

TOPOLOGICAL STABILITY AND PSEUDO-ORBIT TRACING PROPERTY OF GROUP ACTIONS

NHAN-PHU CHUNG AND KEONHEE LEE

(Communicated by Yingfei Yi)

ABSTRACT. In this paper we extend the concept of topological stability from homeomorphisms to group actions on compact metric spaces and prove that if an action of a finitely generated group is expansive and has the pseudo-orbit tracing property, then it is topologically stable. This represents a group action version of P. Walters's stability theorem [Lecture Notes in Math., vol. 668, Springer, 1978, pp. 231–244]. Moreover we give a class of group actions with topological stability or pseudo-orbit tracing property. In particular, we establish a characterization of subshifts of finite type over finitely generated groups in terms of the pseudo-orbit tracing property.

1. INTRODUCTION

In 1970, Walters [15] introduced the notion of *topological stability*, a kind of stability for homeomorphisms in which continuous perturbations are allowed. In that paper he proved that Anosov diffeomorphisms on compact smooth manifolds are not only structurally stable but also topologically stable. Several results dealing with this new kind of stability were then appearing. For instance, Nitecki [10] proved that Axiom *A* diffeomorphisms with the strong transversality condition on compact smooth manifolds are topologically stable. Afterwards, Walters [16] proved that every expansive homeomorphism with pseudo-orbit tracing property on a compact metric space is topologically stable. Very recently, Lee and Morales [9] introduced the notions of topological stability and pseudo-orbit tracing property for Borel measures on compact metric spaces and showed that any expansive measure with pseudo-orbit tracing property is topologically stable.

In this paper we will obtain a group action version of this result. Indeed, we introduce the notion of topological stability for an action of a finitely generated group on a compact metric space and prove that if a group action is expansive and has the pseudo-orbit tracing property, then it is topologically stable. This represents a further contribution to the study of the pseudo-orbit tracing property (or shadowing) of a group action developed elsewhere in the recent literature [11–14]. Furthermore, we establish a characterization of subshifts of finite type over finitely generated groups via the pseudo-orbit tracing property. This characterization extends the main results of [16] and [11] when the acting group is \mathbb{Z} and

Received by the editors September 8, 2016 and, in revised form, December 20, 2016.

2010 *Mathematics Subject Classification.* Primary 37C85, 37C50, 37C75.

Key words and phrases. Expansiveness, group action, pseudo-orbit tracing property, subshift of finite type, topological stability.

The second author is the corresponding author.

\mathbb{Z}^d , respectively. Finally, we show that every equicontinuous action of an infinite, finitely generated group on the Cantor space has the pseudo-orbit tracing property.

We round out the introduction with some notation that we will use in the paper. Let G be a finitely generated discrete group and X be a compact metric space with a metric d . Put $\text{Homeo}(X)$ as the space of all homeomorphisms of X . We denote by $\text{Act}(G, X)$ the set of all continuous actions T of G on X . Let $\text{Homeo}(X)^G = \prod_G \text{Homeo}(X)$ be the set of homomorphisms from G to $\text{Homeo}(X)$ with the product topology. Then $\text{Act}(G, X)$ can be considered as a subset of $\text{Homeo}(X)^G$. Let A be a finitely generating set of G . We define a metric d_A on $\text{Act}(G, X)$ by

$$d_A(T, S) := \sup_{\substack{x \in X \\ a \in A}} d(T_a x, S_a x),$$

for $T, S \in \text{Act}(G, X)$.

2. TOPOLOGICAL SPABILITY AND PSEUDO-ORBIT TRACING PROPERTY

First of all, we introduce the notion of topological stability of a finitely generated group action on a compact metric space.

Definition 2.1. Let A be a finitely generating set of G , and let $T \in \text{Act}(G, X)$. We say that T is *A-topologically stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$, then there exists a continuous map $f : X \rightarrow X$ with $T_g f = f S_g$, for every $g \in G$ and $d(f, Id_X) := \sup_{x \in X} d(f(x), x) \leq \varepsilon$.

It is clear that the definition of *A-topological stability* of T does not depend on the choice of a compatible metric d on X . Note that the definition of topological stability of a homeomorphism introduced in [15] coincides with our definition when $G = \mathbb{Z}$ and $A = \{1\}$. Furthermore, we can see that topological stability of T does not depend on the choice of a symmetric finitely generating set A of G . Recall that A is *symmetric* if for any $a \in A$, $a^{-1} \in A$.

