PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 2, February 2018, Pages 571–580
http://dx.doi.org/10.1090/proc/13872

A CONDITION FOR PURELY ABSOLUTELY CONTINUOUS SPECTRUM FOR CMV OPERATORS USING THE DENSITY OF STATES

JAKE FILLMAN AND DARREN C. ONG

(Communicated by Michael Hitrik)

Abstract. We prove an averaging formula for the derivative of the absolutely continuous part of the density of states measure for an ergodic family of CMV matrices. As a consequence, we show that the spectral type of such a family is almost surely purely absolutely continuous if and only if the density of states is absolutely continuous and the Lyapunov exponent vanishes almost everywhere with respect to the same. Both of these results are CMV operator analogues of theorems obtained by Kotani for Schrödinger operators.

1. Introduction

1.1. **Background and motivation.**

CMV matrices are unitary operators on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$ that have attracted substantial interest in recent years. These operators were introduced in [6] as a bridge between spectral theory and the theory of orthogonal polynomials on the unit circle, an idea extensively explored in the monographs [31, 32]. In [4], CMV operators were also proposed as a model for understanding one-dimensional quantum walks (quantum mechanical analogues of classical random walks). This connection has drawn a lot of attention recently from both mathematicians and physicists. See for instance [2, 3, 5, 7, 9, 11–14, 16–22] in addition to the foregoing references. Moreover, as observed in [13], one can relate the classical ferromagnetic Ising model in one dimension to a suitable CMV operator, a connection that allows one to rigorously prove characteristics such as the absence of phase transitions [10].

From the perspective of quantum walks, understanding the spectral type of CMV operators is particularly important, as the spectral type influences the scattering behavior (or lack thereof) for the associated quantum walk. A more precise relationship is furnished by a discrete-time variant of the RAGE Theorem (a name coined by Barry Simon to reflect the contributions of [1, 15, 28]). Such a discrete-time formulation of the RAGE Theorem suitable for quantum walks (with proofs) may be found in the Appendix of [16]. Roughly, point spectrum is associated with localization; that is, the walker remains close to its starting position. Absolutely continuous spectrum is associated with scattering; that is, the walker flees to infinity; and, most exotically, singular continuous spectrum is associated with recurrent scattering, for which the walker flees to infinity in a time-averaged sense but may

---

Received by the editors December 9, 2016.

2010 *Mathematics Subject Classification.* Primary 47B36.

potentially recur to its initial position along a subsequence of time scales. Thus, determining when a CMV operator possesses purely absolutely continuous spectrum is helpful in determining when scattering occurs in a quantum walk.

To this end, in this paper we adapt to the CMV operator some ideas originally used to solve related problems in the theory of Schrödinger operators in $L^2(\mathbb{R})$, that is, second-order differential operators of the form $H\psi = -\psi'' + V\psi$.

For such operators, *Kotani theory* refers to a far-reaching family of results of Shinichi Kotani regarding the absolutely continuous spectrum of H whenever V is a metrically transitive potential (in more modern parlance: whenever V is an ergodic, dynamically defined potential) [23–26]. Barry Simon generalized Kotani's work to the setting of Jacobi [29] and CMV [32] operators, with one exception. In the Notes for [32, Section 10.11], Simon points out that Kotani proves a stronger version of [32, Theorem 10.11.2], namely, that an ergodic family of Schrödinger operators exhibits purely absolutely continuous spectrum (almost surely with respect to the underlying ergodic measure) if and only if the density of states (DOS) measure is absolutely continuous and the Lyapunov exponent vanishes a.e. with respect to the DOS measure; see [26, Corollary 4.8.2]. Simon notes that such a result also ought to hold true for ergodic families of CMV operators. The aim of this work is to prove exactly this theorem.

The key ingredient in Kotani's proof of this statement for Schrödinger operators is an averaging formula, found in [26, Theorem 4.8]. Broadly speaking, Kotani's formula identifies the absolutely continuous part of the DOS with the average of the absolutely continuous parts of the associated spectral measures. In the present paper, we prove a suitable version of Kotani's averaging formula for CMV matrices and then deduce his characterization of purely absolutely continuous spectrum as a consequence.

