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GENERATION OF SIEGEL MODULAR FUNCTION FIELDS OF EVEN LEVEL

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ABSTRACT. For positive integers g and N, let $\mathcal{F}_N^{(g)}$ be the field of meromorphic Siegel modular functions of genus g and level N whose Fourier coefficients belong to the Nth cyclotomic field. We construct explicit generators of $\mathcal{F}_N^{(g)}$ over $\mathcal{F}_1^{(g)}$ by making use of a quotient of theta constants, when $g \geq 2$ and N is even.

1. INTRODUCTION

For any positive integers g and N, we denote by $\mathcal{F}_N^{(g)}$ the field of meromorphic Siegel modular functions of genus g and level N whose Fourier coefficients lie in the Nth cyclotomic field (Section 2). Then $\mathcal{F}_N^{(g)}$ is a finite Galois extension of $\mathcal{F}_1^{(g)}$ with $\mathrm{Gal}(\mathcal{F}_N^{(g)}/\mathcal{F}_1^{(g)}) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ ([9]). In particular, when $g = 1$ we have

$$\mathcal{F}_N^{(1)} = \mathcal{F}_1^{(1)}\left(f_{\mathbf{v}}(\tau) \mid \mathbf{v} \in \mathbb{Q}^2 \text{ has exact denominator } N\right) \quad \text{for } N \geq 2,$$

where the $f_{\mathbf{v}}(\tau)$ are Fricke functions ([7]).

Unfortunately, we know relatively little about Siegel modular functions of higher genus and level because it is very difficult to find attractive examples that can be handled. Recently, Koo et al. presented explicit generators of $\mathcal{F}_N^{(g)}$ over $\mathcal{F}_1^{(g)}$ by means of the Siegel modular function $\Theta_{\mathbf{v}}(Z)$ developed in [4] when $g \geq 2$ and $N \geq 3$ ([5, Theorem 6.2]). The function $\Theta_{\mathbf{v}}(Z)$ is a generalization of the one variable Siegel function and has nice properties as a generator of the Siegel modular function field, but it becomes identically zero when $N = 2$, and hence we need another generator in this case.

In this paper, we shall focus on explicit generators of $\mathcal{F}_N^{(g)}$ over $\mathcal{F}_1^{(g)}$ when $g \geq 2$ and N is even. We first define the function $\Phi_{\mathbf{v}}(Z)$ as a quotient of theta constants and investigate its properties (Section 3). Lastly, we shall generate $\mathcal{F}_N^{(g)}$ over $\mathcal{F}_1^{(g)}$ in terms of $\Phi_{\mathbf{v}}(Z)$ for some $\mathbf{v} \in (1/N)\mathbb{Z}^{2g}$ by using the order formula for Siegel functions (Theorem 4.2).

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According to the theory of complex multiplication, there is a close connection between $\mathcal{F}_N^{(g)}$ and the field

$$K\mathcal{F}_N^{(g)}(Z_0) = K\left(f(Z_0) \mid f \in \mathcal{F}_N^{(g)} \text{ which is finite at } Z_0\right)$$

for a CM-field K and a CM-point Z_0 ([9, Theorem 9.6]). In particular, if K is an imaginary quadratic field, then the field $K\mathcal{F}_N^{(1)}(Z_0)$ becomes the ray class field of K modulo N ([7, Chapter 10, §1]). We hope that our explicit generators of $\mathcal{F}_N^{(g)}$ will contribute to finding an analogue of this result for arbitrary CM-fields.

2. Action on the Siegel modular function field

In this section we shall explain the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on the field $\mathcal{F}_N^{(g)}$.

Let g be a positive integer. For a commutative ring R with unity 1, we denote by

$$\begin{aligned}\mathrm{GSp}_{2g}(R) &= \left\{\alpha \in \mathrm{GL}_{2g}(R) \mid \alpha^T\eta_g\alpha = \nu(\alpha)\eta_g \text{ with } \nu(\alpha) \in R^\times\right\},\\ \mathrm{Sp}_{2g}(R) &= \left\{\alpha \in \mathrm{GSp}_{2g}(R) \mid \nu(\alpha) = 1\right\},\end{aligned}$$

where $\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}$ and α^T is the transpose of α. Then for $\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_{2g}(R)$ with $A, B, C, D \in M_g(R)$, we have

$$\alpha \in \mathrm{Sp}_{2g}(R) \Longleftrightarrow A^TD - C^TB = I_g,\ A^TC = C^TA,\ B^TD = D^TB. \tag{1}$$

Let

$$\mathbb{H}_g = \left\{Z \in M_g(\mathbb{C}) \mid Z^T = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\right\}$$

be the Siegel upper half-space of degree g. The group

$$\mathrm{GSp}_{2g}(\mathbb{R})_+ = \left\{\alpha \in \mathrm{GSp}_{2g}(\mathbb{R}) \mid \nu(\alpha) > 0\right\}$$

acts on $\mathbb{H}_g$ by

$$\alpha(Z) = (AZ+B)(CZ+D)^{-1} \quad \text{for } \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}_{2g}(\mathbb{R})_+, Z \in \mathbb{H}_g.$$

For a positive integer N, let

$$\Gamma(N) = \left\{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})}\right\}.$$

Let k be an integer. A holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ is called a Siegel modular form of genus g, weight k and level N if it satisfies the following properties:

(i) $f(\gamma(Z)) = \det(CZ+D)^k f(Z)$ for all $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(N)$.

