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"WEAK YET STRONG" RESTRICTIONS OF HINDMAN'S FINITE SUMS THEOREM

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Abstract. We present a natural restriction of Hindman's Finite Sums Theorem that admits a simple combinatorial proof (one that does not also prove the full Finite Sums Theorem) and low computability-theoretic and proof-theoretic upper bounds, yet implies the existence of the Turing Jump, thus realizing the only known lower bound for the full Finite Sums Theorem. This is the first example of this kind. In fact we isolate a rich family of similar restrictions of Hindman's Theorem with analogous properties.

1. Introduction and motivation

The following question was asked by Hindman, Leader and Strauss in [12]:

> *Question 12.* Is there a proof that whenever $\mathbf{N}$ is finitely colored there is a sequence $x_1, x_2, \ldots$ such that all x_i and all $x_i + x_j$ ($i \neq j$) have the same color, that does not also prove the Finite Sums Theorem?

The theorem referred to as the Finite Sums Theorem is the famous result of Hindman's (the original proof is in [11]) stating that whenever $\mathbf{N}$ is finitely colored there is a sequence $x_1, x_2, \ldots$ such that all finite non-empty sums of distinct elements from the sequence have the same color. We will sometimes refer to this statement as Hindman's Theorem, or the full Hindman's Theorem.

In this paper we present some results that are related to Question 12 above. We isolate a rich family $\mathcal{F}$ of natural restrictions of the Finite Sums Theorem with the following two properties:

(1) Each member of the family $\mathcal{F}$ admits a simple combinatorial proof that does not establish Hindman's Theorem, but
(2) Each member of a non-trivial sub-family of $\mathcal{F}$ is strong in the sense of having the same computability-theoretic lower bounds that are known to hold for Hindman's Theorem.

The simplicity of the proof referred to in point (1) above is evident in the sense that all members of $\mathcal{F}$ admit a proof consisting of a finite iteration of the Infinite Ramsey's Theorem and an application of some classical theorem from finite combinatorics. Yet, much more detailed information can be obtained by using the tools of

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Computability Theory and Reverse Mathematics, the areas where the lower bound mentioned in point (2) above come from.

The strength of the Finite Sums Theorem is indeed a major open problem in these areas (see [16], Question 9). A huge gap remains between the known lower and upper bounds on the computability-theoretic and proof-theoretic strength of Hindman's Theorem [11]. Blass, Hirst and Simpson in [2] established the following lower and upper bounds thirty years ago:

(1) There exists a computable coloring $c : \mathbf{N} \to 2$ such that any solution to Hindman's Theorem for c computes $\emptyset'$, the first Turing Jump of the computable sets.
(2) For every computable coloring $c : \mathbf{N} \to 2$ there exists a solution to Hindman's Theorem for coloring c computable from $\emptyset^{(\omega+1)}$, the $(\omega+1)$-th Turing Jump of the computable sets.

By a "solution to Hindman's Theorem for coloring c" we mean an infinite set H such that all finite non-empty sums of elements from H have the same c-color. As often is the case, the above computability-theoretic results have consequences in Reverse Mathematics (see [13, 22] for excellent introductions to the topic). Letting HT denote the natural formalization of Hindman's Finite Sums Theorem in the language of arithmetic, the only known upper and lower bounds on the logical strength of the full Finite Sums Theorem are the following (again from [2]):

$$\mathsf{ACA}_0^+ \geq \mathsf{HT} \geq \mathsf{ACA}_0.$$

Recall that ACA_0 is equivalent to $\mathsf{RCA}_0 + \forall X \exists Y (Y = X')$ and that ACA_0^+ is equivalent to $\mathsf{RCA}_0 + \forall X \exists Y (Y = X^{(\omega)})$. Note that the ACA_0-lower bound already holds for Hindman's Theorem restricted to colorings in 2 colors.

Recently there has been some interest in the computability-theoretic and proof-theoretic strength of restrictions of Hindman's Theorem (see [4, 6, 14]). While [14] deals with a restriction on the sequence of finite sets in the Finite Unions formulation of Hindman's Theorem, both [6] and [4] deal with restrictions on the types of sums that are guaranteed to be colored the same color.

Blass conjectured in [1] that the complexity of Hindman's Theorem might grow with the length of the sums for which homogeneity is guaranteed. Let us denote by $\mathsf{HT}^{\leq n}_r$ the restriction of the Finite Sums Theorem to colorings with r colors and sums of at most n terms. The conjecture discussed in [1] is then that the complexity of $\mathsf{HT}^{\leq n}_r$ is growing with n.

