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ENDOMORPHISMS OF POWER SERIES FIELDS AND RESIDUE FIELDS OF FARGUES-FONTAINE CURVES

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Abstract. We show that for k a perfect field of characteristic p, there exist endomorphisms of the completed algebraic closure of $k((t))$ which are not bijective. As a corollary, we resolve a question of Fargues and Fontaine by showing that for p a prime and $\mathbb{C}_p$ a completed algebraic closure of $\mathbb{Q}_p$, there exist closed points of the Fargues-Fontaine curve associated to $\mathbb{C}_p$ whose residue fields are not (even abstractly) isomorphic to $\mathbb{C}_p$ as topological fields.

1. Introduction

In this short note, we address the following question. By an *analytic field*, we will always mean a field complete with respect to a nonarchimedean multiplicative absolute value (assumed to be real-valued and written multiplicatively); by default, we always allow the trivial absolute value.

Question 1.1. Let K be an analytic field. Let k be a trivially valued subfield of K. Is every continuous k-linear homomorphism from K to itself which induces automorphisms of residue fields and value groups necessarily surjective (and hence an automorphism)?

We will view Question 1.1 as a collection of distinct cases indexed by the choice of K, k. For example, one has affirmative answers in the following cases:

- when K is trivially valued, discretely valued, or more generally spherically complete (Proposition 3.1);
- when $\mathrm{char}(k) = 0$ and K is the completed algebraic closure of a power series field over k (Remark 3.3);

whereas one has negative answers in the following cases:

- in certain cases in characteristic 0 (Example 3.2);
- when $\mathrm{char}(k) > 0$ and K is the completed perfect closure of a power series field over k (see [9]).

Hereafter, fix a prime number p. Our main result is a negative answer to Question 1.1 when $\mathrm{char}(k) = p$ and K is the completed algebraic closure of a power series field over k.

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Theorem 1.2. *Let K be a completed algebraic closure of $k((t))$ for some field k of characteristic p. Then there exists a continuous k-linear homomorphism $\tau : K \to K$ which is not an isomorphism.*

The proof depends on a calculation using completed modules of Kähler differentials of analytic fields, as recently studied by the second author [14]. We develop here the bare minimum of this subject needed for the proof of Theorem 1.2; a more detailed treatment of completed differentials between analytic fields will be given by the second author elsewhere.

Theorem 1.2 was prompted by an application to a foundational question of p-adic Hodge theory, specifically in the *perfectoid correspondence* (commonly known as *tilting*) between nonarchimedean fields in mixed and equal characteristics (generalizing the *field of norms correspondence* of Fontaine and Wintenberger). A nonarchimedean field K of residue characteristic p is *perfectoid* if it is not discretely valued and the Frobenius automorphism on $\mathfrak{o}_K/(p)$ is surjective. Given such a field, let $K^\flat$ be the inverse limit of K under the p-power map; one then shows that $K^\flat$ naturally carries the structure of a perfectoid (and hence perfect) nonarchimedean field of equal characteristic p and that there is a canonical isomorphism between the absolute Galois groups of K and $K^\flat$ [8, 10, 12]. The functor $K \mapsto K^\flat$ is not fully faithful, even on fields of characteristic 0; for instance, one can construct many algebraic extensions of $\mathbb{Q}_p$ whose completions K map to the completed perfect closure of a power series field over $\mathbb{F}_p$ (e.g., the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$ and the Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$). However, Fargues and Fontaine have asked [5, Remark 2.24] (see also [4]) whether this can happen for a completed algebraic closure of $\mathbb{Q}_p$, and using Theorem 1.2 we are able to answer this question.

Theorem 1.3. *Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$. Then there exists a perfectoid field K which is not isomorphic to $\mathbb{C}_p$ as a topological field, but for which there exists an isomorphism $K^\flat \cong \mathbb{C}_p^\flat$.*

This result admits the following geometric interpretation. For each perfectoid field K, Fargues and Fontaine define an associated scheme X_K which is a "complete curve" (i.e., a regular one-dimensional noetherian scheme equipped with a surjection of its Picard group onto $\mathbb{Z}$) in terms of which p-adic Hodge theory over K can be simply formulated. Theorem 1.3 implies that for $K = \mathbb{C}_p$, there exists a closed point of X_K whose residue field is not isomorphic to $\mathbb{C}_p$.