(ii) f is holomorphic at every cusp.

When $g \geq 2$, condition (ii) is not necessary by the Koecher principle. Then f has an expansion of the form

$$f(Z) = \sum_M c(M)e\left(\mathrm{tr}(MZ)/N\right) \quad (c(M) \in \mathbb{C}), \tag{2}$$

where M runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(z) = e^{2\pi iz}$ for $z \in \mathbb{C}$. We call the right-hand side of (2) the Fourier expansion of f, and call the $c(M)$ the Fourier coefficients of f.

For a subfield F of $\mathbb{C}$, we denote by

$$\begin{aligned}
\mathcal{M}_k^{(g)}(\Gamma(N),F) &= \text{the space of Siegel modular forms of genus } g\text{, weight } k \text{ and level } N \text{ with Fourier coefficients in } F,\\
\mathcal{M}_k^{(g)}(F) &= \bigcup_{N=1}^{\infty} \mathcal{M}_k^{(g)}(\Gamma(N),F),\\
\mathcal{A}_0^{(g)}(\Gamma(N),F) &= \text{the field of functions } f \text{ which are invariant under } \Gamma(N) \text{ and } f=g/h \text{ with } g\in\mathcal{M}_k^{(g)}(F) \text{ and } h\in\mathcal{M}_k^{(g)}(F)\setminus\{0\} \text{ for the same weight } k.
\end{aligned}$$

In particular, let

$$\mathcal{F}_N^{(g)} = \mathcal{A}_0^{(g)}(\Gamma(N),\mathbb{Q}(\zeta_N)),$$

where $\zeta_N = e(1/N)$. Then $\mathcal{F}_N^{(g)}$ is a Galois extension of $\mathcal{F}_1^{(g)}$ with

$$\mathrm{Gal}(\mathcal{F}_N^{(g)}/\mathcal{F}_1^{(g)}) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N\cdot \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where $G_N = \left\{\begin{bmatrix} I_g & O_g \\ O_g & \nu I_g\end{bmatrix} \mid \nu\in(\mathbb{Z}/N\mathbb{Z})^\times\right\}$ ([9, Theorem 8.10]). The action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\mathcal{F}_N^{(g)}$ is given as follows: Let $f=f_1/f_2$ be an element of $\mathcal{F}_N^{(g)}$ for some $f_1, f_2\in\mathcal{M}_k^{(g)}(\mathbb{Q}(\zeta_N))$ with Fourier expansions

$$f_i(Z) = \sum_M c_i(M) e\left(\mathrm{tr}(MZ)/N\right) \quad \text{for } i=1,2.$$

(i) An element $\begin{bmatrix} I_g & O_g \\ O_g & tI_g\end{bmatrix}$ of G_N acts on f by

$$f^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g\end{smallmatrix}\right]} = \frac{\sum_M c_1(M)^\sigma e\left(\mathrm{tr}(MZ)/N\right)}{\sum_M c_2(M)^\sigma e\left(\mathrm{tr}(MZ)/N\right)},$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ defined by $\zeta_N^\sigma = \zeta_N^t$.

(ii) An element $\widetilde{\gamma}$ of $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on f by

$$f^{\widetilde{\gamma}} = f\circ\gamma,$$

where γ is any preimage of $\widetilde{\gamma}$ under the reduction

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$

3. Theta constants

Let g be a positive integer. For $\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_{2g}\end{bmatrix}^T \in\mathbb{R}^{2g}$, we denote by

$$\mathbf{v}_u = \begin{bmatrix} v_1 \\ \vdots \\ v_g\end{bmatrix}, \quad \mathbf{v}_l = \begin{bmatrix} v_{g+1} \\ \vdots \\ v_{2g}\end{bmatrix}, \quad \langle\mathbf{v}\rangle = \begin{bmatrix} \langle v_1\rangle \\ \vdots \\ \langle v_{2g}\rangle\end{bmatrix},$$

where $\langle x\rangle$ is the fractional part of $x\in\mathbb{R}$ in the interval $[0,1)$. We define the theta constant $\theta_{\mathbf{v}}(Z)$ by the following infinite series:

$$\theta_{\mathbf{v}}(Z) = \sum_{\mathbf{n}\in\mathbb{Z}^g} e\left(\frac{1}{2}(\mathbf{n}+\mathbf{v}_u)^T Z(\mathbf{n}+\mathbf{v}_u) + (\mathbf{n}+\mathbf{v}_u)^T\mathbf{v}_l\right) \quad \text{for } Z\in\mathbb{H}_g.$$

It is holomorphic on $\mathbb{H}_g$ and we have

$$\theta_{\mathbf{v}}(Z) \equiv 0 \iff \langle \mathbf{v} \rangle \in \{0, 1/2\}^{2g} \setminus S_+, \tag{3}$$

where

$$S_+ = \left\{ \mathbf{a} \in \{0, 1/2\}^{2g} \mid e(2\mathbf{a}_u^T \mathbf{a}_l) = 1 \right\}$$

is the set of even theta characteristics ([3, Theorem 2]).