The main result in [6] is that the above described $\emptyset'$ lower bound known to hold for the full Hindman's Theorem already applies to its restriction to 3 colors and to sums of at most 3 terms ($\mathsf{HT}^{\leq 3}_3$ in the notation introduced above). Note, however, that no upper bound other than the upper bound for the full Hindman's Theorem is known to hold for this restricted version, and the same is true for $\mathsf{HT}^{\leq 2}_2$, the restriction to sums of at most 2 terms! This is obviously related to Question 12 of [12] quoted above.

On the other hand, the variants studied by Hirst in [14] (called Hilbert's Theorem) and by the author in [4] (called the Adjacent Hindman's Theorem) do admit simple proofs, but are very weak and provably fall short of hitting the known lower bounds for the full Hindman's Theorem (they are provable, respectively, from the Infinite Pigeonhole Principle and from Ramsey's Theorem for pairs).

By contrast, the family of natural restriction of the Finite Sums Theorems introduced in the present paper has members that are "weak" in the sense of admitting easy proofs yet "strong" with respect to computability-theoretic and proof-theoretic lower bounds. In terms of Computability Theory and Reverse Mathematics, the properties of our family of restrictions of the Finite Sums Theorem are summarized as follows: *All* members of the family have upper bounds in the Arithmetical Hierarchy for computable instances and proofs in ACA_0. Yet *many* members of this family imply the existence of the Turing Jump. In terms of Reverse Mathematics, they imply ACA_0.

The principles for which we establish a "strong" (i.e., ACA_0) lower bound all feature an extra condition on the solution set, i.e., that the solution set $H \subseteq \mathbf{N}$ is *apart* in the following sense: for any $x < y$ in H, the greatest exponent in the base 2 representation of x is strictly smaller than the smallest exponent in the base 2 representation of y. This condition plays a central role in Hindman's original proof of the Finite Sums Theorem [11] as well as in the lower bound proof by Blass, Hirst and Simpson [2]. It should be stressed that assuming that the solution set is apart does not alter the Finite Sums Theorem. In fact, an apart solution can be easily (and computably) extracted from any solution to the theorem (this was proved by Hindman in [10]). Thus, the Hindman-type principles with apartness imposed on the solution set, as are those that we introduce and study in this paper, are genuine *restrictions* of the Finite Sums Theorem, as opposed to mere *variants* of the latter. This is not the case for other witnesses of the "weak yet strong" phenomenon which can be concocted relatively easily. Consider, for example, the following principle: For each $c : \mathbf{N} \to 2$, there is an infinite set H all of whose elements are powers of 2, such that all non-empty n-term sums of elements from H have the same color. This principle is easily shown to be equivalent to Ramsey's Theorem for coloring n-tuples in 2 colors, and so has the same properties as the "weak yet strong" principles studied in our paper. Yet it does not qualify as a genuine *restriction* of the Finite Sums Theorem: requiring that the solution contains only powers of 2 simply gives, in that case, a false statement. Accordingly, we believe that the witnesses of the "weak yet strong" phenomenon studied in the present paper are much more significant in the perspective of understanding the combinatorics of the Finite Sums Theorem and of answering the many open problems concerning its strength and the strength of its natural restrictions.

2. A family of restrictions of the Finite Sums Theorem

The present section is organized as follows. We first formulate, in section 2.1, a particular restriction of the Finite Sums Theorem, called the Hindman-Brauer Theorem, and prove it by a simple finite iteration of the Infinite Ramsey's Theorem plus finitary combinatorial tools (Theorem 2). The argument is indeed general and in section 2.2 we describe the family of statements that can be proved by exactly the same proof. This proof does not establish the full version of the Finite Sums Theorem. It can be argued that the proof is conceptually simpler than the known proofs of the latter if the knowledge of the Infinite Ramsey's Theorem and of some classical results from finite combinatorics (e.g., Schur's Theorem, Van der Waerden's Theorem, etc.) is presupposed. In the last subsection (section 2.3) we extract computability-theoretic and proof-theoretic upper bounds for each member of the family. The upper bounds obtained are significantly below the only known bounds for the full Finite Sums Theorem, thus giving a precise measure of the simplicity of the underlying proof.

2.1. **An example: the Hindman-Brauer Theorem.** We start with a particular example. We will use the following theorem, due to Alfred Brauer [3], which is a joint strengthening of Van Der Waerden's [23] and Schur's [20] theorems.