We conclude this introduction by pointing out that after we prepared our proof of Theorem 1.2, we learned that this statement is a special case of a result of Matignon and Reversat [11, Théorème 2]. However, since our proof of the special case is somewhat simpler than the more general argument of Matignon–Reversat, we have elected to retain the proof here.

2. Analytic fields and completed differentials

As a technical input into the proof of Theorem 1.2, we review some basic properties of analytic fields and completed differentials.

Definition 2.1. By an *analytic field*, we will mean a field equipped with a multiplicative nonarchimedean absolute value with respect to which the field is complete. By default, we allow the trivial absolute value. When we consider an *extension* L/K of analytic fields, we require that the absolute value on L restricts to the absolute value on K.

Definition 2.2. We say that an extension L/K of analytic fields is *primitive* if there exists $t \in L^\times$ such that $K(t)$ is dense in L; we will write $L = \widehat{K(t)}$ if we need to indicate the choice of t.

With t given, the extension $\widehat{K(t)}/K$ corresponds to a point in the projective line over K in the category of Berkovich nonarchimedean analytic spaces [2]. Without t given, the points associated to L/K are all of the same type 1–4 in Berkovich's classification [2, (1.4.4)]; we thus classify L/K accordingly. Write

$$E_{L/K} = \dim_{\mathbb{Q}}(|L^\times|/|K^\times|) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad F_{L/K} = \operatorname{trdeg}_{\kappa(K)} \kappa(L),$$

where $\kappa(*)$ denotes the residue field of $*$; these are determined by the type of L/K as follows:

Type of L/K	$E_{L/K}$	$F_{L/K}$
1	0	0
2	0	1
3	1	0
4	0	0

In all cases we have $E_{L/K} + F_{L/K} \leq 1$, as per Abhyankar's inequality (e.g., see [13, Lemma 2.1.2]). However, types 1 and 4 cannot be distinguished using $E_{L/K}$ and $F_{L/K}$ alone: one must instead observe that L/K is of type 1 if and only if L embeds into the completed algebraic closure of K.

In order to better distinguish between primitive extensions of types 1 and 4, we will use completed modules of differentials.

Definition 2.3. Let L/K be an extension of analytic fields. As described in [14, §4], the module $\Omega_{L/K}$ admits a maximal seminorm $\|\bullet\|$ (the *Kähler seminorm*) with respect to which $d_{L/K} : L \to \Omega_{L/K}$ is nonexpanding. Let $\widehat{\Omega}_{L/K}$ denote the completion of $\Omega_{L/K}$ with respect to $\|\bullet\|$; it receives an induced derivation $\widehat{d}_{L/K} : L \to \widehat{\Omega}_{L/K}$.

Lemma 2.4. *Let L/K be a primitive extension, and choose $t \in L$ such that $K(t)$ is dense in L.*

- (a) *The module $\widehat{\Omega}_{L/K}$ is generated over L by the single element $\widehat{d}_{L/K}(t)$.*
- (b) *The equality $\widehat{\Omega}_{L/K} = 0$ holds if and only if the separable closure of K in L is dense. (Note that this condition implies that L/K is of type 1, and conversely whenever* $\operatorname{char}(K) = 0$.*)*

Proof. Since $\Omega_{K(t)/K}$ is generated by $d_{L/K}(t)$, (a) is obvious.

Let l be the separable integral closure of K in L. If l is dense in L (which forces L/K to be of type 1), then $\Omega_{l/K} = 0$ and so $\widehat{\Omega}_{L/K} = 0$. This proves the inverse implication in (b).

Suppose that L/K is not of type 1. Let K' be a completed algebraic closure of K and put $L' = l \widehat{\otimes}_K K'$; then the natural map $\Omega_{L/K} \widehat{\otimes}_L L' \to \Omega_{L'/K'}$ sends $\widehat{d}_{L/K}(t) \otimes 1$ to $\widehat{d}_{L'/K'}(t)$. The latter is nonzero by [3, Theorem 2.3.2(i)], so $\widehat{d}_{L/K}(t) \neq 0$.