1.2. **Results.** An *extended CMV matrix* is a pentadiagonal unitary operator on $\ell^2(\mathbb{Z})$ with a repeating 2×4 block structure of the form

$$\mathcal{E} = \mathcal{E}_\alpha = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & & & \\ & \overline{\alpha_0}\rho_{-1} & -\overline{\alpha_0}\alpha_{-1} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & \\ & \rho_0\rho_{-1} & -\rho_0\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & \\ & & & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \\ & & & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \\ & & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1}$$

where $\alpha_n \in \mathbb{D} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$ and $\rho_n = \sqrt{1 - |\alpha_n|^2}$ for all $n \in \mathbb{Z}$. Setting $\alpha_{-1} = -1$, the operator decouples into two half-line operators. The operator on the right half-line takes the form

$$\mathcal{C} = \begin{bmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & \\ & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \\ & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2}$$

and is known as a *standard* or *half-line CMV matrix.*

Let us briefly recall how the density of states is defined. Given a CMV operator $\mathcal{E}$ and $n \in \mathbb{Z}_+$, we denote its restriction to $[-n, n]$ with Dirichlet boundary conditions by $\mathcal{E}_n = \chi_{[-n,n]}\mathcal{E}$. Then, we define $\mathrm{d}k_n$ to be the normalized eigenvalue counting measure; that is, $\mathrm{d}k_n$ puts a Dirac atom of weight $m/(2n+1)$ at ζ whenever ζ is an eigenvalue of $\mathcal{E}_n$ having multiplicity m. Whenever $\mathrm{d}k_n$ enjoys a weak* limit as $n \to \infty$, we refer to said limit as the *density of states measure* (henceforth: DOS) of $\mathcal{E}$ and denote it by $\mathrm{d}k$. This is also closely related to the so-called *density of zeros measure* for orthogonal polynomials on the unit circle (OPUC); cf. [31, Proposition 8.2.1 and Theorem 10.5.21].

Ergodic CMV operators (sometimes called *stochastic* CMV operators) supply an important class of examples for which the DOS exists. Concretely, let $S : \Omega \to \Omega$ be an invertible transformation of a Borel space Ω. Given a measurable function $f : \Omega \to \mathbb{D}$, we may define CMV operators indexed by Ω via $\mathcal{E}_\omega = \mathcal{E}_{\alpha(\omega)}$, where

$$\alpha_n(\omega) = f(S^n\omega), \quad n \in \mathbb{Z},\ \omega \in \Omega.$$

Then, if μ is an S-ergodic measure on Ω and $|f| \le C < 1$ μ-almost everywhere, then the DOS of $\mathcal{E}_\omega$ exists for μ-a.e. $\omega \in \Omega$ by ergodicity. Moreover, as demonstrated in [32, Theorem 10.5.21], the DOS of an element of such an ergodic family is μ-almost surely given by the μ-average of spectral measures:

$$\int g \,\mathrm{d}k = \int_\Omega \langle \delta_0, g(\mathcal{E}_\omega)\delta_0\rangle \,\mathrm{d}\mu(\omega).$$

We will use $\mathbb{E}(\cdot)$ to denote integration against μ, that is,

$$\mathbb{E}(f) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)$$

for $f \in L^1(\Omega, \mathrm{d}\mu)$. We denote the δ_0 spectral measure of $\mathcal{E}_\omega$ by $\mathrm{d}\nu_\omega$, i.e.,

$$\langle \delta_0, g(\mathcal{E}_\omega)\delta_0\rangle = \int_{\mathbb{S}^1} g(z)\,\mathrm{d}\nu_\omega(z), \tag{3}$$

where $\mathbb{S}^1 = \partial\mathbb{D}$ denotes the unit circle. Finally, we define the *Lyapunov exponent* by

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log \|A_z^n\|),$$

where A_z^n denotes the Szegő cocycle at spectral parameter $z \in \mathbb{C}$, that is,

$$A_z^n(\omega) = \begin{bmatrix} z & -\overline{\alpha_{n-1}(\omega)} \\ -\alpha_{n-1}(\omega)z & 1 \end{bmatrix} \times \cdots \times \begin{bmatrix} z & -\overline{\alpha_0(\omega)} \\ -\alpha_0(\omega)z & 1 \end{bmatrix}, \quad n \ge 1,\ \omega \in \Omega.$$

A crucial role is played by the set on which γ vanishes:

$$\mathcal{Z} \overset{\mathrm{def}}{=} \{z \in \mathbb{C} : \gamma(z) = 0\}.$$

We let $\nu_\omega^{(\mathrm{ac})}(z)$ denote the density of the absolutely continuous part of $\mathrm{d}\nu_\omega$ and $k^{(\mathrm{ac})}(z)$ the density of the absolutely continuous part of $\mathrm{d}k$. Our main result is that $k^{(\mathrm{ac})}$ is precisely the μ-average of $\nu_\omega^{(\mathrm{ac})}$ almost everywhere on $\mathcal{Z}$.