Theorem 1 (Brauer's Theorem, [3]). *For all $r, \ell, s \geq 1$ there exists $n = n(r, \ell, s)$ such that if $g : [1, n] \to r$, then there exists $a, b > 0$ such that $\{a, a+b, a+2b, \ldots, a+(\ell-1)b\} \cup \{sb\} \subseteq [1, n]$ is monochromatic.*

Let $B : \mathbf{N}^3 \to \mathbf{N}$ denote the witnessing function for Brauer's Theorem. For $n = 2^{t_1} + \cdots + 2^{t_k}$ with $t_1 < \cdots < t_k$ let $\lambda(n) = t_1$ and $\mu(n) = t_k$. The following Apartness Condition is crucial in what follows.

Definition 1 (Apartness Condition). We say that a set $X \subseteq \mathbf{N}$ satisfies the Apartness Condition (or *is apart*) if for all $x, x' \in X$, if $x < x'$, then $\mu(x) < \lambda(x')$.

Note that the Apartness Condition is inherited by subsets. For a Hindman-type principle P, let "P *with apartness*" denote the corresponding version in which the solution set is required to satisfy the Apartness Condition. Hindman showed how apartness can be ensured (Lemma 2.2 in [11]) by a simple counting argument (Lemma 2.2 in [10]) under the assumption that we have a solution to the Finite Sums Theorem. In our terminology, we have that HT is equivalent to HT *with apartness* over RCA_0. This fact is a key ingredient in the lower bound proof by Blass, Hirst, and Simpson [2]. The Apartness Condition will also play a key role in our lower bound proofs in section 3.

Let us fix some notation. If a is a positive integer and X is a set we denote by $FS^{=a}(X)$ (resp. $FS^{\leq a}(X)$) the set of sums of exactly (resp. at most) a distinct elements from X. More generally, if A and X are sets we denote by $FS^{A}(X)$ the set of all sums of j-many distinct terms from X, for all $j \in A$. Thus, e.g., $FS^{\{1,2,3\}}(X)$ is another name for $FS^{\leq 3}(X)$. By $FS(X)$ we denote $FS^{\mathbf{N}}(X)$, the set of all non-empty finite sums of distinct elements of X. By RT^n_r we denote the Infinite Ramsey's Theorem for r-colorings of n-tuples.

We now state and prove our first example of a weak yet strong principle.

Theorem 2 (Hindman-Brauer Theorem with apartness). *For all $c : \mathbf{N} \to 2$ there exists an infinite and apart set $H \subseteq \mathbf{N}$ such that for some $a, b > 0$ the set $FS^{\{a,a+b,a+2b\}\cup\{b\}}(H)$ is monochromatic.*

Proof. Let $c : \mathbf{N} \to 2$ be given. Let $k = B(2, 3, 1)$. Consider the following construction.

Let H_0 be an infinite (computable) set satisfying the Apartness Condition, e.g., $\{2^t : t \in \mathbf{N}\}$.

Let $H_1 \subseteq H_0$ be an infinite homogeneous set for c, witnessing RT^1_2 relative to H_0.

Let $f_2 : [\mathbf{N}]^2 \to 2$ be defined as $f(x, y) = c(x + y)$. Let $H_2 \subseteq H_1$ be an infinite homogeneous set for f_2, witnessing RT^2_2 relative to H_1.

Let $f_3 : [\mathbf{N}]^2 \to 2$ be defined as $f(x, y, z) = c(x + y + z)$. Let $H_3 \subseteq H_2$ be an infinite homogeneous set for f_3, witnessing RT^3_2 relative to H_2.

We continue in this fashion for k steps. This determines a finite sequence of infinite sets $H_0, H_1, \ldots, H_k$ such that

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_k.$$

Each H_i satisfies the Apartness Condition. Furthermore, for each $i \in [1,k]$ we have that $FS^{=i}(H_j)$ is monochromatic under c for all $j \in [i,k]$. Also, $FS^{=i}(H_k)$ is monochromatic for each $i \in [1,k]$. Let c_i be the color of $FS^{=i}(H_k)$ under c.