It remains to consider the case when L/K is of type 1 but l is not dense in L. (Note that this last step is not needed for the proof of Theorem 1.2, so the uninterested reader can skip it.) Observe that any separable extension of $\widehat{l}$ is the closure of a separable extension of l. Since l is separably closed in L, we obtain that $\widehat{l}$ is separably closed in L too. It suffices to show that $\widehat{d}_{L/\widehat{l}}(t) \neq 0$, so after replacing K by $\widehat{l}$ we can assume that $K = l$.

Fix an embedding of L into K'. Let G be the group of continuous automorphisms of K' fixing K; this group is naturally identified with the absolute Galois group of K. The subgroup H fixing L is closed in G, and hence is the absolute Galois group of some separable extension L_0 of K. If $\operatorname{char} K = 0$, then the Ax-Sen theorem [1] applied to both L_0 and L implies that $(K')^H = \widehat{L_0} = L$, but this contradicts our previous assumption that $K = l \neq L$. We must then have $\operatorname{char} K = p > 0$. By Ax-Sen again, we have $(K')^H = \widehat{L_0^{1/p^\infty}} = \widehat{L^{1/p^\infty}}$. If $L_0 \neq K$, we may choose a separable irreducible polynomial $P \in K[T]$ of degree > 1 with a root in L_0; by Krasner's lemma, P has a root x in L^{1/p^n} for some sufficiently large n. But then $x^{p^n} \in L$ generates a nontrivial separable extension of K, again contradicting our assumption that $K = l \neq L$. We conclude that $L_0 = K$ and so $t \in \widehat{K^{1/p^\infty}} \setminus K$.

Choose $a_0 = 0, a_1, \ldots \in K$ such that the sequence $r_n = |t - a_n^{1/p^n}|$ converges to zero. Then L is the completion of its subalgebra $\bigcup_n k\{r_0^{-1}t, r_n^{-p^n}(t^{p^n} - a_n)\}$; in particular, $k[t]$ is dense in L. Consider the Banach ring $\mathcal{A} := L\widehat{\otimes}_K L$ provided with the tensor product norm $\|\bullet\|$ and note that the ideal $J = \operatorname{Ker}(\mathcal{A} \to L)$ is generated by $T := 1 \otimes t - t \otimes 1$.

We claim that $\|T^{p^n}\| \leq r_n^{p^n}$. Indeed, since $|t^{p^n} - a_n| = r_n^{p^n}$, we have that $\|1 \otimes t^{p^n} - a_n\| \leq r_n^{p^n}$ and $\|t^{p^n} \otimes 1 - a_n\| \leq r_n^{p^n}$. (Note, for the sake of completeness, that T is quasi-nilpotent, i.e. its spectral norm vanishes, and hence L is the uniform completion of $\mathcal{A}$, i.e. the completion with respect to the spectral seminorm. This is a topological extension of the classical fact that T is nilpotent and L is the reduction of $\mathcal{A}$ when L/K is finite and purely inseparable.)

By [14, Remark 4.3.4(ii)], there is an isomorphism $J/J^2 \xrightarrow{\sim} \widehat{\Omega}_{L/K}$ that takes T to $\widehat{d}_{L/K}(t)$. Thus, we should only show that $T \neq aT^2$ in $\mathcal{A}$. Assume, to the contrary, that $T = aT^2$ and set $s = \|a\|$. Then, $\|T\| = \|a^{p^n-1}T^{p^n}\| \leq s^{p^n-1}r_n^{p^n}$ for any n and hence $\|T\| = 0$. Thus $L\widehat{\otimes}_K L = L$ and since $L \otimes_K L$ embeds into $L\widehat{\otimes}_K L$ by [6, 3.2.1(4)], we obtain a contradiction. □

3. Proofs and examples

We now settle the questions raised in the introduction.