Lemma 2.2. *Let A and B be symmetric finitely generating sets of G . For any $T \in \text{Act}(G, X)$, T is *A-topologically stable* if and only if it is *B-topologically stable*.*

Proof. Suppose T is *A-topologically stable*. Then for any $\varepsilon > 0$, there exists $\delta' > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta'$, then there exists a continuous map $f : X \rightarrow X$ with $T_g f = f S_g$, for every $g \in G$ and $d(f, Id_X) \leq \varepsilon$. It suffices to show that there exists $\delta > 0$ such that for any $S \in \text{Act}(G, X)$, if $d_B(T, S) < \delta$, then $d_A(T, S) < \delta'$. Put $m := \max_{a \in A} \ell_B(a)$, where ℓ_B is the word length metric on G induced by B . Choose $\delta_1 > 0$ such that $m\delta_1 < \delta'$. Since X is compact, A and B are finite and the action T is continuous, there exists $\delta > 0$ such that $d(T_h x, T_h y) < \delta_1$ for $x, y \in X$ with $d(x, y) < \delta$ and for $h \in G$ with $\ell_B(h) \leq m$. For any $a \in A$, we write a as $b_1 \cdots b_{\ell(a)}$, where $\ell(a) = \ell_B(a) \leq m$, $b_i \in B$, $i = 1, \dots, \ell(a)$. Then for any $S \in \text{Act}(G, X)$ with

$d_B(T, S) < \delta$, we have

$$\begin{aligned} d(T_a x, S_a x) &= d(T_{b_1 \dots b_{\ell(a)}} x, S_{b_1 \dots b_{\ell(a)}} x) \\ &\leq d(T_{b_1 \dots b_{\ell(a)-1}} T_{b_{\ell(a)}} x, T_{b_1 b_2 \dots b_{\ell(a)-1}} S_{b_{\ell(a)}} x) \\ &\quad + d(T_{b_1 b_2 \dots b_{\ell(a)-2}} T_{b_{\ell(a)-1}} S_{b_{\ell(a)}} x, T_{b_1 b_2 \dots b_{\ell(a)-2}} S_{b_{\ell(a)-1}} S_{b_{\ell(a)}} x) \\ &\quad + \dots + d(T_{b_1} T_{b_2} S_{b_3 \dots b_{\ell(a)-1} b_{\ell(a)}} x, T_{b_1} S_{b_2} S_{b_3 \dots b_{\ell(a)-1} b_{\ell(a)}} x) \\ &\quad + d(T_{b_1} S_{b_2 \dots b_{\ell(a)-1} b_{\ell(a)}} x, S_{b_1 \dots b_{\ell(a)}} x) \\ &< m\delta_1 < \delta'. \end{aligned}$$

This means that $d_A(T, S) < \delta'$, and so completes the proof. □

Definition 2.3. An action $T \in \text{Act}(G, X)$ is said to be *topologically stable* if it is A -topologically stable for a symmetric finitely generating set A of G .

Remark 2.4. Let A be a finitely generating set of G . We define a metric \tilde{d}_A on $\text{Act}(G, X)$ by

$$\tilde{d}_A(T, S) := \sup_{\substack{x \in X \\ a \in A}} \{d(T_a x, S_a x) + d(T_a^{-1} x, S_a^{-1} x)\},$$

for $T, S \in \text{Act}(G, X)$. Clearly d_A and \tilde{d}_A are equivalent. Furthermore, the topology on $\text{Act}(G, X)$ induced by \tilde{d}_A coincides with the product topology on $\text{Act}(G, X)$ inherited from $\text{Homeo}(X)^G$. Hence the space $\text{Act}(G, X)$ is a separable complete metrizable topological space, and so a Polish space.

If T and S are two continuous actions of G on X with $d_A(T, S) < \delta$, then the S -orbit $\{S_g x\}$ of $x \in X$ is nearly a T -orbit in the sense that $d(T_a S_g x, S_{ag} x) < \delta$ for all $a \in A$ and $g \in G$. This observation motivates the following definition.

Definition 2.5. Let A be a finitely generating set of G and $\delta > 0$. A δ *pseudo-orbit* of $T \in \text{Act}(G, X)$ with respect to A is a sequence $\{x_g\}_{g \in G}$ in X such that $d(T_a x_g, x_{ag}) < \delta$ for all $a \in A, g \in G$.

Definition 2.6. Let A be a finitely generating set of G . An action $T \in \text{Act}(G, X)$ is said to have the *pseudo-orbit tracing property (abbreviated POTP)* with respect to A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any δ pseudo-orbit $\{x_g\}_{g \in G}$ for T with respect to A is ε -traced by some point x of X , that is, $d(T_g x, x_g) < \varepsilon$ for all $g \in G$.

Note that the pseudo-orbit tracing property of T does not depend on the choice of a compatible metric d of X . Osipov and Tikhomirov [12] introduced the notion of POTP, which they called shadowing, for actions of finitely generated groups by using symmetric finitely generating sets of the acting groups. They showed that the definition of POTP does not depend on the choice of symmetric finitely generating sets. Indeed, it is not hard to check that it actually does not depend on the choice of general finitely generating sets; see for example [13, Lemma 2.2].

Definition 2.7. We say that an action $T \in \text{Act}(G, X)$ has POTP if T has POTP with respect to A for a finitely generating set A of G .

