ASYMPTOTICS OF RACAH POLYNOMIALS
WITH FIXED PARAMETERS

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ABSTRACT. In this paper, we investigate asymptotic behaviors of Racah polynomials with fixed parameters and scaled variable as the polynomial degree tends to infinity. We start from the difference equation satisfied by the polynomials and derive an asymptotic formula in the outer region via ratio asymptotics. Next, we find the asymptotic formulas in the oscillatory region via a simple matching principle. Unlike the varying parameter case considered in a previous paper, the zeros of Racah polynomials with fixed parameters may not always be real. For this unusual case, we also provide a standard method to determine the oscillatory curve which attracts the zeros of Racah polynomials when the degree becomes large.

1. INTRODUCTION

The Racah polynomials, which lie on the top level of Askey hierarchy for hypergeometric orthogonal polynomials, have the following $_4F_3$ hypergeometric function representation [6, 8]:

$$(1.1) \quad R_n(z) = R_n(z; \alpha, \beta, \gamma, \delta)$$
$$:= {_4F_3}\left(\begin{array}{cccc} -n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & \gamma+1 \end{array} \middle| 1 \right),$$

where n and N are two non-negative integers with $0 \le n \le N$, $z = x(x+\gamma+\delta+1)$, and one of the following three equalities is satisfied: $\alpha+1 = -N$ or $\gamma+1 = -N$ or $\beta+\delta+1 = -N$. The orthogonality relation for the Racah polynomials is [6, 8]

$$\sum_{x=0}^{N} w(x) R_m(z) R_n(z)$$
$$= M \frac{(n+\alpha+\beta+1)_n (\alpha+\beta-\gamma+1)_n (\alpha-\delta+1)_n (\beta+1)_n n!}{(\alpha+\beta+2)_{2n} (\alpha+1)_n (\beta+\delta+1)_n (\gamma+1)_n} \delta_{mn},$$

where $z = x(x+\gamma+\delta+1)$,

$$w(x) = \frac{(\alpha+1)_x (\beta+\delta+1)_x (\gamma+1)_x (\gamma+\delta+1)_x ((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x (-\beta+\gamma+1)_x ((\gamma+\delta+1)/2)_x (\delta+1)_x x!},$$

and

$$
M = \begin{cases}
\dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N}, & \text{if } \alpha+1=-N, \\[3mm]
\dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N}, & \text{if } \beta+\delta+1=-N, \\[3mm]
\dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N}, & \text{if } \gamma+1=-N.
\end{cases}
$$

Here, we have made use of the Pochhammer symbol

$$
(\alpha)_n := \prod_{k=0}^{n-1}(\alpha+k).
$$

The Racah polynomials can also be viewed as birth and death process polynomials in the sense that they satisfy the difference equation [4,6]:

$$
(1.2) \qquad -zR_n(z) = \lambda_n R_{n+1}(z) + \mu_n R_{n-1}(z) - (\lambda_n+\mu_n)R_n(z),
$$

together with initial conditions $R_0(z)=1$ and $R_1(z)=(\lambda_0+\mu_0-z)/\lambda_0$, where the birth and death rates are given by

$$
\begin{aligned}
(1.3) \qquad \lambda_n &:= -\frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\[2mm]
\mu_n &:= -\frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},
\end{aligned}
$$

respectively. By normalizing the leading coefficients, we have the monic Racah polynomials

$$
\pi_n(z) := \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(n+\alpha+\beta+1)_n}R_n(z),
$$

which satisfy the difference equation

$$
(1.4) \qquad \pi_{n+1}(z) = (z-a_n)\pi_n(z) - b_n\pi_{n-1}(z), \ \pi_0(z)=1, \ \pi_1(z)=z-a_0,
$$

where

$$
(1.5) \qquad\qquad\qquad a_n = \lambda_n+\mu_n, \ b_n = \lambda_{n-1}\mu_n.
$$