Theorem 1. *For Lebesgue-almost every $z \in \mathcal{Z}$,*

$$k^{(\mathrm{ac})}(z) = \mathbb{E}\Big(\nu_\omega^{(\mathrm{ac})}(z)\Big). \tag{4}$$

As a consequence, we deduce Kotani's characterization of (almost-sure) pure a.c. spectrum.

Corollary 1. *The spectral type of $\mathcal{E}_\omega$ is purely absolutely continuous for μ-almost every $\omega \in \Omega$ if and only if the density of states measure is absolutely continuous and the Lyapunov exponent vanishes almost everywhere with respect to the density of states measure.*

We note that the hypotheses of this corollary are satisfied in the case where the Verblunsky coefficients form a periodic sequence. We refer the reader to [32, Chapter 11] for further details.

Establishing connections between the DOS measure and the spectral measure is important, because this allows us to extract facts about the spectral measure of a CMV operator based on an eigenvalue analysis of the finite sub-matrices of the CMV matrix. An example where this technique was useful is in [16], where we were able to prove results regarding the spreading behavior of a limit-periodic quantum walk model by using the DOS measure of the CMV matrix to characterize its spectral measure.

2. Proof of main theorem

We will follow Kotani's original argument as presented in the proof of [8, Theorem 5]. The broad strokes of the argument are similar; the main differences arise from the more complicated nature of Weyl–Titchmarsh theory for CMV matrices. Concretely, there are several different analytic functions that play the role of the Weyl–Titchmarsh m-function, which complicates some of the algebraic gymnastics. For more on the various analogues of the m-function for CMV operators, see [30].

We define first Green's function

$$G_\omega(z) = G_\omega(0,0;z) \overset{\text{def}}{=} \left\langle \delta_0, (\mathcal{E}_\omega - z)^{-1}\delta_0 \right\rangle, \quad z \in \mathbb{C} \setminus \sigma(\mathcal{E}_\omega). \tag{5}$$

In view of the definition (3), one immediately has

$$\int_{\mathbb{S}^1} \frac{\mathrm{d}\nu_\omega(\tau)}{\tau - z} = G_\omega(z).$$

The Carathéodory function of ν_ω will be defined by

$$F_\omega(z) = \left\langle \delta_0, (\mathcal{E}_\omega + z)(\mathcal{E}_\omega - z)^{-1}\delta_0 \right\rangle, \quad z \in \mathbb{C} \setminus \sigma(\mathcal{E}_\omega). \tag{6}$$

This defines an analytic function from $\mathbb{D}$ to the right half plane, whose limiting behavior on the unit circle is connected to the behavior of $\mathrm{d}\nu_\omega$. Critically, one can recover the absolutely continuous part of $\mathrm{d}\nu_\omega$ from the boundary values of $\operatorname{Re} F_\omega$. That is, by [32, (1.3.31)] we have

$$\nu_\omega^{(\mathrm{ac})}(e^{i\theta}) = \lim_{r\uparrow 1} \frac{1}{2\pi} \operatorname{Re} F_\omega(re^{i\theta}) \tag{7}$$

for Lebesgue almost every $\theta \in [0, 2\pi)$.

Using (3), (5), and (6) we may relate the Carathéodory and Green functions via

$$F_\omega(z) = \int_{\mathbb{S}^1} \frac{\tau + z}{\tau - z}\,\mathrm{d}\nu_\omega(\tau) = \int_{\mathbb{S}^1} \left(1 + \frac{2z}{\tau - z}\right) \mathrm{d}\nu_\omega(\tau) = 1 + 2zG_\omega(z). \tag{8}$$

Let us define Γ as in [32, (10.11.18)], i.e.,

$$\Gamma(z) = \int_{\mathbb{S}^1} \log\left(\frac{1 - z\overline{\tau}}{\rho_\infty}\right) \mathrm{d}k(\tau), \text{ where } \rho_\infty = \exp\left(\frac{1}{2}\mathbb{E}\big(\log(1 - |f(\omega)|^2)\big)\right).$$

By the ergodic theorem, one has

$$\lim_{n\to\infty}\left(\prod_{j=0}^{n-1}(1-|\alpha_j(\omega)|^2)\right)^{1/2n}=\rho_\infty \text{ for } \mu\text{-almost every } \omega\in\Omega.$$

The Lyapunov exponent satisfies the Thouless formula. That is, we have

$$\gamma(z)=\operatorname{Re}\Gamma(z) \tag{9}$$

for all z.[1]