Note that the definition of POTP of a homeomorphism coincides with our definition when $G = \mathbb{Z}$ and $A = \{1\}$. We recall that an action T of a group G on a compact metric space X is *expansive* if there exists an open subset U in $X \times X$ such that $\Delta_X = \bigcap_{g \in G} g^{-1}U$, where $\Delta_X := \{(x, x) : x \in X\}$ and the action of G on

$X \times X$ is defined by $g(x, y) := (gx, gy)$ for every $x, y \in X, g \in G$. Note that T is expansive if and only if there exists a constant $c > 0$ called an *expansive constant* of T such that for every $x \neq y$, one has $\sup_{g \in G} d(gx, gy) > c$. Now we prove that if an action of a finitely generated group on a compact metric space is expansive and has POTP, then it is topologically stable. This extends the main results of [16, Theorems 4 and 5] to group actions.

Theorem 2.8. *If an action $T \in \text{Act}(G, X)$ is expansive and has POTP, then it is topologically stable. Moreover, for a finitely generating set A of G , for $\varepsilon > 0$ with $\varepsilon < \eta/3$, where η is an expansive constant of T , there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$, then there is a unique map $f : X \rightarrow X$ with $T_g f = f S_g$ for every $g \in G$ and $d(f, Id_X) \leq \varepsilon$. Furthermore, if S is also expansive with an expansive constant $\eta_S \geq 2\varepsilon$, then the conjugating map f is injective.*

To prove the above theorem, we need the following two lemmas.

Lemma 2.9. *Let $T \in \text{Act}(G, X)$ be an expansive action with POTP with respect to a finitely generating set A of G . Let $\varepsilon < \eta/2$ and δ correspond to ε as in Definition 2.6, where η is an expansive constant of T . Then every δ pseudo-orbit of T is ε -traced by a unique point in X .*

Proof. Let $\{x_g\}_{g \in G}$ be a δ pseudo-orbit of T , and let x, y be two points that ε -trace $\{x_g\}_{g \in G}$. Then one has $d(T_g x, T_g y) \leq d(T_g x, x_g) + d(x_g, T_g y) < 2\varepsilon < \eta$ for every $g \in G$. By the expansiveness of T , we get $x = y$. □

Lemma 2.10. *Let $T \in \text{Act}(G, X)$ be an expansive action with an expansive constant η . Then, for any $\varepsilon > 0$, there exists a non-empty finite subset F of G such that whenever $\sup_{g \in F} d(T_g x, T_g y) \leq \eta$, we have $d(x, y) < \varepsilon$.*

Proof. Assume that there exists $\varepsilon > 0$ such that for any non-empty finite subset F of G , there exist $x_F, y_F \in X$ such that

$$\sup_{g \in F} d(T_g x, T_g y) \leq \eta \text{ and } d(x_F, y_F) \geq \varepsilon.$$

Choose a sequence of non-empty finite subsets F_n of G satisfying

$$F_1 \subset F_2 \subset \dots \text{ and } G = \bigcup_{n \in \mathbb{N}} F_n.$$

Then for every $n \in \mathbb{N}$, there exist $x_n, y_n \in F_n$ such that

$$\sup_{g \in F_n} d(T_g x_n, T_g y_n) \leq \eta \text{ and } d(x_n, y_n) \geq \varepsilon.$$

After taking a subsequence, we can assume that $x_n \rightarrow x$ and $y_n \rightarrow y$. Then we have $d(T_g x, T_g y) \leq \eta$ for all $g \in G$ and $d(x, y) \geq \varepsilon$, which contradicts the expansiveness of T . □

Proof of Theorem 2.8. Let η be an expansive constant of T and let $\varepsilon < \eta/3$. Let A be a finitely generating set of G . Choose δ corresponding to ε as in Definition 2.6. Let S be a continuous action of G on X with $d_A(T, S) < \delta$. For any $x \in X$, we see that the S -orbit $\{S_g x\}_{g \in G}$ of x is a δ pseudo-orbit for T . By Lemma 2.9, there is a unique point denoted by $f(x)$ whose T -orbit ε -traces $\{S_g x\}_{g \in G}$. Then we have the map $f : X \rightarrow X$ satisfying

(*)
$$d(T_g f(x), S_g x) < \varepsilon \text{ for all } g \in G, x \in X.$$

In particular, we have $d(f(x), x) < \varepsilon$ for every $x \in X$, and hence $d(f, Id_X) \leq \varepsilon$.

Now we will prove that $T_g f(x) = f S_g(x)$ for every $x \in X, g \in G$. In fact, we have

$$d(T_h f(S_g x), S_{hg} x) = d(T_h f(S_g x), S_h S_g x) < \varepsilon$$

for $x \in X$ and $g, h \in G$. On the other hand, applying (*) again, we obtain

$$d(T_h T_g f(x), S_{hg} x) = d(T_{hg} f(x), S_{hg} x) < \varepsilon.$$

Then we get $T_g f(x) = f S_g(x)$ by Lemma 2.9.