Let $N \geq 2$ be a positive integer and put

$$\mathcal{I}_N = \left\{ \mathbf{v} \in \mathbb{Q}^{2g} \mid N \text{ is the smallest positive integer so that } N\mathbf{v} \in \mathbb{Z}^{2g} \right\}.$$

For $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, we let

$$\mathbf{c}_\gamma = \frac{1}{2} \begin{bmatrix} \{A^T C\} \\ \{B^T D\} \end{bmatrix}$$

where $\{S\}$ denotes the g-vector whose components are the diagonal entries of $S \in M_g(\mathbb{Z})$.

Proposition 3.1. *Let* $\mathbf{v}, \mathbf{w} \in \mathcal{I}_N$.

(i) $\theta_{-\mathbf{v}}(Z) = \theta_{\mathbf{v}}(Z)$.
(ii) $\theta_{\mathbf{v}+\mathbf{n}}(Z)^N = \theta_{\mathbf{v}}(Z)^N$ *for* $\mathbf{n} \in \mathbb{Z}^{2g}$.
(iii) *For* $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, *we have*

$$\theta_{\mathbf{v}}(\gamma(Z))^{4N} = \kappa(\gamma)^{2N} \det(CZ+D)^{2N} e\left(2N(\mathbf{v}_u^T \mathbf{v}_l - (\mathbf{v}'_u)^T \mathbf{v}'_l)\right) \theta_{\gamma^T \mathbf{v} + \mathbf{c}_\gamma}(Z)^{4N},$$

where $\mathbf{v}' = \gamma^T \mathbf{v}$ *and* $\kappa(\gamma)$ *is a 4th root of unity which depends only on* γ.

(iv) *The function* $\theta_{\mathbf{v}}(Z)/\theta_{\mathbf{w}}(Z)$ *belongs to* $\mathcal{F}_{2N^2}^{(g)}$. *Furthermore, if*

$$t \in (\mathbb{Z}/2N^2\mathbb{Z})^\times,$$

then

$$\left(\frac{\theta_{\mathbf{v}}(Z)}{\theta_{\mathbf{w}}(Z)} \right)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]} = \frac{\theta_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]\mathbf{v}}(Z)}{\theta_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]\mathbf{w}}(Z)}.$$

Proof. [8, (13), Propositions 1.3 and 1.7] and [1, Theta Transformation Formula 8.6.1]] □

From now on, we assume that N is even. For $\mathbf{v} \in \mathcal{I}_N$, we define

$$\Phi_{\mathbf{v}}(Z) = e\left(-2N|S_+|\mathbf{v}_u^T \mathbf{v}_l\right) \frac{\theta_{\mathbf{v}}(Z)^{4N|S_+|}}{\prod_{\mathbf{a} \in S_+} \theta_{\mathbf{a}}(Z)^{4N}} \quad \text{for } Z \in \mathbb{H}_g.$$

Lemma 3.2. *Let* $\mathbf{v} \in \mathcal{I}_N$.

(i) $\Phi_{-\mathbf{v}}(Z) = \Phi_{\mathbf{v}}(Z)$.
(ii) $\Phi_{\mathbf{v}+\mathbf{n}}(Z) = \Phi_{\mathbf{v}}(Z)$ *for* $\mathbf{n} \in \mathbb{Z}^{2g}$.
(iii) *For* $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$, *we have*

$$\Phi_{\mathbf{v}}(\gamma(Z)) = \Phi_{\gamma^T \mathbf{v} + \mathbf{c}_\gamma}(Z).$$

(iv) *The function* $\Phi_{\mathbf{v}}(Z)$ *belongs to* $\mathcal{F}_N^{(g)}$.
(v) *If* $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, *then we have*

$$\Phi_{\mathbf{v}}(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]} = \Phi_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]\mathbf{v}}(Z).$$

Proof. (i), (ii) are immediate consequences of Proposition 3.1 (i), (ii).
(iii) For $\mathbf{w} \in \mathbb{Q}^{2g}$, we let

$$\begin{aligned} \mathbf{w}' &= \gamma^T \mathbf{w}, \\ \mathbf{w}'' &= \gamma^T \mathbf{w} + \mathbf{c}_\gamma. \end{aligned}$$