Proposition 3.1. *Question* 1.1 *admits an affirmative answer if K is spherically complete.*

Proof. Let $\tau : K \to K$ be a homomorphism as in Question 1.1. Suppose by way of contradiction that there exists $x \in K$ with $x \notin \tau(K)$. Since K is spherically complete, the set of possible valuations of $x - \tau(y)$ for $y \in K$ has a least element. If y realizes this valuation, then by the matching of value groups, we can find $y' \in K$ such that $\tau(y')$ and $x - \tau(y)$ have the same valuation; by the matching of residue fields, we can further choose y' such that $(x - \tau(y))/\tau(y')$ maps to 1 in k. But then $x - \tau(y + y')$ has smaller valuation than $x - \tau(y)$, a contradiction. □

Example 3.2. Let k be an analytic field whose absolute value is nondiscrete, and choose a sequence $x_1, x_2, \ldots \in k^\times$ such that $|x_i| < 1$ and $\lim_n |x_1 \cdots x_n| > 0$. (For a more concrete example, take k to be a completed algebraic closure of $\mathbb{C}((t))$ and take $x_n = t^{2^{-n}}$.) Let K be the completion of $k(t_1, t_2, \ldots)$ for the Gauss valuation (i.e., the valuation of a nonzero polynomial is the maximum valuation of its coefficients); then K admits a unique valuation-preserving endomorphism τ fixing k and taking t_n to $t_n - x_n t_{n+1}$ for each n. We will show that the image of τ does not contain t_1, and hence τ is not an isomorphism.

Suppose to the contrary that there exists $y \in K$ with $\tau(y) = t_1$. By hypothesis, there exists some $\lambda \in k$ such that $|\lambda| < |x_1 \cdots x_n|$ for all n. We may then choose $y' \in K_0(t_1, \dots, t_n)$ for some positive integer n in such a way that $|y - y'| < |\lambda|$. Put $y'' = t_1 + x_1 t_2 + \cdots + x_1 \cdots x_n t_{n+1}$; then $\tau(y'') = t_1 - x_1 \cdots x_{n+1} t_{n+2}$, so $|y'' - y| = |\tau(y'' - y)| = |x_1 \cdots x_{n+1}| > |\lambda|$. Hence $|y'' - y'| = |x_1 \cdots x_{n+1}|$, but $y'' - y'$ equals $x_1 \cdots x_n t_{n+1}$ plus an element of $k(t_1, \dots, t_n)$ and so cannot have valuation less than $|x_1 \cdots x_n|$. This yields the desired contradiction.

Remark 3.3. Let k be a field of characteristic 0. For each positive integer n, the derivation $\frac{d}{dt}$ on $k((t))$ extends to the derivation $\partial_n = n^{-1} t^{1/n-1} \frac{d}{dt^{1/n}}$ on $k((t^{1/n}))$ satisfying $|\partial_n f| \leq |t|^{-1} |f|$ for any $f \in k((t^{1/n}))$. Let K be a completed algebraic closure of $k((t))$; by Puiseux's theorem, K is the completion of $\bigcup_{n=1}^{\infty} \overline{k}((t^{1/n}))$ for $\overline{k}$ the algebraic closure of k in K, so the derivation $\frac{d}{dt}$ extends uniquely to a continuous derivation on K. Consequently, $\widehat{\Omega}_{K/k}$ is generated by $\widehat{d}_{K/k}(x)$ for any $x \in K - \overline{k}$. For any k-linear automorphism τ of K, let L be the completion of $\tau(K)(t)$ within K; taking $x = \tau(t)$ in the previous discussion shows that $\widehat{\Omega}_{L/\tau(K)} = 0$. By Lemma 2.4(b) we conclude that $L/\tau(K)$ is of type 1. Since $\tau(K)$ is algebraically closed, it follows that $L = \tau(K)$ and hence τ is an isomorphism.

Proof of Theorem 1.2. Choose a sequence $\{d_i\}_{i=1}^{\infty}$ of positive integers in such a way that:

(a) d_i is not divisible by p;
(b) $\lim_{i\to\infty}(d_{i+1} - p d_i) = \infty$; and
(c) the sequence $\{p^{-i} d_i\}_{i=1}^{\infty}$ is strictly increasing (for large i, this follows from (b)) and bounded.