Next we will show that f is continuous. Let $\varepsilon_1 > 0$. By Lemma 2.10, there exists a non-empty finite subset F of G such that whenever $\sup_{g \in F} d(T_g x, T_g y) \leq \eta$ one has $d(x, y) < \varepsilon_1$. Choose $\delta_1 > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta_1$, one has $d(S_g x, S_g y) < \eta/3$ for every $g \in F$. Then, for any $x, y \in X$ with $d(x, y) < \delta_1$ and $g \in F$, we get

$$\begin{aligned} d(T_g f(x), T_g f(y)) &= d(f S_g(x), f S_g(y)) \\ &\leq d(f S_g(x), S_g(x)) + d(S_g(x), S_g(y)) + d(S_g(y), f S_g(y)) \\ &< \varepsilon + \eta/3 + \varepsilon < \eta. \end{aligned}$$

Thus $d(f(x), f(y)) < \varepsilon_1$ for $x, y \in X$ with $d(x, y) < \delta_1$. This means that f is continuous.

Assume that there is another map f_1 such that $T_g f_1 = f_1 S_g$ for all $g \in G$ and $d(f_1, Id_X) \leq \varepsilon$. Then, for any $g \in G$, we have

$$\begin{aligned} d(T_g f(x), T_g f_1(x)) &\leq d(T_g f(x), S_g(x)) + d(S_g(x), T_g f_1(x)) \\ &= d(f S_g(x), S_g(x)) + d(S_g(x), f_1 S_g(x)) < 2\varepsilon < \eta. \end{aligned}$$

By the expansiveness of T , we get $f(x) = f_1(x)$. Finally, we will prove the last assertion. Let $f(x) = f(y)$. Then for any $g \in G$, we have

$$\begin{aligned} d(S_g x, S_g y) &\leq d(S_g x, f(S_g x)) + d(f(S_g x), f(S_g y)) + d(f(S_g y), S_g y) \\ &= d(S_g x, f(S_g x)) + d(T_g f(x), T_g f(y)) + d(f(S_g y), S_g y) < 2\varepsilon \leq \eta_S, \end{aligned}$$

and so $x = y$ by the expansiveness of S . □

Next we will provide a class of topologically stable actions. First, we recall the definition of nilpotent group. Let G be a countable group. We denote by e the unit of G . The *lower central series* of G is the sequence $\{G_i\}_{i \geq 0}$ of subgroups of G defined by $G_0 = G$ and $G_{i+1} = [G_i, G]$, where $[G_i, G]$ is the subgroup of G generated by all commutators $[a, b] := aba^{-1}b^{-1}$, $a \in G_i, b \in G$. The group G is said to be *nilpotent* if there exists $n \geq 0$ such that $G_n = \{e\}$. The smallest such n is called the *nilpotent degree* of G .

Theorem 2.11. *Let G be a finitely generated virtually nilpotent group; i.e. there exists a nilpotent subgroup H of G with finite index. Let T be a continuous action of G on X . If there exists an element $g \in G$ such that T_g is expansive and has POTP, then T is topologically stable.*

The following two lemmas were proved implicitly in ([12, Lemmas 1 and 2]). For convenience, we provide those implicit proofs here.

Lemma 2.12. *Let G be a finitely generated group and H be a finitely generated normal subgroup of G . Let T be a continuous action of G on X . If the restriction action T_H of T to H is expansive and has POTP, then T has POTP.*

Proof. Let A be a finitely generating set of H . We can add more elements to A to get a symmetric finitely generating set B of G . Let c be an expansive constant of T_H . Since X is compact and B is finite, there exists $0 < \eta < c/3$ such that $d(T_b x, T_b y) < c/3$ for every $b \in B$ and every $x, y \in X$ with $d(x, y) < \eta$. Let $\varepsilon > 0$ be a constant with $\varepsilon < \eta$. We can choose $0 < \delta < \varepsilon$ such that every δ pseudo-orbit for T_H with respect to A is ε -traced by some point of X . Let $\{x_g\}_{g \in G}$ be a δ pseudo-orbit of T with respect to B . For every $g \in G$, the sequence $\{x_{hg}\}_{h \in H}$ is a δ pseudo-orbit of T_H with respect to A . Since T_H is expansive, by Lemma 2.9, there exists a unique point $y_g \in X$ such that

$$(1) \quad d(x_{hg}, T_h y_g) < \varepsilon, \text{ for every } h \in H.$$

Now we will prove that $y_g = T_g y_e$ for $g \in G$. Fix $g \in G$ and $b \in B$. For any $h \in H$, there exists $h' \in H$ such that $hb = bh'$. Then we have

$$d(x_{bh'g}, T_h y_{bg}) = d(x_{hb g}, T_h y_{bg}) < \varepsilon \text{ and } d(T_b x_{h'g}, T_b T_{h'} y_g) < c/3.$$