We observe from Proposition 3.1 (iii) and (3) that the map

$$\mathbf{a} \mapsto \langle \mathbf{a}'' \rangle \tag{4}$$

is a permutation of the set S_+. Hence we achieve

$$\begin{aligned} \Phi_{\mathbf{v}}(\gamma(Z)) &= e\left(-2N|S_+|\mathbf{v}_u^T \mathbf{v}_l\right) \frac{\theta_{\mathbf{v}}(\gamma(Z))^{4N|S_+|}}{\prod\limits_{\mathbf{a}\in S_+} \theta_{\mathbf{a}}(\gamma(Z))^{4N}} \\ &= e\left(-2N|S_+|(\mathbf{v}_u')^T \mathbf{v}_l'\right) \frac{\theta_{\mathbf{v}''}(Z)^{4N|S_+|}}{\prod\limits_{\mathbf{a}\in S_+} e\left(2N(\mathbf{a}_u^T \mathbf{a}_l - (\mathbf{a}_u')^T \mathbf{a}_l')\right) \theta_{\mathbf{a}''}(Z)^{4N}} \\ &\qquad \text{by Proposition 3.1 (iii)} \\ &= e\left(-2N|S_+|(\mathbf{v}_u'')^T \mathbf{v}_l''\right) \frac{\theta_{\mathbf{v}''}(Z)^{4N|S_+|}}{\prod\limits_{\mathbf{a}\in S_+} \theta_{\langle \mathbf{a}'' \rangle}(Z)^{4N}} \\ &\qquad \text{by Proposition 3.1 (ii) and the fact that } 4 \,|\, 2N = \Phi_{\mathbf{v}''}(Z) \quad \text{by (4).} \end{aligned}$$

(iv) By Proposition 3.1 (iv), the function $\Phi_{\mathbf{v}}(Z)$ lies in $\mathcal{F}_{2N^2}^{(g)}$.
Let $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(N)$. Since N is even, we obtain

$$\gamma^T \mathbf{v} + \mathbf{c}_\gamma \equiv \mathbf{v} \pmod{\mathbb{Z}^{2g}}.$$

Hence we get $\Phi_{\mathbf{v}}(\gamma(Z)) = \Phi_{\mathbf{v}}(Z)$ by (ii) and (iii), which implies

$$\Phi_{\mathbf{v}}(Z) \in \mathcal{A}_0^{(g)}(\Gamma(N), \mathbb{Q}(\zeta_{2N^2})).$$

Now, take $t \in (\mathbb{Z}/2N^2\mathbb{Z})^\times$ such that $t \equiv 1 \pmod N$. Since t is odd, the map

$$\mathbf{a} \mapsto \left\langle \begin{bmatrix} I_g & O_g \\ O_g & tI_g \end{bmatrix} \mathbf{a} \right\rangle \tag{5}$$

is a permutation of S_+. Thus we deduce

$$\begin{aligned}\Phi_{\mathbf{v}}(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]} &= e\left(-2Nt|S_+|\mathbf{v}_u^T\mathbf{v}_l\right)\frac{\theta_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]\mathbf{v}}(Z)^{4N|S_+|}}{\prod_{\mathbf{a}\in S_+}\theta_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & tI_g \end{smallmatrix}\right]\mathbf{a}}(Z)^{4N}} \\ &\qquad \text{by Proposition 3.1 (iv)} \\ &= e\left(-2N|S_+|\mathbf{v}_u^T\mathbf{v}_l\right)\frac{\theta_{\mathbf{v}}(Z)^{4N|S_+|}}{\prod_{\mathbf{a}\in S_+}\theta_{\mathbf{a}}(Z)^{4N}} \quad \text{by (ii) and (5)} \\ &= \Phi_{\mathbf{v}}(Z),\end{aligned} \tag{6}$$

which indicates $\Phi_{\mathbf{v}}(Z) \in \mathcal{F}_N^{(g)}$.

In the same manner as used in (6), we can verify (v). □

For $\tau_1, \ldots, \tau_g \in \mathbb{H}_1$, we denote by $\mathrm{diag}(\tau_1, \ldots, \tau_g)$ the $g \times g$ diagonal matrix with diagonal entries $\tau_1, \ldots, \tau_g$. Then $\mathrm{diag}(\tau_1, \ldots, \tau_g)$ belongs to $\mathbb{H}_g$ and

$$\begin{aligned}&\theta_{\mathbf{v}}\left(\mathrm{diag}(\tau_1, \ldots, \tau_g)\right) \\ &= \begin{cases} \prod_{k=1}^{g} \xi_k \dfrac{g_{\left[\begin{smallmatrix} 1/2-v_k \\ 1/2-v_{k+g} \end{smallmatrix}\right]}(\tau_k)\theta_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]}(\tau_k)}{g_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau_k)} & \text{if } \begin{bmatrix} \langle v_k \rangle \\ \langle v_{k+g} \rangle \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \text{ for all } k = 1, \ldots, g, \\ 0 & \text{otherwise,} \end{cases}\end{aligned} \tag{7}$$