In addition to the applications in birth and death processes, the Racah polynomials are also related to the so-called 6-j symbols in the quantum mechanical theory of angular momentum [1,8]. Asymptotic behaviors of Racah polynomials for large degree n were studied by Chen, Ismail and Simeonov [2], where both the variable and parameters were fixed. Their formulas were given in terms of $_3F_2$ or $_2F_1$ hypergeometric functions. Ismail and Simeonov [5] studied inequalities and asymptotics for a terminating $_4F_3$ series. Their results can be applied to a special case of Racah polynomials with $\alpha=\beta=0$ and $\delta=-\gamma=N+1$. Note that for this special case, two parameters are fixed, while the other two are large and varying. In a recent work [7], we derived explicit asymptotic formulas, in terms of elementary functions such as logarithmic, exponential and rational functions, for Racah polynomials with scaled variable and large varying parameters. A simple formula for the fixed variable was also obtained. It remains to consider the case with scaled variable and fixed parameters, which is the main focus of this paper. To be specific, we will provide asymptotic analysis of the monic Racah polynomials $\pi_n(N^2y)$ as

$N \to \infty$, where $n/N = p$ is a fixed positive number. The following three cases will be investigated respectively: $\alpha + 1 = -N$, $\gamma + 1 = -N$, and $\beta + \delta + 1 = -N$.

2. CASE I: $\alpha + 1 = -N$

In this section, we consider the case $\alpha + 1 = -N$ and β, δ, γ are fixed. First, we let $z = N^2 y$ and introduce the ratios

$$(2.1) \qquad w_k(z) := \frac{\pi_k(z)}{\pi_{k-1}(z)}$$

for $k = 1, \cdots, n$. It is readily seen from (1.4) that $w_1(z) = z - a_0$ and

$$(2.2) \qquad w_{k+1}(z) = z - a_k - \frac{b_k}{w_k(z)},$$

where a_k and b_k are obtained by substituting $\alpha + 1 = -N$ in (1.3) and using (1.5):

$$a_k = -\frac{(k-N)(k-N+\beta)(k+\beta+\delta+1)(k+\gamma+1)}{(2k-N+\beta)(2k-N+\beta+1)}$$
$$- \frac{k(k-N+\beta-\gamma-1)(k-N-\delta-1)(k+\beta)}{(2k-N+\beta-1)(2k-N+\beta)};$$

$$b_k = \frac{k(k-N+\beta-\gamma-1)(k-N-\delta-1)(k+\beta)}{(2k-N+\beta-1)(2k-N+\beta)}$$
$$\times \frac{(k-N-1)(k-N+\beta-1)(k+\beta+\delta)(k+\gamma)}{(2k-N+\beta-2)(2k-N+\beta-1)}.$$

If β, δ, γ are positive, then for sufficiently large N the coefficient b_k with $1 \le k \le n$ is always positive as long as $n/N = p < 1/2$. By Favard's theorem [3, §§1.4 & 1.5], the zeros of $\pi_k(z)$ are all real and simple. Put $t := k/N$. It follows from the above expressions for a_k and b_k that

$$(2.3)$$
$$\frac{a_k}{N^2} = -2f(t) - \frac{f(t)h_1(t)}{N} + O\left(\frac{1}{N^2}\right); \quad \frac{b_k}{N^4} = f(t)^2 + \frac{f(t)^2 h_2(t)}{N} + O\left(\frac{1}{N^2}\right),$$

and

$$(2.4) \qquad \frac{a_{k+1} - a_k}{N} = -2f(t)h_0(t) + O\left(\frac{1}{N}\right); \quad \frac{b_{k+1} - b_k}{N^3} = 2f(t)^2 h_0(t) + O\left(\frac{1}{N}\right),$$

where

$$(2.5) \qquad f(t) := \frac{t^2(t-1)^2}{(2t-1)^2};$$

$$(2.6) \qquad h_1(t) := \frac{2\beta - \gamma - \delta - 2}{t-1} + \frac{2\beta + \gamma + \delta + 2}{t} - \frac{4\beta}{2t-1};$$

$$(2.7) \qquad h_2(t) := \frac{2\beta - \gamma - \delta - 4}{t-1} + \frac{2\beta + \gamma + \delta}{t} - \frac{4\beta - 4}{2t-1};$$

$$(2.8) \qquad h_0(t) := \frac{f'(t)}{f(t)} = \frac{2}{t-1} + \frac{2}{t} - \frac{4}{2t-1} = h_1(t) - h_2(t).$$

Formula (2.4) can be obtained from (2.3) by simply taking derivative with respect to t. When $n/N = p < 1/2$, the asymptotic formulas (2.3)-(2.4) are uniform in $k = 0, 1, \cdots, n$. Throughout this section, we shall always assume that $p \in (0, 1/2)$.