Hence we get

$$d(T_h y_{bg}, T_h T_b y_g) \leq d(T_h y_{bg}, x_{bh'g}) + d(x_{bh'g}, T_b x_{h'g}) + d(T_b x_{h'g}, T_{bh'} y_g) < c.$$

Since T_H is expansive, we have $T_b y_g = y_{bg}$ for every $b \in B$. As B is a symmetric generating set of G , we get $T_g y_e = y_g$ for every $g \in G$. On the other hand, by applying $h = e_G$ for (1), we have $d(x_g, y_g) < \varepsilon$ for every $g \in G$ and hence $d(x_g, T_g y_e) < \varepsilon$ for $g \in G$. \square

Lemma 2.13. *Let G be a finitely generated nilpotent group and T be a continuous action of G on X . If there exists $g \in G$ such that T_g is expansive and has POTP, then T has POTP.*

Proof. We prove by induction on the nilpotent degree n of G . If $n = 1$, then the group G is abelian and hence $H = \langle g \rangle$ is a normal subgroup of G . Thus, T has POTP by Lemma 2.12. Let $n > 1$ and assume that the statement of the lemma is true for all nilpotent groups with nilpotent degree less than or equal to $n - 1$. Put $G_1 := [G, G]$ and $K := \langle G_1, g \rangle$. Then K has the nilpotent degree at most $n - 1$ [12, Proposition 2]. It is known that G_1 is finitely generated [4, Lemma 6.8.4] and hence K is finitely generated. Thus, from the induction assumption, we know T_K has POTP. Since T_g is expansive and $g \in K$, we have T_K is expansive. As K is a normal subgroup of G [12, Proposition 2], we complete the proof by Lemma 2.12. \square

Proof of Theorem 2.11. Let H be a nilpotent normal subgroup of G with finite index. Then H is finitely generated [4, Proposition 6.6.2]. Since H has finite index in G , there exists $n \in \mathbb{N}$ such that $g^n \in H$. Because T_g is expansive and has POTP, $T_{g^n} = T_g^n$ is also expansive and has POTP. Thus, from Lemma 2.13, the action T_H has POTP. Since $g^n \in H$ and T_{g^n} is expansive, one has T_H is expansive. Applying Theorem 2.8 and Lemma 2.12, we complete the proof. \square

In the following example given in [7, Example 1.1], we see that the integral Heisenberg group H induces an action on the torus \mathbb{T}^{3n} which is topologically stable.

Example 2.14. Let H be the integral Heisenberg group, i.e. $H = \langle a, b, c | ac = ca, bc = cb, ab = bac \rangle$. Then H is a nilpotent group. For every $n \in \mathbb{N}$, we let $x, y \in SL(n, \mathbb{Z})$ be such that $xy = yx$. Put

$$a = \begin{pmatrix} x & I_n & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \quad b = \begin{pmatrix} y & 0 & 0 \\ 0 & y & I_n \\ 0 & 0 & y \end{pmatrix}, \quad c = \begin{pmatrix} I_n & 0 & x^{-1}y^{-1} \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

Then a, b, c satisfy the relations in H . Let T be the natural action of $H \leq SL(3n, \mathbb{Z})$ on \mathbb{T}^{3n} . If x has no eigenvalues of modulus 1, then a also has no eigenvalues of modulus 1. Then we see that T_a is expansive and has POTP. Applying Theorems 2.8 and 2.11, we know that the action T is topologically stable. Similarly the case y has no eigenvalues of modulus 1.

3. SHIFTS OF FINITE TYPE AND POTP

Let A be a finitely generating set of G . For any $k \in \mathbb{N}$, we put $B(k) := \{g \in G : \ell_A(g) \leq k\}$. Let S be a non-empty finite set. We denote by S^G the product space $\prod_G S$ endowed with the product topology. We consider the action of G on S^G by right shifts, i.e. $(gx)_h = x_{hg}$ for every $g, h \in G$ and $x \in S^G$. For any non-empty finite subset F of G and $x \in S^G$, we denote by x_F the restriction of x to F and by $\pi_F : S^G \rightarrow S^F$ the natural projection map. An element $f \in S^F$ is called a *pattern*. If $F = \{g\}$ for some $g \in G$, we write x_F simply as x_g . A closed subset X of S^G is called a *subshift* if it is G -invariant, i.e. $gx \in X$ for every $g \in G, x \in X$. A pattern $f \in S^F$ is said to be *allowed* for the shift X if there exists x in X such that $f = x_F$. For every $k \in \mathbb{N}$, a pattern $f \in S^{B(k)}$ is called a $B(k)$ -*block*, and we denote by $B_k(S)$ and $B_k(X)$ the set of all $B(k)$ -blocks and the set of all $B(k)$ -blocks allowed in X , respectively. Then, we put $B(S) = \bigcup_{k \in \mathbb{N}} B_k(S)$ and $B(X) = \bigcup_{k \in \mathbb{N}} B_k(X)$. Let W be a set of patterns. We define

$$X_W := \{x \in S^G : (gx)_F = x_{Fg} \notin W \text{ for all } g \in G \text{ and all finite subsets } F(\neq \emptyset) \text{ of } G\}.$$