We define also the Carathéodory function corresponding to the DOS by

$$K(z)=\int_{\mathbb{S}^1}\frac{\tau+z}{\tau-z}\,\mathrm{d}k(\tau),\quad z\in\mathbb{D}.$$

Just as with F_ω, we may use the boundary values of K to recover the absolutely continuous part of $\mathrm{d}k$ as in (7). One has

$$k^{(\mathrm{ac})}(e^{i\theta})=\frac{1}{2\pi}\lim_{r\uparrow 1}\operatorname{Re}K(re^{i\theta}) \tag{10}$$

for a.e. $\theta\in[0,2\pi)$. By the definitions of the functions K and Γ, it is straightforward to calculate that they are connected via

$$K(z)=1-2z\frac{\mathrm{d}\Gamma}{\mathrm{d}z}(z), \tag{11}$$

which may be viewed as an analogue of (8).

Proof of Theorem 1. The "$\geq$" direction follows immediately from [32, Theorem 10.11.11] and the fact that the average of absolutely continuous measures is absolutely continuous.

For the other direction, we use (10.11.28) of [32], which states that

$$\mathbb{E}(G_\omega(z))=\int_{\mathbb{S}^1}\frac{\mathrm{d}k(\tau)}{\tau-z}. \tag{12}$$

For convenience, let us introduce $\mathcal{Z}^\circ\subseteq[0,2\pi)$ by insisting that $\theta\in\mathcal{Z}^\circ$ if and only if $e^{i\theta}\in\mathcal{Z}$. By (9), (11), and standard facts about Carathéodory functions (e.g. [31, Section 1.3]), one has

$$\lim_{r\uparrow 1}\frac{\partial\gamma}{\partial r}(re^{i\theta})=\lim_{r\uparrow 1}\frac{\gamma(e^{i\theta})-\gamma(re^{i\theta})}{1-r}=-\lim_{r\uparrow 1}\frac{\gamma(re^{i\theta})}{1-r} \tag{13}$$

for Lebesgue almost every $\theta\in\mathcal{Z}^\circ$. Since it is critical to our proof, let us point out that there is a sign error in [32, (10.11.22)], which is why it does not match our equation (13) for the boundary values of $\partial\gamma/\partial r$ on $\mathcal{Z}$.

---

[1] There is a small subtlety here: we are using γ to denote the averaged Lyapunov exponent, which always exists and obeys the Thouless formula for all z. The behavior of the non-averaged Lyapunov exponent can be somewhat delicate on $\mathbb{S}^1$; see the discussion in the Remark following [32, Theorem 10.5.26] for additional details.

We can then write

$$\begin{aligned}
k^{(\mathrm{ac})}(e^{i\theta}) &= \lim_{r\uparrow 1}\frac{1}{2\pi}\operatorname{Re}K(re^{i\theta}) \text{ by (10)}\\
&= \lim_{r\uparrow 1}\frac{1}{2\pi}\operatorname{Re}\left(1-2re^{i\theta}\frac{\mathrm{d}\Gamma}{\mathrm{d}z}(re^{i\theta})\right) \text{ by (11)}\\
&= \frac{1}{2\pi}-\lim_{r\uparrow 1}\frac{1}{\pi}\frac{\partial\gamma}{\partial r}(re^{i\theta}) \text{ by (9) and Cauchy–Riemann}\\
&= \frac{1}{2\pi}+\lim_{r\uparrow 1}\frac{1}{\pi}\left(\frac{\gamma(re^{i\theta})}{1-r}\right) \text{ by (13)} \qquad (14)
\end{aligned}$$

for Lebesgue a.e. $\theta \in \mathcal{Z}^\circ$.

Let $f_+(z) = f_+(z,\omega)$ and $f_-(z) = f_-(z,\omega)$ be the Schur functions corresponding to the half-line CMV matrices with Verblunsky coefficient sequences given by $\alpha_0(\omega), \alpha_1(\omega), \dots$ and $-\overline{\alpha_{-1}(\omega)}, -\overline{\alpha_{-2}(\omega)}, \dots$ respectively. A bit more precisely, these coefficient sequences determine half-line operators $\mathcal{C}_\pm(\omega)$ as in (2) with cyclic vector δ_0 and associated spectral measures $\mathrm{d}\mu_{\pm,\omega}$. The Schur functions are analytic functions from $\mathbb{D}$ to itself defined in terms of the half-line Carathéodory functions $K_{\pm,\omega}$ by

$$f_\pm(z,\omega) = \frac{1}{z}\frac{K_{\pm,\omega}(z)-1}{K_{\pm,\omega}(z)+1}, \quad K_{\pm,\omega}(z) = \int_{\mathbb{S}^1}\frac{\tau+z}{\tau-z}\,\mathrm{d}\mu_{\pm,\omega}(\tau).$$

For further discussion of the role of the Schur function in the spectral theory of CMV matrices, we refer the interested reader to [31, Section 1.3].