Note that if $p \geq 1/2$, then the asymptotic formulas (2.3)-(2.4) are no longer valid. It would be a challenge to derive the asymptotic formula of $\pi_n(N^2 y)$ in this case.

Lemma 2.1. *Let $n/N = p$ be a fixed number in $(0, 1/2)$ and $z = N^2 y$. For $y \in \mathbf{C} \setminus [y_p, 0]$ with $y_p = -4p^2(p-1)^2/(2p-1)^2$, we have*

$$(2.9) \qquad w_k = w_k^0[1 + w_k^1 + O(1/N^2)]$$

as $N \to \infty$, where the leading term is

$$(2.10) \qquad w_k^0 = \frac{z - a_k + \sqrt{(z - a_k)^2 - 4b_k}}{2},$$

and the first-order term is

$$(2.11) \qquad w_k^1 = \frac{a_{k+1} - a_k}{2\sqrt{(z - a_k)^2 - 4b_k}} + \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{2[(z - a_k)^2 - 4b_k]}.$$

The error estimate is uniform for all $1 \leq k \leq n$.

Proof. Formally, we can obtain the leading term by substituting w_k^0 for both w_{k+1} and w_k in (2.2) and then solving the resulting quadratic equation. To give a rigorous argument, we need to estimate the error bound; namely, we will show that if w_k^0 is defined as in (2.10), then the first-order error term $\varepsilon_k^1 := w_k/w_k^0 - 1$ is of order $O(1/N)$. Furthermore, if w_k^1 is defined as in (2.11), then we shall prove that the second-order error term $\varepsilon_k^2 := w_k/w_k^0 - 1 - w_k^1$ is of order $O(1/N^2)$.

We substitute $w_k = w_k^0(1 + \varepsilon_k^1)$ in (2.2) and use (2.10) to find a difference equation for the first-order error term:

$$\varepsilon_{k+1}^1 = \frac{z - a_k - w_{k+1}^0 - (z - a_k - w_k^0)(1 + \varepsilon_k^1)^{-1}}{w_{k+1}^0}.$$

For $z = N^2 y$ with $y \in \mathbf{C} \setminus [y_p, 0]$, we shall prove by induction that $\varepsilon_k^1 = O(1/N)$ uniformly for all $k = 1, \cdots, n$. First, we rewrite the above equation as

$$(2.12) \qquad \varepsilon_{k+1}^1 = \frac{w_k^0 - w_{k+1}^0}{w_{k+1}^0} + \frac{N^2 y - a_k - w_k^0}{w_{k+1}^0}[1 - (1 + \varepsilon_k^1)^{-1}].$$

As $N \to \infty$, we have from (2.3), (2.4), (2.10)

$$\varepsilon_1^1 = \frac{w_1 - w_1^0}{w_1^0} = O\left(\frac{1}{N}\right),$$

$$\frac{w_k^0 - w_{k+1}^0}{w_{k+1}^0} = O\left(\frac{1}{N}\right),$$

and

$$\frac{N^2 y - a_k - w_k^0}{w_{k+1}^0} = \frac{y + 2f(t) - \sqrt{y^2 + 4yf(t)}}{y + 2f(t) + \sqrt{y^2 + 4yf(t)}} + O\left(\frac{1}{N}\right).$$

By the definition of $f(t)$ in (2.5), we observe that $0 \leq f(t) \leq f(p)$ for all $0 \leq t \leq p < 1/2$. Moreover, $y_p = -4p^2(p-1)^2/(2p-1)^2 = -4f(p)$. Note that $|u + \sqrt{u^2 - 1}| > 1$ for all u bounded away from $[-1, 1]$. If we take $u = [y + 2f(t)]/[2f(t)]$, then we have for any $y \in \mathbf{C} \setminus [-4f(p), 0]$,

$$\sup_{t \in [0,p]} \left| \frac{y + 2f(t) - \sqrt{y^2 + 4yf(t)}}{y + 2f(t) + \sqrt{y^2 + 4yf(t)}} \right| < 1.$$