The construction can be seen as defining a coloring $C : [1,k] \to 2$, setting $C(i) = c_i$. Since $k = B(2,3,1)$, by Brauer's Theorem there exists $a, b > 0$ in $[1,k]$ such that $\{a, a+b, a+2b\} \cup \{b\} \subseteq [1,k]$ is monochromatic for C. Let $i < 2$ be the color. Then $FS^{\{a,a+b,a+2b\}\cup\{b\}}(H_k)$ is monochromatic of color i for the original coloring c. $\square$

Note that nothing in the above construction is special about 2 colors and 3-term arithmetic progressions. A comment is in order: it is informally clear that the above proof does not establish the full Finite Sums Theorem. The proof is arguably conceptually simpler than any known proof establishing Hindman's Theorem: it consists of a straightforward finite iteration of the infinite Ramsey's Theorem with an application of a theorem from finite combinatorics on top. Below we will measure the simplicity of the proof in terms of Computability Theory and Reverse Mathematics.

2.2. **A family of restrictions of the Finite Sums Theorem admitting simple proofs.** The proof of Theorem 2 is easily adapted to arbitrary values r for number of colors and ℓ for the length of the arithmetic progression. More importantly one can substitute Brauer's Theorem by *virtually any* theorem about finite colorings of numbers from the literature (Schur's Theorem [20], Van der Waerden's Theorem [23], Folkman's Theorem [9, 18], just to name a few), yielding a rich family of Hindman-type theorems.

The general form of the restrictions of the Finite Sums Theorem obtained by the proof of Theorem 2 is the following:

> For all $c : \mathbf{N} \to r$ there exists an infinite $H \subseteq \mathbf{N}$ and there exists a finite A, satisfying some specific conditions, such that $FS^A(H)$ is monochromatic.

For each set $A \subseteq \mathbf{N}$ and positive integer $r > 0$, we let HT_r^A denote such a statement. As indicated above, HT_r^A *with apartness* then indicates the corresponding version in which the solution set is required to be apart. We describe the family by presenting a list of some of its typical members, grouped by sub-families. The general pattern will be clear enough.

Schur Family: For each positive integer r let $\mathsf{HT}_r^{\{a,b,a+b\}}$ denote the following statement.

> Whenever $\mathbf{N}$ is colored in r colors there is an infinite set $X = \{x_1, x_2, \dots\}$ and positive integers a, b such that all elements of $FS^{\{a,b,a+b\}}(\{x_1, x_2, \dots\})$ have the same color.

Van der Waerden Family: For each pair of positive integers r, ℓ let $\mathsf{HT}_r^{\{a,b,a+b,\dots,a+(\ell-1)b\}}$ denote the following statement.

> Whenever $\mathbf{N}$ is colored in r colors there is an infinite set $X = \{x_1, x_2, \dots\}$ and positive integers a, b such that all elements of $FS^{\{a,a+b,a+2b,\dots,a+(\ell-1)b\}}(\{x_1, x_2, \dots\})$ have the same color.

Brauer Family: For each pair of positive integers r, ℓ, let $\mathsf{HT}_r^{\{a,b,a+b,\dots,a+(\ell-1)b\}\cup\{b\}}$ denote the following statement.

> Whenever $\mathbf{N}$ is colored in r colors there is an infinite set $X = \{x_1, x_2, \dots\}$ and positive integers a, b such that all elements of $FS^{\{a,a+b,a+2b,\dots,a+(\ell-1)b\}\cup\{b\}}(\{x_1, x_2, \dots\})$ have the same color.

Folkman Family: For each pair of positive integers r, ℓ, let $\mathsf{HT}_r^{FS(\{i_1,\dots,i_\ell\})}$ denote the following statement.

> Whenever $\mathbf{N}$ is colored in r colors there is an infinite set $X = \{x_1, x_2, \dots\}$ and positive integers $i_1, \dots, i_\ell$ such that all elements of $FS^{FS(\{i_1,\dots,i_\ell\})}(\{x_1, x_2, \dots\})$ have the same color.

It is clear that the family can be extended at leisure by considering different combinatorial principles about finite colorings of $\mathbf{N}$.

2.3. **Computability-theoretic and proof-theoretic upper bounds.** The observable simplicity of the proof of Theorem 2 can be measured by extracting from it computability-theoretic and proof-theoretic upper bounds. From the finite iteration argument given above one can glean upper bounds that are much below the known upper bounds for the full Finite Sums Theorem.