A subshift X of S^G is said to be of *finite type* if there are a non-empty finite subset F of G and a subset P of S^F such that

$$X = \{x \in S^G : \pi_F(gx) \in P, \forall g \in G\} =: S(F, P).$$

The subsets F and P are called a *defining window* and a *set of allowed words* for X , respectively. It is clear that if X is a subshift of finite type defined by a defining window F and a set of allowed words P , then for any $F' \supset F$, it is also a subshift of finite type defined by F' and $P' = \{x \in S^{F'} : \pi_F(x) \in P\}$. We refer readers to [6, Chapter 2, Section 3] for details of subshifts of finite type over a countable group. The following lemma is clear from the definitions of a subshift of finite type and X_W .

Lemma 3.1. *Let X be a subset of S^G . Then X is a subshift of finite type if and only if there exist a non-empty finite subset F of G and $W \subset S^F$ such that $X = X_W$.*

Now we establish a characterization of subshifts of finite type over finitely generated groups via POTP. This characterization extends the main results of [16] and [11] when the acting group is \mathbb{Z} and \mathbb{Z}^d , respectively.

Theorem 3.2. *Let G be a finitely generated group and S be a non-empty finite set. Let T be the right action of G on S^G and X be a subshift of S^G . Then X is of finite type if and only if it has POTP; i.e. T has POTP on X .*

Proof. Let A be a symmetric finitely generating set of G . We define a metric d on X by

$$d(x, y) = 2^{-k}, \quad k := \sup\{j \in \mathbb{N} : x_g = y_g, \text{ for all } g \in B(j)\}$$

for $x, y \in X$, where $B(j) = \{g \in G : \ell_A(g) \leq j\}$. Then d is a compatible metric on X . Assume that X is a subshift of finite type. Then we can choose $M \in \mathbb{N}$ such that $F = B(M)$ is a defining window for X . Let W be the set of allowed words for X with respect to the window F . Let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$ and $m > M$. Put $\delta = 2^{-(m+1)}$. Then we will prove that every δ pseudo-orbit $\{x^{(g)}\}_{g \in G}$ for T on X with respect to A will be ε -traced by some point $x \in X$. Let $\{x^{(g)}\}_{g \in G}$ be a δ pseudo-orbit for T on X with respect to A , i.e. $d(T_a x^{(g)}, x^{(ag)}) < 2^{-(m+1)}$ for every $g \in G, a \in A$. Then for any $h \in B(m), g \in G$ and $a \in A$, we have

$$(**) \quad x_{ha}^{(g)} = (T_a x^{(g)})_h = x_h^{(ag)}.$$

Fix $h \in B(m)$. Then we can write h as $a_1 \cdots a_n$, where $a_i \in A$ for every $1 \leq i \leq n$ and $n \leq m$. Put $x := (x_g)_{g \in G}$, where $x_g := x_e^{(g)}$ for every $g \in G$. Applying (**), one has

$$(T_g x)_h = x_{hg} = x_e^{(hg)} = x_e^{(a_1 \cdots a_n g)} = x_{a_1}^{(a_2 \cdots a_n g)} = \cdots = x_{a_1 \cdots a_n}^{(g)} = x_h^{(g)},$$

for every $g \in G$. Thus $d(T_g x, x^{(g)}) \leq 2^{-m} < \varepsilon$ for every $g \in G$. Now we show that $x \in X$. Since $F = B(M) \subset B(m)$, we have $(T_g x)_f = x_f^{(g)}$ for $f \in F$ and $g \in G$. Hence we get

$$\pi_F(T_g x) = \pi_F(x^{(g)}) \in W$$

for every $g \in G$. This implies $x \in X$.

