'s are independent. Since $V_{m,n}\in L(X)$, $(m,n)\in\mathbb{Z}^2$, $V_{m,n}$ is represented by some $f_{m,n}\in L^\alpha(\mathbb{T}^2,\mu)$, $(m,n)\in\mathbb{Z}^2$. Now, as observed above, the mutual independence of the $V_{m,n}$'s implies that the $f_{m,n}$'s have mutually disjoint supports. We will use $E_{m,n}$ to denote the support of $f_{m,n}$, $(m,n)\in\mathbb{Z}^2$. By the correspondence between $L(X)$ and $L^\alpha(\mathbb{T}^2,\mu)$ and the moving average representation, it follows that

$$e^{-(im\lambda+in\theta)} = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} f_{m-k,n-l}(e^{i\lambda},e^{i\theta}) \text{ in } L^\alpha(\mathbb{T}^2,\mu).$$

Since $X_{m,n}$, $(m,n)\in\mathbb{Z}^2$ is non-trivial, some $f_{m,n}$, say f_{m_0,n_0}, is not identically zero. Then, for all $k,l\ge 0$,

$$e^{-i((k+m_0)\lambda+(l+n_0)\theta)} = a_{k,l} f_{m_0,n_0}(e^{i\lambda},e^{i\theta}) \quad [\mu]\text{-a.e. on } E_{m_0,n_0}.$$

It follows that $a_{k,l}\ne 0$ for all $k,l\ge 0$. Now, putting $k,l=0$ and then $k,l=1$, we get $a_{1,1} = e^{-i(\lambda+\theta)}a_{0,0}$ $[\mu]$-a.e. on E_{m_0,n_0} and since $\mu(E_{m_0,n_0})>0$, we get a contradiction. □

We now want to obtain conditions on μ to get a four-fold Wold-type decomposition. Let

$$d\mu(e^{i\lambda},e^{i\theta}) = f(e^{i\lambda},e^{i\theta})\, d\sigma^2(e^{i\lambda},e^{i\theta}) + d\mu^s(e^{i\lambda},e^{i\theta})$$

be the Lebesgue decomposition of μ with respect to σ^2 and let Δ^s be the support of μ^s and $\Delta^a=(\Delta^s)^c$. Then, we get

$$X_{m,n} = X^a_{m,n} + X^s_{m,n}, \tag{3}$$

where

$$X^a_{m,n} = \int\int_{\Delta^a} e^{im\lambda + in\theta}\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta}),$$

and

$$X^s_{m,n} = \int\int_{\Delta^s} e^{im\lambda + in\theta}\, d\mathcal{Z}(e^{i\lambda}, e^{i\theta}).$$

Because their representing functions have disjoint support, $X^a_{m,n}$ and $X^s_{m,n}$ are independent stationary random fields. Our next theorem gives a decomposition of $X^s_{m,n}$, $(m,n) \in \mathbb{Z}^2$.

Theorem 8. *The random field $X^s_{m,n}$, $(m,n) \in \mathbb{Z}^2$ admits the following unique decomposition*

$$X^s_{m,n} = X^{r,s}_{m,n} + X^{s,r}_{m,n} + X^{s,s}_{m,n},$$

where

(1) *$X^{r,s}_{m,n}$, $X^{s,r}_{m,n}$, and $X^{s,s}_{m,n}$ are independent random fields.*
(2) *$X^{r,s}_{m,n}$ is horizontally regular and vertically singular.*
(3) *$X^{s,r}_{m,n}$ is horizontally singular and vertically regular.*
(4) *$X^{s,s}_{m,n}$ is horizontally singular and vertically singular.*

Proof. Using Theorem 3, we get that

$$X^s_{m,n} = X^{s,(1)}_{m,n} + X^{s,(2)}_{m,n},$$

where $X^{s,(1)}_{m,n}$ and $X^{s,(2)}_{m,n}$ are independent random fields with $X^{s,(1)}_{m,n}$ horizontally regular and $X^{s,(2)}_{m,n}$ horizontally singular. Now, using Theorem 6, we get

$$X^{s,(1)}_{m,n} = X^{s,(1,1)}_{m,n} + X^{s,(1,2)}_{m,n},$$

and

$$X^{s,(2)}_{m,n} = X^{s,(2,1)}_{m,n} + X^{s,(2,2)}_{m,n},$$

where $X^{s,(1,1)}_{m,n}$ and $X^{s,(1,2)}_{m,n}$ are independent random fields, $X^{s,(1,1)}_{m,n}$ is vertically regular and $X^{s,(1,2)}_{m,n}$ is vertically singular; and $X^{s,(2,1)}_{m,n}$ and $X^{s,(2,2)}_{m,n}$ are independent random fields, $X^{s,(2,1)}_{m,n}$ is vertically regular and $X^{s,(2,2)}_{m,n}$ is vertically singular. Next, we observe that the support for the representing functions for $X^{s,(1,1)}_{m,n}$, $X^{s,(1,2)}_{m,n}$, $X^{s,(2,1)}_{m,n}$, and $X^{s,(2,2)}_{m,n}$ are disjoint. Therefore, they are all independent and $X^{s,(1,1)}_{m,n}$ and $X^{s,(1,2)}_{m,n}$ are also horizontally regular and $X^{s,(2,1)}_{m,n}$ and $X^{s,(2,2)}_{m,n}$ are horizontally singular. We finally observe that $X^{s,(1,1)}_{m,n}$ must equal zero. For otherwise, since we have shown that it is strongly regular, and the proof of Theorem 7 guarantees that its spectral measure would be absolutely continuous with respect to σ^2, we would get a contradiction. Now, if we set $X^{r,s}_{m,n} = X^{s,(1,2)}_{m,n}$, $X^{s,r}_{m,n} = X^{s,(2,1)}_{m,n}$, and $X^{s,s}_{m,n} = X^{s,(2,2)}_{m,n}$, we get our desired decomposition. $\square$

Combining this result with Theorem 7 gives us the following four-fold Wold-type decomposition.

Theorem 9. *Let $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ be a harmonizable $S\alpha S$ field with $1 < \alpha \le 2$ and spectral measure μ. If $d\mu(e^{i\lambda}, e^{i\theta}) = |\varphi(e^{i\lambda}, e^{i\theta})|^\alpha\, d\sigma^2(e^{i\lambda}, e^{i\theta}) + d\mu^s(e^{i\lambda}, e^{i\theta})$ is the Lebesgue decomposition of μ with respect to σ^2 and φ is a strongly outer* *function in $H^\alpha(\mathbb{T}^2)$, then the random field $X_{m,n}$, $(m,n) \in \mathbb{Z}^2$ admits the following unique decomposition:*

$$X_{m,n} = X^{r,r}_{m,n} + X^{r,s}_{m,n} + X^{s,r}_{m,n} + X^{s,s}_{m,n},$$

where

(1) *$X^{r,r}_{m,n}$, $X^{r,s}_{m,n}$, $X^{s,r}_{m,n}$, and $X^{s,s}_{m,n}$ are independent random fields.*
(2) *$X^{r,r}_{m,n}$ is horizontally regular and vertically regular.*
(3) *$X^{r,s}_{m,n}$ is horizontally regular and vertically singular.*
(4) *$X^{s,r}_{m,n}$ is horizontally singular and vertically regular.*
(5) *$X^{s,s}_{m,n}$ is horizontally singular and vertically singular.*

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