These half-line Schur functions are connected to the Lyapunov exponent via [32, Proposition 10.11.7], which gives us

$$\gamma(z) = \frac{1}{2}\mathbb{E}\left(\log\left(\frac{1-|zf_+|^2}{1-|f_+|^2}\right)\right). \qquad (15)$$

Additionally, by [32, Proposition 10.11.12], the Schur functions are connected to Green's function in the following way:

$$G_\omega(z) = \frac{f_+(z,\omega)f_-(z,\omega)}{1-zf_+(z,\omega)f_-(z,\omega)}. \qquad (16)$$

In view of (8), this implies that

$$F_\omega(z) = \frac{1+zf_+(z,\omega)f_-(z,\omega)}{1-zf_+(z,\omega)f_-(z,\omega)}. \qquad (17)$$

Moreover, by [32, Theorem 10.11.16], $\mathcal{E}_\omega$ is reflectionless on $\mathcal{Z}$ for μ a.e. $\omega \in \Omega$. That is, for μ a.e. ω, we have

$$f_+(z_0,\omega) = \overline{z_0 f_-(z_0,\omega)} \qquad (18)$$

for Lebesgue a.e. $z_0 \in \mathcal{Z}$. Consequently, for μ a.e. ω, the following calculation holds for Lebesgue a.e. $\theta \in \mathcal{Z}^\circ$:

$$\begin{aligned}\nu_\omega^{(\mathrm{ac})}(e^{i\theta}) &= \lim_{r\uparrow 1} \frac{1}{2\pi} \operatorname{Re} F_\omega(re^{i\theta}) \text{ by (7)}\\ &= \lim_{r\uparrow 1} \frac{1}{2\pi} \operatorname{Re}\left(\frac{1+re^{i\theta}f_+(re^{i\theta})f_-(re^{i\theta})}{1-re^{i\theta}f_+(re^{i\theta})f_-(re^{i\theta})}\right) \text{ by (17)}\\ &= \lim_{r\uparrow 1} \frac{1}{2\pi}\left(\frac{1+|f_+(re^{i\theta})|^2}{1-|f_+(re^{i\theta})|^2}\right) \text{ by (18).} \end{aligned} \tag{19}$$

Let P_R be the Poisson kernel for the unit disk; that is, for $R \in [0,1)$ and $\tau \in \mathbb{S}^1$, put

$$P_R(\tau) = \operatorname{Re}\left(\frac{1+R\tau}{1-R\tau}\right).$$

Throughout the rest of the argument, we will freely use some basic facts about Poisson integrals and boundary values of harmonic functions. The reader is invited to consult [27, Chapter 11] for further information. Let us also define

$$C_R(\phi) = \int_{\mathcal{Z}^\circ} P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi'$$

and

$$\widetilde{P}_R(\phi,\phi') = P_R(e^{i(\phi-\phi')})C_R(\phi)^{-1}.$$

By Jensen's inequality, we have

$$\begin{aligned}\int_0^{2\pi} \nu_\omega^{(\mathrm{ac})}(e^{i\phi'})P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi' &\geq \int_{\mathcal{Z}^\circ} \nu_\omega^{(\mathrm{ac})}(e^{i\phi'})P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi'\\ &= C_R(\phi)\int_{\mathcal{Z}^\circ} \nu_\omega^{(\mathrm{ac})}(e^{i\phi'})\widetilde{P}_R(\phi,\phi')\,\mathrm{d}\phi'\\ &\geq C_R(\phi)\left(\int_{\mathcal{Z}^\circ}\left(\nu_\omega^{(\mathrm{ac})}(e^{i\phi'})\right)^{-1}\widetilde{P}_R(\phi,\phi')\,\mathrm{d}\phi'\right)^{-1}\\ &\geq C_R(\phi)^2\left(\int_0^{2\pi}\left(\nu_\omega^{(\mathrm{ac})}(e^{i\phi'})\right)^{-1}P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi'\right)^{-1}\end{aligned}$$

for every ω and every ϕ. Then, by (19), we get

$$\int_0^{2\pi} \nu_\omega^{(\mathrm{ac})}(e^{i\phi'})P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi' \geq \frac{C_R(\phi)^2}{2\pi}\left(\frac{1+|f_+(Re^{i\phi})|^2}{1-|f_+(Re^{i\phi})|^2}\right)$$

for μ a.e. ω and Lebesgue a.e. $\phi \in \mathcal{Z}^\circ$. Consequently, for a.e. $\phi \in \mathcal{Z}^\circ$, we get

$$\int_0^{2\pi} \mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(e^{i\phi'})\right)P_R(e^{i(\phi-\phi')})\,\mathrm{d}\phi' \geq \frac{C_R(\phi)^2}{2\pi}\mathbb{E}\left(\frac{1+|f_+(Re^{i\phi})|^2}{1-|f_+(Re^{i\phi})|^2}\right).$$