To assess the Computability and Reverse Mathematics corollaries, it may be convenient to reformulate the general argument of Theorem 2 as follows (again, we give the details only for the case of the Hindman-Brauer Theorem):

Second proof of Theorem 2. Let n be a positive integer. Given $c : \mathbf{N} \to 2$ let $g_n : [\mathbf{N}]^n \to 2^n$ be defined as follows:

$$g_n(x_1, \dots, x_n) = \langle c(x_1), c(x_1 + x_2), \dots, c(x_1 + \dots + x_n)\rangle.$$

Fix an infinite and apart set H_0 of positive integers. By $\mathsf{RT}^n_{2^n}$ relativized to H_0 we get an infinite apart set $H \subseteq H_0$ monochromatic for g_n. Let the color be $\sigma = (c_1, \dots, c_n)$, a binary sequence of length n. Then, for each $i \in [1, n]$, g_n restricted to $FS^{=i}(H)$ is monochromatic of color c_i. The sequence σ is a coloring of n in 2 colors. If $n = B(2, 3, 1)$, then, by the finite Brauer's Theorem, there exists $a, b > 0$ such that $\{a, a + b, a + 2b\} \cup \{b\} \subseteq [1, n]$ and

$$c_a = c_b = c_{a+b} = c_{a+2b}.$$

Then $FS^{\{a,a+b,a+2b,b\}}(H)$ is monochromatic for c of color c_a. □

The above argument shows that $\mathsf{RT}^{B(2,3,1)}_{2^{B(2,3,1)}}$ implies $\mathsf{HT}_2^{\{a,a+b,a+2b\}\cup\{b\}}$ *with apartness*. The difference from the previously given argument is that we have used only one instance of Ramsey's Theorem, albeit for a larger number of colors.

We can then quote the following classical result of Jockusch about upper bounds on the computability-theoretic content of Ramsey's Theorem (see [15]).

Theorem 3 (Jockusch, [15]). *Every computable $f : [\mathbf{N}]^n \to r$ has an infinite homogeneous set H such that $H' \leq_T \emptyset^{(n)}$.*

Then we have the following proposition as an immediate corollary, where $\leq_T$ denotes Turing reducibility.

Proposition 1. *Every computable $c : \mathbf{N} \to 2$ has an infinite and apart set H such that for some $a, b > 0$ the set $FS^{\{a,b,a+b,a+2b\}}(H)$ is monochromatic and such that $H' \leq_T \emptyset^{(B(2,3,1))}$.*

Analogously we get arithmetical upper bounds for the other theorems admitting a similar proof, some of which are apparently quite strong (e.g., the one derived from Folkman's Theorem described above). This should be contrasted with the fact that there are no similar upper bounds on the computability-theoretic content of Hindman's Theorem, not even when restricted to sums of at most two terms! Again, for the latter two theorems, the only upper bound for general computable solutions is $\emptyset^{(\omega+1)}$. Even if the optimality of the bounds is not our main concern here, we note in passing that using Theorem 12.1 of [5] in place of Theorem 3 we can improve Proposition 1 to H''.

We now comment on Reverse Mathematics implications. The argument described above is formalizable in ACA_0 (note that Brauer's Theorem, as well as the other finite combinatorial theorems quoted above, is provable in RCA_0). We then get, for every standard $k \in \mathbf{N}$, that

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^{B(2,k,1)}_{2^k} \to \mathsf{HT}_2^{\{a,a+b,\dots,a+(k-1)b\}\cup\{b\}} \text{ with apartness.}$$

Thus we have the following proposition.

Proposition 2. *For each standard $k \in \mathbf{N}$:*

$$\mathsf{ACA}_0 \vdash \mathsf{HT}_2^{\{a,a+b,a+2b,\dots,a+(k-1)b\}\cup\{b\}} \textit{ with apartness.}$$

Again, this should be contrasted with the ACA_0^+ upper bound that is known to hold for the full Finite Sums Theorem, as well as for its restriction to sums of at most two terms.

Obviously, similar proof-theoretic upper bounds hold for many other members of the family by the same argument, as long as the underlying finite combinatorial principle does not itself require strong axioms. Note that Schur's Theorem, Van der Waerden's Theorem, Brauer's Theorem and Folkman's Theorem are all provable in RCA_0.[1]

3. A lower bound on the Hindman-Brauer Theorem

Let K denote the (computably enumerable but not computable) Halting Set or, equivalently, the first Turing Jump $\emptyset'$. We show that there exists a computable coloring $c : \mathbf{N} \to 2$ such that K is computable from any apart solution H of $\mathsf{HT}_2^{\{a,a+b,a+2b,b\}}$ for the instance c, i.e., from any infinite and apart H such that for some $a, b > 0$, the set $FS^{\{a,a+b,a+2b,b\}}(H)$ is monochromatic.