where $\xi_k = e((2v_kv_{k+g} + v_k - v_{k+g})/4)$ and $g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau)$ is the Siegel function defined by

$$g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau) = -q^{(1/2)\mathbf{B}_2(r)}e(s(r-1)/2)(1-q^re(s))\prod_{n=1}^{\infty}(1-q^{n+r}e(s))(1-q^{n-r}e(-s))$$

with $q = e(\tau)$ and the second Bernoulli polynomial $\mathbf{B}_2(r) = r^2 - r + 1/6$ ([2, Example 4.3]). Here we note that

$$\mathrm{ord}_q\ g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau) = (1/2)\mathbf{B}_2(\langle r \rangle) \tag{8}$$

([6, §2.1]).

Lemma 3.3. *Let* $\mathbf{v} = [v_k]_{k=1}^{2g}, \mathbf{w} = [w_k]_{k=1}^{2g} \in \mathcal{I}_N$. *Assume that* $\begin{bmatrix} \langle v_k \rangle \\ \langle v_{k+g} \rangle \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ *for all* $k = 1, \ldots, g$. *If* $\Phi_{\mathbf{v}}$ *and* $\Phi_{\mathbf{w}}$ *are equal as meromorphic functions, then* $v_k \equiv \pm w_k \pmod{\mathbb{Z}}$ *for all* $k = 1, \ldots, 2g$.

Proof. Since $\Phi_{\mathbf{v}}(Z) = \Phi_{\mathbf{w}}(Z)$, we obtain

$$\Phi_{\mathbf{v}}(\mathrm{diag}(\tau_1, \ldots, \tau_g)) = \Phi_{\mathbf{w}}(\mathrm{diag}(\tau_1, \ldots, \tau_g)) \quad \text{for } \tau_1, \ldots, \tau_g \in \mathbb{H}_1.$$

Then we see from (7) that $\begin{bmatrix} \langle w_k \rangle \\ \langle w_{k+g} \rangle \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ for all $k = 1, \ldots, g$ and

$$\prod_{k=1}^{g} g_{\left[\begin{smallmatrix} 1/2-v_k \\ 1/2-v_{k+g} \end{smallmatrix}\right]}(\tau_k)^{4N|S_+|} \doteq \prod_{k=1}^{g} g_{\left[\begin{smallmatrix} 1/2-w_k \\ 1/2-w_{k+g} \end{smallmatrix}\right]}(\tau_k)^{4N|S_+|}$$

where $\doteq$ stands for the equality up to a root of unity. Comparing the orders with respect to $e(\tau_k)$ by using (8), we derive

$$\mathbf{B}_2(\langle 1/2 - v_k \rangle) = \mathbf{B}_2(\langle 1/2 - w_k \rangle) \tag{9}$$

for $k = 1, \ldots, g$.

On the other hand, acting $\begin{bmatrix} O_g & I_g \\ -I_g & O_g \end{bmatrix}^T \in \mathrm{Sp}_{2g}(\mathbb{Z})$ on both sides of $\Phi_{\mathbf{v}}(Z) = \Phi_{\mathbf{w}}(Z)$, we deduce by Proposition 3.2 (iii) that

$$\Phi_{\left[\begin{smallmatrix} \mathbf{v}_l \\ -\mathbf{v}_u \end{smallmatrix}\right]}(Z) = \Phi_{\left[\begin{smallmatrix} \mathbf{w}_l \\ -\mathbf{w}_u \end{smallmatrix}\right]}(Z).$$

In a similar way as above, one can show that (9) also holds for $k = g+1, \ldots, 2g$.

Now, we may assume by Proposition 3.2 (ii) that

$$0 \le v_k, w_k < 1 \quad \text{for all } k = 1, \ldots, 2g. \tag{10}$$

We then consider the following four cases for each index $1 \le k \le 2g$:

Case 1 : $0 \le v_k, w_k < 1/2$

We achieve by (9)

$$\mathbf{B}_2(1/2 - v_k) = \mathbf{B}_2(1/2 - w_k),$$

which implies $v_k^2 = w_k^2$. Hence $v_k = w_k$ by the assumption (10).

Case 2 : $0 \le v_k < 1/2$ and $1/2 \le w_k < 1$.

By (9), we have

$$\mathbf{B}_2(1/2 - v_k) = \mathbf{B}_2(3/2 - w_k).$$

Thus $v_k^2 = (w_k - 1)^2$ and so $v_k = 1 - w_k$.

Case 3 : $1/2 \le v_k < 1$ and $0 \le w_k < 1/2$.

In a similar fashion to **Case 2**, one can further show that $v_k = 1 - w_k$.

Case 4 : $1/2 \le v_k, w_k < 1$.

We derive by (9)

$$\mathbf{B}_2(3/2 - v_k) = \mathbf{B}_2(3/2 - w_k),$$

which indicates $(v_k - w_k)(v_k + w_k - 2) = 0$. Since $v_k + w_k < 2$, we get $v_k = w_k$.