Observe that the quantity inside the absolute-value signs above is $(u-\sqrt{u^2-1})/(u+\sqrt{u^2-1})$. Hence, there exist $\delta \in (0,1)$, $L > 0$ and $N_1 > L$ such that $|\varepsilon_1^1| \leq L/N$ and

$$\left| \frac{N^2 y - a_k - w_k^0}{w_{k+1}^0} \right| \leq 1 - \delta,$$

$$\left| \frac{w_k^0 - w_{k+1}^0}{w_{k+1}^0} \right| \leq \frac{\delta^2 L}{N},$$

$$\sup_{|s| \leq L/N} \frac{|1 - (1+s)^{-1}|}{|s|} \leq 1 + \delta,$$

for all $k = 1, \cdots, n$ and $N > N_1$. It is readily seen from (2.12) that if $|\varepsilon_k^1| \leq L/N$ for some $k \geq 1$, then

$$|\varepsilon_{k+1}^1| \leq \frac{\delta^2 L}{N} + \frac{(1-\delta)(1+\delta)L}{N} = \frac{L}{N}.$$

By induction, we have $|\varepsilon_k^1| \leq L/N$ for all $k = 1, \cdots, n$ and $N > N_1$.

To derive the explicit formula (2.11) for the first-order term w_k^1, we substitute $w_k = w_k^0(1 + w_k^1 + \varepsilon_k^2)$ into (2.2) to obtain a difference equation for the second-order error term:

$$(2.13) \qquad w_{k+1}^0(1 + w_{k+1}^1 + \varepsilon_{k+1}^2) = z - a_k - \frac{b_k}{w_k^0}(1 + w_k^1 + \varepsilon_k^2)^{-1}.$$

By (2.10), we can rewrite b_k/w_k^0 as $z - a_k - w_k^0$. Formally, let us for the moment treat $\varepsilon_k^2, \varepsilon_{k+1}^2, (w_k^1)^2$ and $w_{k+1}^1 - w_k^1$ as being of order $O(1/N^2)$. Equation (2.13) then becomes

$$w_{k+1}^0\left[1 + w_k^1 + O(1/N^2)\right] = z - a_k - (z - a_k - w_k^0)\left[1 - w_k^1 + O(1/N^2)\right].$$

Upon rearranging the terms, we obtain

$$(2.14) \qquad \left[w_{k+1}^0 - (z - a_k - w_k^0)\right]w_k^1 = w_k^0 - w_{k+1}^0 + O(1),$$

where we have made use of the facts that

$$z - a_k - w_k^0 = \frac{z - a_k - \sqrt{(z - a_k)^2 - 4b_k}}{2} = O(N^2)$$

and

$$w_{k+1}^0 = \frac{z - a_{k+1} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}}{2} = O(N^2).$$

The coefficient of w_k^1 on the left-hand side of (2.14) can be expressed as

$$w_{k+1}^0 - (z - a_k - w_k^0) = \frac{a_k - a_{k+1} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}} + \sqrt{(z - a_k)^2 - 4b_k}}{2}.$$

It is easily seen that the first square root on the right-hand side of the last equation is equal to

$$\sqrt{(z - a_k)^2 - 4b_k - 4(b_{k+1} - b_k) - 2(z - a_k)(a_{k+1} - a_k) + (a_{k+1} - a_k)^2}.$$

Using binomial expansion, we can also show that the above quantity is equal to

$$\sqrt{(z - a_k)^2 - 4b_k} - \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{\sqrt{(z - a_k)^2 - 4b_k}} + O(1).$$

Thus

$$\sqrt{(z-a_{k+1})^2-4b_{k+1}} = \sqrt{(z-a_k)^2-4b_k}$$
$$- \frac{2(b_{k+1}-b_k)+(z-a_k)(a_{k+1}-a_k)}{\sqrt{(z-a_k)^2-4b_k}} + O(1).$$

Coupling the last two equations gives

$$\frac{w_{k+1}^0-(z-a_k-w_k^0)}{\sqrt{(z-a_k)^2-4b_k}} = 1 + \frac{a_{k+1}-a_k}{2\sqrt{(z-a_k)^2-4b_k}}$$