We adapt the beautiful proof of the lower bound for the full Hindman's Theorem by Blass, Hirst and Simpson (Theorem 2.2 in [2]). Gaps and short gaps of numbers are defined as in [2]. We recall the definitions for convenience. Fix an enumeration of the computably enumerable set K and denote by $K[k]$ the set enumerated in k steps of computation by this algorithm. If $n = 2^{t_1} + \dots + 2^{t_k}$ with $t_1 < \dots < t_k$ we refer to pairs (t_i, t_{i+1}) as the *gaps* of n. A gap (a, b) of n is *short in n* if there exists $x \le a$ such that $x \in K$ but $x \notin K[b]$. A gap (a, b) of n is *very short in n* if

[1]Provability in RCA_0 of these theorems is folklore or can be gleaned from inspection of the classical proofs. Schur's Theorem can be proved from the finite Ramsey's Theorem (see, e.g., [17], Theorem 2.2). Provability of Van der Waerden's Theorem in RCA_0 follows from a formalization of Shelah's proof [21] (see [7]). Brauer's Theorem follows easily from Van der Waerden's Theorem (see, e.g., Theorem 2.4 of [17] or Lemma 4.1 of [19]). Folkman's Theorem can also be obtained from Van der Waerden's Theorem; see Theorem 4.3' in [19], and see [8] for a Reverse Mathematics analysis.

there exists $x \leq a$ such that $x \in K[\mu(n)]$ but $x \notin K[b]$. A gap of n that is short in n is called *a short gap of* n. Let $SG(n)$ denote the cardinality of the set of short gaps of n. A gap of n that is very short in n is called *a very short gap of* n. Let $VSG(n)$ denote the cardinality of the set of very short gaps of n. Notice that given n one can effectively compute $VSG(n)$ but not $SG(n)$.

Theorem 4. *There exists a computable coloring $c : \mathbf{N} \to 2$ such that if $H \subseteq \mathbf{N}$ is an apart solution to the Hindman-Brauer Theorem for instance c, then K is computable from H.*

Proof. Consider the following computable coloring of $\mathbf{N}$ in 2 colors:

$$c(n) = VSG(n) \mod 2.$$

Let $H \subseteq \mathbf{N}$ and $a, b > 0$ be such that H is infinite, satisfies the Apartness Condition, and is such that all sums of length $a, b, a+b, a+2b$ of elements from H have the same color under c.

Claim 1. For every $m \in FS^{=a}(H)$, $SG(m)$ is even.

Proof. Pick n in $FS^{=b}(H)$ so large that the following three points are satisfied:

(1) $\mu(m) < \lambda(n)$,
(2) for all $x \leq \mu(m)$, $x \in K$ if and only if $x \in K[\lambda(n)]$,
(3) $\mu(m+n) = \mu(n)$.

This choice is legitimate since H satisfies the Apartness Condition and is infinite. Since $m \in FS^{=a}(H)$ there exists $t_1 < t_2 < \cdots < t_a$ elements of H such that $m = t_1 + t_2 + \cdots + t_a$. Since H satisfies the Apartness Condition, we have that $\mu(m) = \mu(t_a)$ and $\lambda(m) = \lambda(t_1)$. (Analogous equations hold for sums of type b, $a+b$, $a+2b$). Now observe that elements of $FS^{=b}(H)$ are unbounded with respect to their λ-projection; i.e., for all d there exists $q \in FS^{=b}(H)$ such that $\lambda(q) > d$. This follows from the fact that H satisfies the Apartness Condition and by the previous observations on λ-projections of sums. So requirements (1) and (2) above can be met. Requirement (3) follows from requirement (1).

We now compute the number of very short gaps of $m+n$, arguing as in [2]. We consider separately the gaps of m, the gaps of n and the gap $(\mu(m), \lambda(n))$.

The gap $(\mu(m), \lambda(n))$ is not very short, by choice of n (item (2) above).

A gap of n is very short in $m+n$ if and only if it is very short in n, since $\mu(m+n) = \mu(n)$.

A gap (a, b) of m is very short in $m+n$ if and only if it is short (not necessarily very short) as a gap of m: Suppose that (a, b) is a gap of m very short in $m+n$. By definition there exists $x \leq a$ such that $x \in K[\mu(m+n)]$ but $x \notin K[b]$. Then there exists $x \leq a$ such that $x \in K$ but $x \notin K[b]$; hence (a, b) is short in m. For the other direction suppose (a, b) is short in m; that is, there exists $x \leq a$ such that $x \in K$ but $x \notin K[b]$. Then by choice of n ($\mu(m) < \lambda(n)$ by item (1) above and $\lambda(n) < \mu(n)$) we have that $x \leq a$ and $x \in K$ implies that $x \in K[\mu(n)]$. But $\mu(n) = \mu(m+n)$ by item (3) above. Hence (a, b) is very short in m.