Now we will prove the converse. Assume that X has POTP. Take $\delta > 0$ such that every δ pseudo-orbit for T on X with respect to A is $1/2$ -traced. Choose $m \in \mathbb{N}$ such that $2^{-m} < \delta$. Let $W \subset B_{m+1}(S) \setminus B_{m+1}(X)$. Then from the definition of X_W , one has $X \subset X_W$. Now we claim that $X_W \subset X$. Let $y \in X_W$. Then for every $g \in G$, one has $(gy)_{B(m+1)} \in B_{m+1}(X)$, and hence there exists $x^{(g)} \in X$ such that $x_{B(m+1)}^{(g)} = (gy)_{B(m+1)}$. Then for any $a \in A$ and $g \in G$, we get

$$(ax^{(g)})_{B(m)} = x_{B(m)a}^{(g)} = y_{B(m)a g} = x_{B(m)}^{(ag)}$$

because $B(m)a \subset B(m+1)$. Therefore, we have $d(ax^{(g)}, x^{(ag)}) \leq 2^{-m} < \delta$, and hence $\{x^{(g)}\}_{g \in G}$ is a δ pseudo-orbit for T on X with respect to A . As X has POTP, there exists $x \in X$ such that $\{x^{(g)}\}_{g \in G}$ is $1/2$ -traced by x , i.e. $d(gx, x^{(g)}) < 1/2$ for every $g \in G$. Hence we get

$$x_{B(1)g} = x_{B(1)}^{(g)} = y_{B(1)g}$$

for $g \in G$, and thus $y = x \in X$. Applying Lemma 3.1, we complete the proof. \square

4. EQUICONTINUOUS ACTIONS AND POTP

We say that an action $T \in \text{Act}(G, X)$ is *equicontinuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$, one has $d(gx, gy) < \varepsilon$ for every $g \in G$. T is called *distal* if for every $x \neq y \in X$, one has $\inf_{g \in G} d(gx, gy) > 0$.

It is known that every compact, totally disconnected, metrizable space without isolated points is homeomorphic to the Cantor set and every equicontinuous action of \mathbb{Z} on the Cantor space has POTP. In the following theorem, we can see that it still holds for actions of general groups.

Theorem 4.1. *Let G be an infinite, finitely generated group and let T be an equicontinuous action of G on the Cantor space X . Then T has POTP.*

Proof. Since $\{0, 1\}^G$ is a compact, totally disconnected, metrizable space without isolated points, we can assume that $X = \{0, 1\}^G$. Let A be a symmetric finitely generating set of G . We enumerate elements of G by $e = g_0, g_1, \dots$ and define a compatible metric d on X by

$$d(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^i} d_S(x_{g_i}, y_{g_i})$$

for $x, y \in X$, where d_S is the metric on $\{0, 1\}$ defined by $d_S(a, b)$ is 1 if $a \neq b$ and 0 otherwise. Put $F_n := \{g_0, g_1, \dots, g_n\}$. Let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$. Since the action T is equicontinuous, there exists $k \in \mathbb{N}$ such that whenever $d(x, y) < 2^{-k}$, one has $d(gx, gy) < 2^{-m}$ for every $g \in G$. This means that for any $x, y \in X$ with $d(x, y) < 2^{-k}$, we have $(gx)_{F_m} = (gy)_{F_m}$ for $g \in G$. Let $\{x^{(h)}\}_{h \in G}$ be a 2^{-k} pseudo-orbit for T with respect to A ; i.e. for $a \in A, h \in G$, we have $d(ax^{(h)}, x^{(ah)}) < 2^{-k}$. Then $(gax^{(h)})_{F_m} = (gx^{(ah)})_{F_m}$ for every $a \in A$ and $h, g \in G$. For $g \in G$, we write g as $a_1 \cdots a_n$ for some $n \in \mathbb{N}$ and $a_i \in A$. Then we have

$$\begin{aligned} (gx^{(e)})_{F_m} &= (a_1 \cdots a_n x^{(e)})_{F_m} = (a_1 \cdots a_{n-1} x^{(a_n)})_{F_m} \\ &= (a_1 \cdots a_{n-2} x^{(a_{n-1} a_n)})_{F_m} = \cdots = x_{F_m}^{(a_1 \cdots a_n)} = x_{F_m}^{(g)}. \end{aligned}$$

Consequently we get $d(gx^{(e)}, x^{(g)}) < 2^{-m} < \varepsilon$, and so complete the proof. □

Corollary 4.2. *Every distal action of a finitely generated group on the Cantor space has POTP.*

Proof. From [3, Corollary 1.9], we know that every distal action of a finitely generated group on the Cantor space is equicontinuous. □

Remark 4.3. We cannot leave out the assumption of Cantor space in Theorem 4.1, because even in the case $G = \mathbb{Z}$, every distal homeomorphism on a connected space does not have POTP [1, Theorem 2.3.2].