Therefore, $v_k \equiv \pm w_k \pmod{\mathbb{Z}}$ for all cases. This completes the proof. □

4. Generation of the Siegel modular function field of even level

Let g and N be positive integers such that $g \ge 2$ and N is even. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2g}\}$ be the standard basis for $\mathbb{R}^{2g}$, and let

$$\begin{aligned} \mathbf{e}_{ij} &= \mathbf{e}_i + \mathbf{e}_j \quad \text{for } 1 \le i < j \le 2g, \\ \mathbf{e} &= \sum_{k=1}^{2g} \mathbf{e}_k. \end{aligned}$$

Lemma 4.1. *Let* $\mathbf{v} \in \mathcal{I}_N$.

(i) *If* $\Phi_{\mathbf{v}}(Z) = \Phi_{(1/N)\mathbf{e}_i}(Z)$ *with* $1 \le i \le 2g$, *then we have*

$$\mathbf{v} \equiv \pm(1/N)\mathbf{e}_i \pmod{\mathbb{Z}^{2g}}.$$

(ii) *If* $\Phi_{\mathbf{v}}(Z) = \Phi_{(1/N)\mathbf{e}_{ij}}(Z)$ *with* $1 \le i < j \le 2g$ *such that* $j \ne i + g$ *when* $N = 2$, *then we have* $\mathbf{v} \equiv \pm(1/N)\mathbf{e}_{ij} \pmod{\mathbb{Z}^{2g}}$.

(iii) *If* $N \neq 2$ *and* $\Phi_{\mathbf{v}}(Z) = \Phi_{(1/N)\mathbf{e}}(Z)$*, then we have* $\mathbf{v} \equiv \pm(1/N)\mathbf{e} \pmod{\mathbb{Z}^{2g}}$.

Proof. Observe that if $\mathbf{w} = [w_k]_{k=1}^{2g}$ is one of the vectors $(1/N)\mathbf{e}_i$, $(1/N)\mathbf{e}_{ij}$ and $(1/N)\mathbf{e}$ given in (i)∼(iii), then

$$\begin{bmatrix} \langle w_k \rangle \\ \langle w_{k+g} \rangle \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad \text{for all } k = 1, \ldots, g.$$

Hence by Lemma 3.3, it suffices to prove (ii), (iii) when $N \neq 2$. Now, we assume $N \neq 2$. For each $1 \leq r, s \leq g$, let E_{rs} be the $g \times g$ matrix with 1 at the entry (r, s) and zeros everywhere else, and let

$$E'_{rs} = \begin{cases} E_{rs} + E_{sr} & \text{if } r \neq s, \\ E_{rr} & \text{if } r = s. \end{cases}$$

(ii) By Lemma 3.3, we have

$$v_k \equiv \begin{cases} \pm 1/N \pmod{\mathbb{Z}} & \text{if } k = i \text{ or } k = j, \\ 0 \pmod{\mathbb{Z}} & \text{otherwise.} \end{cases}$$

Suppose that $v_i \equiv -v_j \pmod{\mathbb{Z}}$. Let

$$\alpha = \begin{cases} \begin{bmatrix} I_g + E_{ij} & O_g \\ O_g & I_g - E_{ji} \end{bmatrix}^T & \text{if } 1 \leq i, j \leq g, \\ \begin{bmatrix} I_g & E'_{i\,j-g} \\ O_g & I_g \end{bmatrix}^T & \text{if } 1 \leq i \leq g \text{ and } g+1 \leq j \leq 2g, \\ \begin{bmatrix} I_g - E_{j-g\,i-g} & O_g \\ O_g & I_g + E_{i-g\,j-g} \end{bmatrix}^T & \text{if } g+1 \leq i, j \leq 2g, \end{cases}$$

which belongs to $\mathrm{Sp}_{2g}(\mathbb{Z})$ by (1). Acting α on both sides of $\Phi_{\mathbf{v}}(Z) = \Phi_{(1/N)\mathbf{e}_{ij}}(Z)$, we obtain by Lemma 3.2 (iii)

$$\Phi_{\alpha^T\mathbf{v}+\mathbf{c}_\alpha}(Z) = \Phi_{(1/N)\alpha^T\mathbf{e}_{ij}+\mathbf{c}_\alpha}(Z).$$