Sending $R \uparrow 1$, we note that $|C_R(\phi)| \leq 1$ for all ϕ and $C_R(\phi) \to 1$ for Lebesgue a.e. $\phi \in \mathcal{Z}^\circ$ by standard properties of Poisson integrals. Consequently, after re-labeling R as r, we get

$$\mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(e^{i\phi})\right) \geq \frac{1}{2\pi}\limsup_{r\uparrow 1}\mathbb{E}\left(\frac{1+|f_+(re^{i\phi})|^2}{1-|f_+(re^{i\phi})|^2}\right) \tag{20}$$

for Lebesgue a.e. $\phi \in \mathcal{Z}^\circ$.

At last, we put everything together. By (14) and (15), we have

$$k^{(\mathrm{ac})}(e^{i\phi}) = \frac{1}{2\pi} + \lim_{r\uparrow 1} \frac{1}{2\pi(1-r)} \mathbb{E}\left(\log\left(1 + \frac{|f_+(re^{i\phi})|^2(1-r^2)}{1-|f_+(re^{i\phi})|^2}\right)\right)$$

for a.e. $\phi \in \mathcal{Z}^\circ$. Since $\log(1+x) \le x$ for $x \ge 0$, we get

$$\begin{aligned} k^{(\mathrm{ac})}(e^{i\phi}) &\le \frac{1}{2\pi} + \frac{1}{2\pi} \limsup_{r\uparrow 1} \mathbb{E}\left(\frac{|f_+(re^{i\phi})|^2(1+r)}{1-|f_+(re^{i\phi})|^2}\right) \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \limsup_{r\uparrow 1} \mathbb{E}\left(\frac{|f_+(re^{i\phi})|^2}{1-|f_+(re^{i\phi})|^2}\right) \\ &= \frac{1}{2\pi} \limsup_{r\uparrow 1} \mathbb{E}\left(\frac{1+|f_+(re^{i\phi})|^2}{1-|f_+(re^{i\phi})|^2}\right) \end{aligned}$$

for a.e. $\phi \in \mathcal{Z}^\circ$. Combining this with (20) we get

$$k^{(\mathrm{ac})}(e^{i\phi}) \le \frac{1}{2\pi} \limsup_{r\uparrow 1} \mathbb{E}\left(\frac{1+|f_+(re^{i\phi})|^2}{1-|f_+(re^{i\phi})|^2}\right) \le \mathbb{E}\Big(\nu_\omega^{(\mathrm{ac})}(e^{i\phi})\Big)$$

for a.e. $\phi \in \mathcal{Z}^\circ$, which completes the proof. □

Proof of Corollary 1. Suppose first that the spectrum of $\mathcal{E}_\omega$ is purely absolutely continuous for μ-almost every $\omega \in \Omega$. Then, as it is the average of absolutely continuous measures, it follows that $\mathrm{d}k$ is absolutely continuous (cf. [32, (10.5.50)–(10.5.51)]). Moreover, since the almost-sure absolutely continuous spectrum is given by the essential closure of the set upon which the Lyapunov exponent vanishes (e.g. by [32, Theorem 10.11.1]), it follows that the Lyapunov exponent vanishes Lebesgue-a.e. (hence $\mathrm{d}k$-a.e.) on the spectrum.

Conversely, if $\mathrm{d}k$ is purely a.c. and γ vanishes $\mathrm{d}k$-a.e. on $\mathbb{S}^1$, then

$$\begin{aligned} 1 &= \int_\Sigma \mathrm{d}k(z) \\ &= \int_{\mathcal{Z}} \mathrm{d}k(z) \\ &= \int_{\mathcal{Z}} k^{(\mathrm{ac})}(z)\, \mathrm{d}\lambda(z) \\ &= \int_{\mathcal{Z}} \mathbb{E}(\nu_\omega^{(\mathrm{ac})}(z))\, \mathrm{d}\lambda(z) \\ &= \mathbb{E}\left(\int_{\mathcal{Z}} \nu_\omega^{(\mathrm{ac})}(z)\, \mathrm{d}\lambda(z)\right) \\ &\le \mathbb{E}\left(\int_\Sigma \nu_\omega^{(\mathrm{ac})}(z)\, \mathrm{d}\lambda(z)\right) \\ &\le 1, \end{aligned}$$