Let G be a countable group. A *chain* of G is a sequence of subgroups $\{G_n\}_{n \geq 0}$ such that $G = G_0 \geq G_1 \geq G_2 \geq \dots$ such that $[G : G_n] < \infty$ for every $n \geq 0$. For a chain $\{G_n\}_{n \geq 0}$, we define tree structure $T(G, \{G_n\})$ as follows. The vertices are $\{gG_n : n \geq 0, g \in G\}$, and (g_1G_n, g_2G_m) is an edge if $m = n + 1$ and $g_2G_m \subset g_1G_n$. The boundary $\overleftarrow{G}_{\{G_n\}}$ of $T(G, \{G_n\})$ consists of all sequences (x_0, x_1, \dots) of vertices with x_n adjacent to x_{n+1} for each $n \in \mathbb{Z}_+$. Then $\overleftarrow{G}_{\{G_n\}}$ is a compact metrizable space endowed with the topology generated by the open basis consisting of all

subsets $O_x = \{(x_0, x_1, \dots) \in \overleftarrow{G}_{\{G_n\}} : x_N = x\}$ with $x \in G/G_N$ and $N \in \mathbb{Z}_+$. The natural left actions of G on G/G_n induce the *profinite action* $(\overleftarrow{G}_{\{G_n\}}, G)$, an action of G on $\overleftarrow{G}_{\{G_n\}}$ by homeomorphisms. In this case the profinite action $(\overleftarrow{G}_{\{G_n\}}, G)$ is equicontinuous, since for any $x = gG_n \in G/G_n$ and $h \in G$ we have $hO_x = O_y$ with $y = hgG_n \in G/G_n$.

Profinite actions have been studied extensively in rank gradient, orbit equivalence, operator algebras, and sofic entropy theory [2, 5, 8]. Since every profinite action of G on $\overleftarrow{G}_{\{G_n\}}$ is equicontinuous and every infinite boundary space $\overleftarrow{G}_{\{G_n\}}$ is a Cantor space, from Theorem 4.1 we get the following corollary.

Corollary 4.4. *Every profinite action of G on an infinite boundary space $\overleftarrow{G}_{\{G_n\}}$ has POTP.*

ACKNOWLEDGEMENTS

The first author was supported by the NRF grant funded by the Korea government (MEXT) (No. NRF-2016R1A5A1008055), and the second author was supported by the NRF grant funded by the Korea government (MSIP) (No. NRF-2015R1A2A2A01002437).

REFERENCES

- [1] N. Aoki and K. Hiraide, *Topological theory of dynamical systems*, Recent advances, North-Holland Mathematical Library, vol. 52, North-Holland Publishing Co., Amsterdam, 1994. MR1289410
- [2] Miklós Abért and Nikolay Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1657–1677. MR2966663
- [3] Joseph Auslander, Eli Glasner, and Benjamin Weiss, *On recurrence in zero dimensional flows*, Forum Math. **19** (2007), 107–114.
- [4] Tullio Ceccherini-Silberstein and Michel Coornaert, *Cellular automata and groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. MR2683112
- [5] Nhan-Phu Chung and Guohua Zhang, *Weak expansiveness for actions of sofic groups*, J. Funct. Anal. **268** (2015), no. 11, 3534–3565, DOI 10.1016/j.jfa.2014.12.013. MR3336733
- [6] Michel Coornaert and Athanase Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Mathematics, vol. 1539, Springer-Verlag, Berlin, 1993. MR1222644
- [7] Hu Huyi, Shi Enhui, and Zhenqi Jenny Wang, *Some ergodic and rigidity properties of discrete Heisenberg group actions*, preprint, arXiv:1405.1120.
- [8] Adrian Ioana, *Cocycle superrigidity for profinite actions of property (T) groups*, Duke Math. J. **157** (2011), no. 2, 337–367. MR2783933
- [9] Keonhee Lee and C. A. Morales, *Topological stability and pseudo-orbit tracing property for expansive measures*, J. Differential Equations **262** (2017), no. 6, 3467–3487, DOI 10.1016/j.jde.2016.04.029. MR3592647
- [10] Z. Nitecki, *On semi-stability for diffeomorphisms*, Invent. Math. **14** (1971), 83–122. MR0293671
- [11] Piotr Oprocha, *Shadowing in multi-dimensional shift spaces*, Colloq. Math. **110** (2008), 451–460. MR2353915
- [12] Alexey V. Osipov and Sergey B. Tikhomirov, *Shadowing for actions of some finitely generated groups*, Dyn. Syst. **29** (2014), 337–351. MR3227777
- [13] Sergei Yu. Pilyugin, *Inverse shadowing in group actions*, to appear in Dyn. Syst., <http://dx.doi.org/10.1080/14689367.2016.1173651>.
- [14] Sergei Yu. Pilyugin and Sergei B. Tikhomirov, *Shadowing in actions of some abelian groups*, Fund. Math. **179** (2003), 83–96. MR2028929
- [15] Peter Walters, *Anosov diffeomorphisms are topologically stable*, Topology **9** (1970), 71–78. MR0254862

- [16] Peter Walters, *On the pseudo-orbit tracing property and its relationship to stability*, Lecture Notes in Math., vol. 668, Springer, Berlin, 1978, pp. 231–244. MR0518563

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, REPUBLIC OF KOREA

E-mail address: phuchung@skku.edu; phuchung82@gmail.com

DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, DAEJEON 305-764, REPUBLIC OF KOREA

E-mail address: khlee@cnu.ac.kr