(2.15)
$$+ \frac{\sqrt{(z-a_{k+1})^2-4b_{k+1}}-\sqrt{(z-a_k)^2-4b_k}}{2\sqrt{(z-a_k)^2-4b_k}}$$

$$= 1 + O\left(\frac{1}{N}\right)$$

and
(2.16)
$$w_k^0-w_{k+1}^0 = \frac{z-a_k+\sqrt{(z-a_k)^2-4b_k}}{2} - \frac{z-a_{k+1}+\sqrt{(z-a_{k+1})^2-4b_{k+1}}}{2}$$

$$= \frac{a_{k+1}-a_k}{2} + \frac{2(b_{k+1}-b_k)+(z-a_k)}{2\sqrt{(z-a_k)^2-4b_k}} + O(1).$$

Substituting (2.15) and (2.16) in (2.14) yields

$$\left[1+O(1/N)\right]w_k^1 = \frac{a_{k+1}-a_k}{2\sqrt{(z-a_k)^2-4b_k}}$$

$$+ \frac{2(b_{k+1}-b_k)+(z-a_k)(a_{k+1}-a_k)}{2\left[(z-a_k)^2-4b_k\right]}$$

$$+ \frac{O(1)}{\sqrt{(z-a_k)^2-4b_k}}.$$

By ignoring the $O(1/N)$ term in the last equation, we have at least formally derived the explicit formula in (2.11). This formula would have been proved rigorously, if we can show that the error term ε_k^2 is $O(1/N^2)$, which is what we will do next.

Now we return to equation (2.13). By adding and subtracting, the right-hand side of equation (2.13) becomes

$$w_k^0 + z - a_k - w_k^0 - (z-a_k-w_k^0)(1+w_k^1)^{-1}$$
$$+ (z-a_k-w_k^0)\left[(1+w_k^1)^{-1}-(1+w_k^1+\varepsilon_k^2)^{-1}\right].$$

Thus, we obtain

$$w_{k+1}^0(1+w_{k+1}^1+\varepsilon_{k+1}^2) = w_k^0 + (z-a_k-w_k^0)\left[1-(1-w_k^1)^{-1}\right]$$
$$+ (z-a_k-w_k^0)\left[(1+w_k^1)^{-1}-(1+w_k^1+\varepsilon_k^2)^{-1}\right].$$

Solving for ε_{k+1}^2 gives

$$\varepsilon_{k+1}^2 = \frac{w_k^0-w_{k+1}^0+(z-a_k-w_k^0)\left[1-(1+w_k^1)^{-1}\right]-w_{k+1}^0 w_{k+1}^1}{w_{k+1}^0}$$
(2.17)
$$+ \frac{z-a_k-w_k^0}{w_{k+1}^0}\left[(1+w_k^1)^{-1}-(1+w_k^1+\varepsilon_k^2)^{-1}\right].$$

We claim that the first term on the right-hand side of (2.17) is $O(1/N^2)$. By adding and subtracting, this term can be broken into three fractions

(2.18)
$$\frac{w_k^0 - w_{k+1}^0 + (z - a_k - w_k^0)w_k^1 - w_k^0 w_k^1}{w_{k+1}^0}$$
$$+ \frac{(z - a_k - w_k^0)\left[1 - w_k^1 - (1 + w_k^1)^{-1}\right]}{w_{k+1}^0} + \frac{w_k^0 w_k^1 - w_{k+1}^0 w_{k+1}^1}{w_{k+1}^0}.$$

Making use of $(2.3)-(2.4)$ and $(2.10)-(2.11)$, we have the following estimates for the last two fractions:

(2.19)
$$\frac{w_k^0 w_k^1 - w_{k+1}^0 w_{k+1}^1}{w_{k+1}^0} = \frac{(w_k^0 - w_{k+1}^0)}{w_{k+1}^0}w_k^1 + (w_k^1 - w_{K+1}^1)$$
$$= O\left(\frac{1}{N}\right)O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) = O\left(\frac{1}{N^2}\right)$$

and

(2.20)
$$\frac{(z - a_k - w_k^0)}{w_{k+1}^0}\left[1 - w_k^1 - (1 + w_k)^{-1}\right] = O(1)O\left(\frac{1}{N^2}\right) = O\left(\frac{1}{N^2}\right).$$