Observe that

$$\mathbf{c}_\alpha = \begin{cases} (1/2)\mathbf{e}_i & \text{if } 1 \leq i \leq g \text{ and } j = i+g, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Since $N \neq 2$ and

$$((1/N)\alpha^T\mathbf{e}_{ij})_k \equiv \begin{cases} 0 \pmod{\mathbb{Z}} & \text{if } k \neq i, j, \\ 2/N \pmod{\mathbb{Z}} & \text{if } k = i, \\ 1/N \pmod{\mathbb{Z}} & \text{if } k = j, \end{cases}$$

it follows from Lemma 3.3 that

$$(\alpha^T\mathbf{v})_k \equiv \pm((1/N)\alpha^T\mathbf{e}_{ij})_k \pmod{\mathbb{Z}}$$

for all $k = 1, \ldots, 2g$. However, we have

$$(\alpha^T\mathbf{v})_i \equiv 0 \not\equiv \pm 2/N \equiv \pm((1/N)\alpha^T\mathbf{e}_{ij})_i \pmod{\mathbb{Z}},$$

which is a contradiction. Therefore, $v_i \equiv v_j \pmod{\mathbb{Z}}$ and so we conclude $\mathbf{v} \equiv \pm(1/N)\mathbf{e}_{ij} \pmod{\mathbb{Z}^{2g}}$.

(iii) By Lemma 3.3, we get $v_k \equiv \pm 1/N \pmod{\mathbb{Z}}$ for all $k = 1, \dots, 2g$. Suppose that $v_i \equiv -v_j \pmod{\mathbb{Z}}$ for some $1 \le i < j \le 2g$. Let $\alpha \in \mathrm{Sp}_{2g}(\mathbb{Z})$ be as above. Acting α on both sides of $\Phi_{\mathbf{v}}(Z) = \Phi_{(1/N)\mathbf{e}}(Z)$, we deduce by Lemma 3.2 (iii)

$$\Phi_{\alpha^T \mathbf{v} + \mathbf{c}_\alpha}(Z) = \Phi_{(1/N)\alpha^T \mathbf{e} + \mathbf{c}_\alpha}(Z).$$

Thus we see from Lemma 3.3 that

$$(\alpha^T \mathbf{v})_k \equiv \pm((1/N)\alpha^T \mathbf{e})_k \pmod{\mathbb{Z}}$$

for all $k = 1, \dots, 2g$. On the other hand, one can easily check that

$$(\alpha^T \mathbf{v})_i \equiv 0 \not\equiv \pm 2/N \equiv \pm((1/N)\alpha^T \mathbf{e})_i \pmod{\mathbb{Z}},$$

which gives a contradiction. Hence $v_i \equiv v_j \pmod{\mathbb{Z}}$ for any $1 \le i < j \le 2g$, from which (iii) is proved. $\square$

Theorem 4.2. *Assume that $g \ge 2$ and N is even. Then we have*

$$\mathcal{F}_N^{(g)} = \begin{cases} \mathcal{F}_1^{(g)}\left(\Phi_{(1/N)\mathbf{e}_1}(Z), \dots, \Phi_{(1/N)\mathbf{e}_{2g}}(Z), \Phi_{(1/N)\mathbf{e}}(Z)\right) & \text{if } N \ne 2, 4, \\ \mathcal{F}_1^{(g)}\left(\Phi_{(1/N)\mathbf{e}_1}(Z), \dots, \Phi_{(1/N)\mathbf{e}_{2g}}(Z), \Phi_{(1/N)\mathbf{e}_{12}}(Z)\right) & \text{if } N = 2, \\ \mathcal{F}_1^{(g)}\left(\Phi_{(1/N)\mathbf{e}_1}(Z), \dots, \Phi_{(1/N)\mathbf{e}_{2g}}(Z), \Phi_{(1/N)\mathbf{e}}(Z), \Phi_{(1/N)\mathbf{e}_{12}}(Z), \Phi_{(1/N)\mathbf{e}_{34}}(Z)\right) & \text{if } N = 4. \end{cases} \tag{11}$$

Proof. Let E_N be the field on the right-hand side of (11) which is a subfield of $\mathcal{F}_N^{(g)}$ by Lemma 3.2 (iv). Let $\beta \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ which fixes the field E_N. By a strong approximation theorem, there exists a matrix $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that

$$\beta \equiv \begin{bmatrix} I_g & O_g \\ O_g & \nu(\beta) I_g \end{bmatrix} \gamma \pmod{N \cdot M_{2g}(\mathbb{Z})}. \tag{12}$$

Hence for $\mathbf{v} \in \mathcal{I}_N$ we derive

$$\begin{aligned} \Phi_{\mathbf{v}}(Z)^\beta &= \Phi_{\mathbf{v}}(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu(\beta) I_g \end{smallmatrix}\right]\gamma} && \text{by (12)} \\ &= \Phi_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu(\beta) I_g \end{smallmatrix}\right]\mathbf{v}}(\gamma(Z)) && \text{by Lemma 3.2 (v)} \\ &= \Phi_{\gamma^T \left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu(\beta) I_g \end{smallmatrix}\right]\mathbf{v} + \mathbf{c}_\gamma}(Z) && \text{by Lemma 3.2 (iii)} \\ &= \Phi_{\beta^T \mathbf{v} + \mathbf{c}_\gamma}(Z) && \text{by Lemma 3.2 (ii).} \end{aligned}$$