by Theorem 1 and Fubini's Theorem. In the calculation above, $\mathrm{d}\lambda$ denotes normalized 1D Lebesgue measure on $\mathbb{S}^1$. Since the chain of inequalities begins and ends with one, all inequalities are equalities, so the absolutely continuous part of ν_ω has full weight for μ-a.e. ω. The transformation S preserves μ, so one also has $\nu_{S\omega}^{(\mathrm{ac})}(\Sigma) = 1$ for μ-a.e. ω. Since $\nu_{S\omega}$ is the spectral measure of $\mathcal{E}_\omega$ corresponding to the vector δ_1 and the pair $\{\delta_0, \delta_1\}$ is cyclic for $\mathcal{E}_\omega$, taking the intersection of those two full-measure sets yields a full-measure set for which $\mathcal{E}_\omega$ has purely a.c. spectrum. $\square$

Acknowledgments

We are thankful to David Damanik for helpful conversations. The first author was supported in part by an AMS–Simons Travel Grant 2016–2018.

References

[1] W. O. Amrein and V. Georgescu, *On the characterization of bound states and scattering states in quantum mechanics*, Helv. Phys. Acta **46** (1973/74), 635–658. MR0363267

[2] Joachim Asch, Olivier Bourget, and Alain Joye, *Spectral stability of unitary network models*, Rev. Math. Phys. **27** (2015), no. 7, 1530004, 22, DOI 10.1142/S0129055X15300046. MR3396615

[3] J. Bourgain, F. A. Grünbaum, L. Velázquez, and J. Wilkening, *Quantum recurrence of a subspace and operator-valued Schur functions*, Comm. Math. Phys. **329** (2014), no. 3, 1031–1067, DOI 10.1007/s00220-014-1929-9. MR3212879

[4] María-José Cantero, F. Alberto Grünbaum, Leandro Moral, and Luis Velázquez, *Matrix-valued Szegő polynomials and quantum random walks*, Comm. Pure Appl. Math. **63** (2010), no. 4, 464–507. MR2604869

[5] M. J. Cantero, F. A. Grünbaum, L. Moral, and L. Velázquez, *One-dimensional quantum walks with one defect*, Rev. Math. Phys. **24** (2012), no. 2, 1250002, 52, DOI 10.1142/S0129055X1250002X. MR2902157

[6] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56, DOI 10.1016/S0024-3795(02)00457-3. MR1955452

[7] C. Cedzich, F. A. Grünbaum, L. Velázquez, A. H. Werner, and R. F. Werner, *A quantum dynamical approach to matrix Khrushchev's formulas*, Comm. Pure Appl. Math. **69** (2016), no. 5, 909–957, DOI 10.1002/cpa.21579. MR3481284

[8] David Damanik, *Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: a survey of Kotani theory and its applications*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, Proc. Sympos. Pure Math., vol. 76, Amer. Math. Soc., Providence, RI, 2007, pp. 539–563, DOI 10.1090/pspum/076.2/2307747. MR2307747

[9] David Damanik, Jon Erickson, Jake Fillman, Gerhardt Hinkle, and Alan Vu, *Quantum intermittency for sparse CMV matrices with an application to quantum walks on the half-line*, J. Approx. Theory **208** (2016), 59–84, DOI 10.1016/j.jat.2016.04.001. MR3506927

[10] David Damanik, Jake Fillman, Milivoje Lukic, and William Yessen, *Uniform hyperbolicity for Szegő cocycles and applications to random CMV matrices and the Ising model*, Int. Math. Res. Not. IMRN **16** (2015), 7110–7129, DOI 10.1093/imrn/rnu158. MR3428956

[11] David Damanik, Jake Fillman, and Darren C. Ong, *Spreading estimates for quantum walks on the integer lattice via power-law bounds on transfer matrices* (English, with English and French summaries), J. Math. Pures Appl. (9) **105** (2016), no. 3, 293–341, DOI 10.1016/j.matpur.2015.11.002. MR3465806

[12] David Damanik, Jake Fillman, and Robert Vance, *Dynamics of unitary operators*, J. Fractal Geom. **1** (2014), no. 4, 391–425, DOI 10.4171/JFG/12. MR3299818

[13] David Damanik, Paul Munger, and William N. Yessen, *Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, II. Applications*, J. Stat. Phys. **153** (2013), no. 2, 339–362, DOI 10.1007/s10955-013-0830-9. MR3101200