For the first fraction in (2.18), we have

(2.21)
$$\frac{w_k^0 - w_{k+1}^0 + (z - a_k - 2w_k^0)w_k^1}{w_{k+1}^0} = \frac{(z - a_k - 2w_k^0)}{w_{k+1}^0}\left[\frac{(w_k^0 - w_{k+1}^0)}{z - a_k - 2w_k^0} + w_k^1\right]$$
$$= O(1)\left[\frac{w_k^0 - w_{k+1}^0}{z - a_k - 2w_k^0} + w_k^1\right].$$

From (2.10) and (2.11), it also follows that the quantity inside the square brackets is equal to

$$\frac{-\sqrt{(z - a_k)^2 - 4b_k} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}}{2\sqrt{(z - a_k)^2 - 4b_k}} + \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{2\left[(z - a_k)^2 - 4b_k\right]}.$$

Upon simplification, the last quantity becomes

$$\frac{(a_k - a_{k+1})^2 + 2(z - a_k)(a_k - a_{k+1}) + 4(b_k - b_{k+1})}{2\sqrt{(z - a_k)^2 - 4b_k}\left[\sqrt{(z - a_k)^2 - 4b_k} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}\right]}$$
$$+ \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{2\left[(z - a_k)^2 - 4b_k\right]},$$

and it can be shown that it is equal to

$$O\left(\frac{1}{N^2}\right) + \frac{(z - a_k)(a_k - a_{k+1}) + 2(b_k - b_{k+1})}{\sqrt{(z - a_k)^2 - 4b_k}\left[\sqrt{(z - a_k)^2 - 4b_k} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}\right]}$$
$$+ \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{2\left[(z - a_k)^2 - 4b_k\right]}.$$

Further calculation shows that this quantity is equal to

$$O\left(\frac{1}{N^2}\right) + \left(\frac{2(b_k - b_k) + (z - a_k)(a_{k+1} - a_k)}{2\left[(z - a_k)^2 - 4b_k\right]}\right)$$
$$\times \left(\frac{-\sqrt{(z - a_k)^2 - 4b_k} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}}{\sqrt{(z - a_k)^2 - 4b_k} + \sqrt{(z - a_{k+1})^2 - 4b_{k+1}}}\right).$$

Since each factor in the above product is $O(1/N)$, the calculation from (2.17) on-wards gives

$$(2.22) \qquad \frac{w_k^0 - w_{k+1}^0 + (z - a_k - 2w_k^0)w_k^1}{w_{k+1}^0} = O\left(\frac{1}{N^2}\right).$$

A combination of (2.14), (2.15), (2.16) and (2.18) shows that the first term on the right-hand side of (2.13) is indeed $O(1/N^2)$, thus proving our claim made earlier.

Following a similar argument as in the estimation of the first-order error term, we can now find constants $\delta \in (0,1)$, $L > 0$ and $N_1 > L$ such that $|\varepsilon_1^2| \leqslant L/N^2$ and

$$\left|\frac{z - a_k - w_k^0}{w_{k+1}^0}\right| \leqslant 1 - \delta,$$

$$\left|\frac{w_k^0 - w_{k+1}^0 + (z - a_k - w_k^0)[1 - (1 + w_k^1)^{-1}] - w_{k+1}^0 w_{k+1}^1}{w_{k+1}^0}\right| \leqslant \frac{\delta^2 L}{N^2},$$

$$\sup_{|s| \leqslant L/N^2} \frac{\left|(1 + w_k^1)^{-1} - (1 + w_k^1 + s)^{-1}\right|}{|s|} \leqslant 1 + \delta,$$

for all $k = 1, \cdots, n$ and $N > N_1$. On account of (2.13), it is easily verified that if $|\varepsilon_k^2| \leqslant L/N^2$ for some $k \geqslant 1$, then

$$|\varepsilon_{k+1}^2| \leqslant \frac{\delta^2 L}{N^2} + \frac{(1 - \delta)(1 + \delta)L}{N^2} = \frac{L}{N^2}.$$