Therefore we have the following equation:

$$VSG(m+n) = SG(m) + VSG(n).$$

By hypothesis on H, $VSG(m+n)$ and $VSG(n)$ have the same parity, since $m+n \in FS^{=a+b}(H)$. □

Claim 2. For every $m \in FS^{=b}(H)$, $SG(m)$ is even.

Proof. Pick n in $FS^{=a}(H)$ so large that the following three points are satisfied:

(1) $\mu(m) < \lambda(n)$,
(2) for all $x \leq \mu(m)$, $x \in K$ if and only if $x \in K[\lambda(n)]$,
(3) $\mu(m+n) = \mu(n)$.

This choice is legitimate since H satisfies the Apartness Condition and is infinite. Then argue as previously. We end up with

$$VSG(m+n) = SG(m) + VSG(n).$$

By hypothesis on H, $VSG(m+n)$ and $VSG(n)$ have the same parity, since $m+n \in FS^{=a+b}(H)$. $\square$

Claim 3. For every $m \in FS^{=a+b}(H)$, $SG(m)$ is even.

Proof. Pick n in $FS^{=b}(H)$ so large that the following three points are satisfied:

(1) $\mu(m) < \lambda(n)$,
(2) for all $x \leq \mu(m)$, $x \in K$ if and only if $x \in K[\lambda(n)]$,
(3) $\mu(m+n) = \mu(n)$.

This choice is legitimate since H satisfies the Apartness Condition and is infinite. Then argue as previously. We end up with

$$VSG(m+n) = SG(m) + VSG(n).$$

By hypothesis on H, $VSG(m+n)$ and $VSG(n)$ have the same parity, since $m+n \in FS^{=a+2b}(H)$. $\square$

Claim 4. For all $m \in FS^{=a}(H)$ and all $n \in FS^{=b}(H)$ such that $\mu(m) < \lambda(n)$ we have

$$\forall x \leq \mu(m)(x \in K \leftrightarrow x \in K[\lambda(n)]).$$

Proof. By way of contradiction suppose that $(\mu(m), \lambda(n))$ is short. Then

$$SG(m+n) = SG(m) + SG(n) + 1.$$

But $SG(m+n), SG(n), SG(m)$ are all even by the previous claims. Contradiction. $\square$

We now describe an algorithm showing that K is computable from H. Given an input x, use the oracle to find an $m \in FS^{=a}(H)$ such that $x \leq \mu(m)$ and an $n \in FS^{=b}(H)$ such that $m < n$ and $\mu(m) < \lambda(n)$.

Then run the algorithm enumerating K for $\lambda(n)$ steps to decide membership of $x \in K[\lambda(n)]$. By Claim 4 this also decides membership in K. $\square$

As in [2] a straightforward relativization of the above proof gives the following proposition.

Proposition 3. *Over* RCA_0, $\mathsf{HT}_2^{\{a,a+b,a+2b\}\cup\{b\}}$ *with apartness implies* ACA_0.

Note that the above proof works for any member HT_2^A of our family such that A is guaranteed to contain a set of the form $\{x, x+y, x+2y\} \cup \{y\}$ for some positive integers x, y, and the Apartness Condition is assumed.

4. Conclusions

We have introduced a family of natural restrictions of Hindman's Finite Sums Theorem such that each member of the family admits a fairly simple proof, has arithmetical upper bounds for computable instances, yet many members of the family imply the existence of the Halting Set. These are the first examples with these properties. In fact, Hindman's Theorem restricted to sums of at most 3 terms and 3-colorings $\mathsf{HT}_3^{\leq 3}$ shares the same $\emptyset'$ lower bound (by the main result of [6]) but has no other proof (resp. upper bound) apart from the proof (resp. upper bound) known for the full Finite Sums Theorem. Of all members HT_2^A of our family we showed how to prove that they achieve the only lower bounds known for the full Finite Sums Theorem *provided that* the Apartness Condition is assumed and the set A of lengths of sums for which homogeneity is guaranteed contains a 3-term arithmetic progression and its difference. It is an interesting question to characterize the members in the family that imply ACA_0.