In particular, we have

$$\Phi_{(1/N)\mathbf{e}_k}(Z) = \Phi_{(1/N)\mathbf{e}_k}(Z)^\beta = \Phi_{(1/N)\beta^T \mathbf{e}_k + \mathbf{c}_\gamma}(Z) \quad \text{for } k = 1, 2, \dots, 2g.$$

Thus we see from Lemma 4.1

$$\beta^T \mathbf{e}_k + N\mathbf{c}_\gamma \equiv \pm \mathbf{e}_k \pmod{N\mathbb{Z}^{2g}} \quad \text{for each } k = 1, 2, \dots, 2g,$$

which gives

$$\beta^T \equiv \begin{bmatrix}\beta^T\mathbf{e}_1 & \cdots & \beta^T\mathbf{e}_{2g}\end{bmatrix} \equiv \begin{bmatrix}\pm1 & 0 & \cdots & 0\\ 0 & \pm1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \pm1\end{bmatrix} + \begin{bmatrix}N\mathbf{c}_\gamma & \cdots & N\mathbf{c}_\gamma\end{bmatrix} \pmod{N\cdot M_{2g}(\mathbb{Z})}. \tag{13}$$

Here we note that $N\mathbf{c}_\gamma \equiv -N\mathbf{c}_\gamma \pmod{N\mathbb{Z}^{2g}}$ because $N\mathbf{c}_\gamma \in (N/2)\mathbb{Z}^{2g}$. In particular, if $N = 2$ we obtain

$$\beta^T \equiv \pm I_{2g} + \begin{bmatrix}N\mathbf{c}_\gamma & \cdots & N\mathbf{c}_\gamma\end{bmatrix} \pmod{N\cdot M_{2g}(\mathbb{Z})}. \tag{14}$$

First, consider the case $N \neq 2$. Since β also fixes $\Phi_{(1/N)\mathbf{e}}(Z)$, we achieve by (13)

$$\pm\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \equiv \pm\mathbf{e} \equiv \beta^T\mathbf{e} + N\mathbf{c}_\gamma \equiv \begin{bmatrix}\pm1\\\pm1\\\vdots\\\pm1\end{bmatrix} + N\mathbf{c}_\gamma \pmod{N\mathbb{Z}^{2g}}.$$

Write $N\mathbf{c}_\gamma = \begin{bmatrix}c_1 & \cdots & c_{2g}\end{bmatrix}^T$. Then $c_k \equiv 0$ or $2 \pmod{N}$, and $c_k \equiv 0 \pmod{N/2}$ for each $k = 1, 2, \ldots, 2g$. Hence if $N \neq 4$, we get $N\mathbf{c}_\gamma \equiv \mathbf{0} \pmod{N\mathbb{Z}^{2g}}$ and so $\beta \equiv \pm I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})}$. Therefore, β represents the identity element of $\mathrm{Gal}(\mathcal{F}_N^{(g)}/\mathcal{F}_1^{(g)}) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$, which yields $\mathcal{F}_N^{(g)} = E_N$.

Next, assume that $N = 4$. Since β fixes $\Phi_{(1/N)\mathbf{e}_{12}}(Z)$, it follows from (13) that

$$\pm\begin{bmatrix}1\\1\\0\\\vdots\\0\end{bmatrix} \equiv \pm\mathbf{e}_{12} \equiv \beta^T\mathbf{e}_{12} + N\mathbf{c}_\gamma \equiv \begin{bmatrix}\pm1\\\pm1\\0\\\vdots\\0\end{bmatrix} + N\mathbf{c}_\gamma \pmod{N\mathbb{Z}^{2g}}.$$

Thus $c_k \equiv 0 \pmod{N}$ if $k \neq 1, 2$. Similarly, we can derive $c_k \equiv 0 \pmod{N}$ if $k \neq 3, 4$ by using the fact that β fixes $\Phi_{(1/N)\mathbf{e}_{34}}(Z)$. Hence $N\mathbf{c}_\gamma \equiv \mathbf{0} \pmod{N\mathbb{Z}^{2g}}$ and $\beta \equiv \pm I_{2g} \pmod{N\cdot M_{2g}(\mathbb{Z})}$, which proves $\mathcal{F}_4^{(g)} = E_4$.

Lastly, suppose $N = 2$. Since β fixes $\Phi_{(1/N)\mathbf{e}_{12}}(Z)$, we see from (14) that

$$\pm\mathbf{e}_{12} \equiv \beta^T\mathbf{e}_{12} + N\mathbf{c}_\gamma \equiv \pm\mathbf{e}_{12} + N\mathbf{c}_\gamma \pmod{N\mathbb{Z}^{2g}}.$$

Thus $N\mathbf{c}_\gamma \equiv \mathbf{0} \pmod{N\mathbb{Z}^{2g}}$ and so $\beta \equiv \pm I_{2g} \pmod{N\cdot M_{2g}(\mathbb{Z})}$. This shows that $\mathcal{F}_2^{(g)} = E_2$. □

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