[14] T. Endo, N. Konno, H. Obuse, and E. Segawa, Sensitivity of quantum walks to boundary of two-dimensional lattices: approaches from the CGMV method and topological phases, to appear in J. Phys. A.: `http://iopscience.iop.org/article/10.1088/1751-8121/aa8c5e/meta`

[15] Volker Enss, *Asymptotic completeness for quantum mechanical potential scattering. I. Short range potentials*, Comm. Math. Phys. **61** (1978), no. 3, 285–291. MR0523013

[16] Jake Fillman and Darren C. Ong, *Purely singular continuous spectrum for limit-periodic CMV operators with applications to quantum walks*, J. Funct. Anal. **272** (2017), no. 12, 5107–5143, DOI 10.1016/j.jfa.2017.01.021. MR3639523

[17] Jake Fillman, Darren C. Ong, and Zhenghe Zhang, *Spectral characteristics of the unitary critical almost-Mathieu operator*, Comm. Math. Phys. **351** (2017), no. 2, 525–561, DOI 10.1007/s00220-016-2775-8. MR3613513

[18] F. A. Grünbaum, L. Velázquez, A. H. Werner, and R. F. Werner, *Recurrence for discrete time unitary evolutions*, Comm. Math. Phys. **320** (2013), no. 2, 543–569, DOI 10.1007/s00220-012-1645-2. MR3053772

[19] Alain Joye and Marco Merkli, *Dynamical localization of quantum walks in random environments*, J. Stat. Phys. **140** (2010), no. 6, 1025–1053, DOI 10.1007/s10955-010-0047-0. MR2684498

[20] Norio Konno, *Quantum walks and elliptic integrals*, Math. Structures Comput. Sci. **20** (2010), no. 6, 1091–1098, DOI 10.1017/S0960129510000393. MR2735829

[21] Norio Konno and Etsuo Segawa, *Localization of discrete-time quantum walks on a half line via the CGMV method*, Quantum Inf. Comput. **11** (2011), no. 5-6, 485–495. MR2847871

[22] Norio Konno and Etsuo Segawa, *One-dimensional quantum walks via generating function and the CGMV method*, Quantum Inf. Comput. **14** (2014), no. 13-14, 1165–1186. MR3242430

[23] Shinichi Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, Stochastic analysis (Katata/Kyoto, 1982), North-Holland Math. Library, vol. 32, North-Holland, Amsterdam, 1984, pp. 225–247, DOI 10.1016/S0924-6509(08)70395-7. MR780760

[24] Shinichi Kotani, *Support theorems for random Schrödinger operators*, Comm. Math. Phys. **97** (1985), no. 3, 443–452. MR778625

[25] Shinichi Kotani, *One-dimensional random Schrödinger operators and Herglotz functions*, Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), Academic Press, Boston, MA, 1987, pp. 219–250. MR933826

[26] S. Kotani, *Generalized Floquet theory for stationary Schrödinger operators in one dimension*, Chaos Solitons Fractals **8** (1997), no. 11, 1817–1854, DOI 10.1016/S0960-0779(97)00042-8. MR1477262

[27] Walter Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR924157

[28] D. Ruelle, *A remark on bound states in potential-scattering theory* (English, with Italian summary), Nuovo Cimento A (10) **61** (1969), 655–662. MR0246603

[29] Barry Simon, *Kotani theory for one-dimensional stochastic Jacobi matrices*, Comm. Math. Phys. **89** (1983), no. 2, 227–234. MR709464

[30] Barry Simon, *Analogs of the m-function in the theory of orthogonal polynomials on the unit circle*, J. Comput. Appl. Math. **171** (2004), no. 1-2, 411–424, DOI 10.1016/j.cam.2004.01.022. MR2077215

[31] Barry Simon, *Orthogonal polynomials on the unit circle. Part 1*, Classical theory, American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2005. MR2105088

[32] Barry Simon, *Orthogonal polynomials on the unit circle. Part 2*, Spectral theory, American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2005. MR2105089

[33] Gerald Teschl, *Mathematical methods in quantum mechanics, with applications to Schrödinger operators*, 2nd ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. MR3243083

Mathematics (MC0123), Virginia Tech, 225 Stanger Street, Blacksburg, Virginia 24061

E-mail address: `fillman@vt.edu`

Department of Mathematics, Xiamen University Malaysia, Jalan Sunsuria, Bandar Sunsuria, 43900 Sepang, Selangor Darul Ehsan, Malaysia

E-mail address: `darrenong@xmu.edu.my`