For a concrete example, take

$$d_i := 1 + pi + p^2(i-1) + p^3(i-2) + \cdots + p^i.$$

Choose a sequence $\{c_i\}_{n=1}^{\infty}$ of elements of k such that each field $\mathbb{F}_p(c_i)$ is finite, but the field $\mathbb{F}_p(c_1, c_2, \dots)$ is infinite (in fact any $c_i \neq 0$ will do, but this assumption shortens the argument). Set

$$\alpha_n := \sum_{i=0}^{n} c_i t^{p^{-i} d_i} \in K$$

and

$$r_n := |\alpha_{n+1} - \alpha_n| = |t|^{p^{-n-1} d_{n+1}};$$

by construction, $\{r_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence with nonzero limit. Consider the Berkovich affine line $\mathbb{A}^1_{k((t))}$ with coordinate x and let E_n be the closed disc of radius r_n centered at α_n. The intersection of the E_n does not contain any element of any finite extension of $k((t))$, so it consists of a single point z of type 4. (Otherwise, by [7, Lemma 10.1, Corollary 11.9] the generalized power series $x = \sum_{i=0}^{\infty} c_i t^{p^{-i} d_i}$ would be algebraic over $k(t)$, hence over $\overline{\mathbb{F}}_p(t)$ because the minimal polynomial must be invariant under coefficientwise automorphisms, hence over $\mathbb{F}_p(t)$; but then [7, Corollary 11.9] would force the c_i to belong to a finite extension of $\mathbb{F}_p$.) The completed residue field $L = \mathcal{H}(z)$ of this point is a primitive extension of $k((t))$ topologically generated by x, and the conditions $z \in E_n$ mean that $\left|x^{p^n} - \alpha_n^{p^n}\right| = r_n^{p^n}$ for each n. Since $\widehat{d}_{L/k}$ is nonexpanding, we have $\left\|\widehat{d}_{L/k}(\alpha_n^{p^n})\right\| \leq r_n^{p^n}$. Furthermore, in the expression $\alpha_n^{p^n} = \sum_{i=0}^n t^{p^{n-i}d_i}$ only the term t^{d_n} is not a p-th power, so

$$\widehat{d}_{L/k}(\alpha_n^{p^n}) = \widehat{d}_{L/k}(t^{d_n}) = d_n t^{d_n - 1}\widehat{d}_{L/k}(t)$$

and hence (since d_n is not divisible by p)

$$\left\|\widehat{d}_{L/k}(t)\right\| \leq r_n^{p^n}|t|^{1-d_n} = |t|^{p^{-1}d_{n+1}-d_n+1}.$$

Since this holds for all n, we conclude that $\left\|\widehat{d}_{L/k}(t)\right\| = 0$.

By the previous paragraph, $\widehat{d}_{L/k}(t) = 0$ and hence $\widehat{d}_{L/k((x))}(t) = 0$. By Lemma 2.4, $L/k((x))$ is a primitive extension of type 1; the inclusion $k((x)) \to L$ thus induces an isomorphism of completed algebraic closures. That is, t belongs to the completed algebraic closure of $k((x))$, but x does not belong to the completed algebraic closure of $k((t))$. If we write K' for a completed algebraic closure of $k((x))$, we then have a strict inclusion $K \to K'$; composing this with an identification $K' \cong K$ yields the desired endomorphism. □

Proof of Theorem 1.3. We use [8, Theorem 1.5.6] as our blanket reference concerning the perfectoid correspondence. By [8, Example 1.3.5], there is an algebraic extension of $\mathbb{Q}_p$ whose completion is perfectoid with tilt isomorphic to the completed perfect closure of $\mathbb{F}_p((t))$; hence $\mathbb{C}_p$ is perfectoid and $\mathbb{C}_p^\flat$ is isomorphic to the completed algebraic closure of $\mathbb{F}_p((t))$. By Theorem 1.2, there exists an endomorphism $\tau : \mathbb{C}_p^\flat \to \mathbb{C}_p^\flat$ which is not surjective; this corresponds to a morphism $\mathbb{C}_p \to K$ of perfectoid fields which is not surjective either. In particular, the integral closure of $\mathbb{Q}_p$ in K is not dense, so K cannot admit any isomorphism to $\mathbb{C}_p$ in the category of topological fields. □

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