Some members of our family are apparently strong when compared to the family of restrictions of Hindman's Theorem based on the mere number of terms in the sums studied in [6]. Compare, e.g., $\mathsf{HT}_2^{\{a,b,a+b,a+2b,a+3b,\ldots,a+100b\}}$ with $\mathsf{HT}_2^{\leq 3}$. Yet this superficial impression might be misleading. It is an easy observation that HT_2^A for an A such that $A \supseteq \{a, 2a\}$ for some $a > 0$ implies $\mathsf{HT}_2^{\leq 2}$. Analogous relations hold for $A \supseteq \{a, 2a, 3a\}$ and $\mathsf{HT}_2^{\leq 3}$. Yet it doesn't seem easy to prove any of those HT^As by the methods of the present paper. Many more non-trivial implications can be established and will be reported elsewhere.

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References

[1] Andreas Blass, *Some questions arising from Hindman's theorem*, Sci. Math. Jpn. **62** (2005), no. 2, 331–334. MR2179959

[2] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, Logic and combinatorics (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 125–156, DOI 10.1090/conm/065/891245. MR891245

[3] A. T. Brauer, *Über Sequenzen von Potenzresten*, Sitz. Ber. Preuss. Akad. Wiss., Phys. -math. Klasse (1928), pp. 9–16.

[4] L. Carlucci. A weak variant of Hindman's Theorem stronger than Hilbert's Theorem, Archive for Mathematical Logic, to appear.

[5] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman, *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic **66** (2001), no. 1, 1–55, DOI 10.2307/2694910. MR1825173

[6] D. Dzhafarov, C. G. Jockusch, R. Solomon, and L. B. Westrick, Effectiveness of Hindman's Theorem for bounded sums. In A. Day, M. Fellows, N. Greenberg, B. Khoussainov, and A. Melnikov, eds., *Proceedings of the International Symposium on Computability and Complexity* (in honour of Rod Downey's 60th birthday), Lecture Notes in Computer Science, Springer, pp. 134–142. Springer (2017).

[7] Harvey Friedman and Stephen G. Simpson, *Issues and problems in reverse mathematics*, Computability theory and its applications (Boulder, CO, 1999), Contemp. Math., vol. 257, Amer. Math. Soc., Providence, RI, 2000, pp. 127–144, DOI 10.1090/conm/257/04031. MR1770738

[8] Emanuele Frittaion, *Brown's lemma in second-order arithmetic*, Fund. Math. **238** (2017), no. 3, 269–283, DOI 10.4064/fm221-9-2016. MR3654099

[9] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1980. MR591457

[10] Neil Hindman, *The existence of certain ultra-filters on N and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. **36** (1972), 341–346, DOI 10.2307/2039156. MR0307926

[11] Neil Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combinatorial Theory Ser. A **17** (1974), 1–11. MR0349574

[12] Neil Hindman, Imre Leader, and Dona Strauss, *Open problems in partition regularity*, Special issue on Ramsey theory, Combin. Probab. Comput. **12** (2003), no. 5-6, 571–583, DOI 10.1017/S0963548303005716. MR2037071

[13] Denis R. Hirschfeldt, *Slicing the truth*, On the computable and reverse mathematics of combinatorial principles. Edited and with a foreword by Chitat Chong, Qi Feng, Theodore A. Slaman, W. Hugh Woodin and Yue Yang, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. MR3244278

[14] Jeffry L. Hirst, *Hilbert versus Hindman*, Arch. Math. Logic **51** (2012), no. 1-2, 123–125, DOI 10.1007/s00153-011-0257-4. MR2864401

[15] Carl G. Jockusch Jr., *Ramsey's theorem and recursion theory*, J. Symbolic Logic **37** (1972), 268–280, DOI 10.2307/2272972. MR0376319

[16] Antonio Montalbán, *Open questions in reverse mathematics*, Bull. Symbolic Logic **17** (2011), no. 3, 431–454, DOI 10.2178/bsl/1309952320. MR2856080

[17] Hans Jürgen Prömel, *Ramsey theory for discrete structures*, with a foreword by Angelika Steger, Springer, Cham, 2013. MR3157030

[18] Jon Henry Sanders, *A generalization of Schur's Theorem*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Yale University, 1968. MR2617864

[19] Y. Setyawan, *Combinatorial number theory: Results of Hilbert, Schur, Folkman, and Hindman.*, Thesis (M.Sc.), Simon Fraser University, 1998.

[20] I. Schur, *Über die Kongruenze $x^m + y^m \cong z^m (\mathrm{mod}\, p)$*, Jber. Deutsch. Math.-Verein. **25** (1916), 114–117.

[21] Saharon Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc. **1** (1988), no. 3, 683–697, DOI 10.2307/1990952. MR929498

[22] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009. MR2517689

[23] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk. **15** (1927), 212–216.

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