By induction, we have $|\varepsilon_k^2| \leqslant L/N^2$ for all $k = 1, \cdots, n$ and $N > N_1$. This completes the proof of Lemma 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

On account of $\pi_0 = 1$ in (1.4), we can multiply (2.1) from $k = 1$ to $k = n$ and arrive at $\pi_n = w_1 \cdots w_n$. By taking logarithms on both sides of this equation and using Lemma 2.1, we obtain

$$(2.23) \qquad \ln \pi_n = \sum_{k=1}^n \ln w_k = \sum_{k=1}^n \ln w_k^0 + \sum_{k=1}^n \ln(1 + w_k^1) + O\left(\frac{1}{N}\right)$$

$$= \sum_{k=1}^n \ln w_k^0 + \sum_{k=1}^n w_k^1 + O\left(\frac{1}{N}\right).$$

To find an asymptotic formula for π_n, we only need to approximate $\sum_{k=1}^n \ln w_k^0$ and $\sum_{k=1}^n w_k^1$, respectively. Applying (2.3) and (2.4) to (2.10) and (2.11) gives

$$\frac{2w_k^0}{N^2} = T(y;t) + \frac{T_1(y;t)}{N} + O\left(\frac{1}{N^2}\right),$$

and

$$Nw_k^1 = -\frac{f(t)h_0(t)}{\sqrt{y^2 + 4yf(t)}} - \frac{f(t)h_0(t)}{y + 4f(t)} + O\left(\frac{1}{N}\right),$$

where

$$(2.24) \qquad T(y;t) := y + 2f(t) + \sqrt{y^2 + 4yf(t)},$$

and

$$(2.25) \qquad T_1(y;t) := f(t)h_1(t) + \frac{yf(t)h_1(t) + 2f(t)^2 h_0(t)}{\sqrt{y^2 + 4yf(t)}}.$$

Recall that $n/N = p$. It is easily seen from trapezoidal's rule that

(2.26)
$$\sum_{k=1}^{n} \ln w_k^0 = n \ln \frac{N^2}{2} + N \int_0^p \ln T(y; t) dt + \frac{1}{2} \ln \frac{T(y; p)}{T(y; 0)} + \int_0^p \frac{T_1(y; t)}{T(y; t)} dt + O\left(\frac{1}{N}\right),$$

and

$$(2.27) \qquad \sum_{k=1}^{n} w_k^1 = - \int_0^p \frac{f(t) h_0(t)}{\sqrt{y^2 + 4yf(t)}} dt - \int_0^p \frac{f(t) h_0(t)}{y + 4f(t)} dt + O\left(\frac{1}{N}\right).$$

Finally, we have the following result.

Theorem 2.2. *Let $\pi_n(z)$ be the monic Racah polynomials with $\alpha + 1 = -N$. Assume $n/N = p$ is fixed in $(0, 1/2)$. Denote $y_p = -4p^2(p-1)^2/(2p-1)^2$. For any $y \in \mathbf{C} \setminus [y_p, 0]$, we have*

$$(2.28) \qquad \pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n e^{Ng(y)+r(y)} \left[1 + O\left(\frac{1}{N}\right)\right],$$

as $N \to \infty$, where the main function is given by

$$(2.29) \qquad g(y) := \int_0^p \ln T(y; t) dt,$$

and the correction term is given by
(2.30)
$$r(y) := \frac{1}{2} \ln \frac{T(y; p)}{T(y; 0)} + \int_0^p \frac{T_1(y; t)}{T(y; t)} dt - \int_0^p \frac{f(t) h_0(t)}{\sqrt{y^2 + 4yf(t)}} dt - \int_0^p \frac{f(t) h_0(t)}{y + 4f(t)} dt.$$

Here, the functions $f(t), h_1(t), h_2(t), h_0(t), T(y; t), T_1(y; t)$ are defined in (2.5)-(2.8) and (2.24)-(2.25), respectively.

Proof. Use a combination of (2.23), (2.26) and (2.27). $\qquad \square$

Next, we investigate the asymptotic behavior of $\pi_n(N^2 y)$ for y in the oscillatory interval $(y_p, 0)$. Recall that $0 < p < 1/2$. By the definition in (2.5), the function $f(t)$ is monotonically increasing for $t \in [0, p]$. Moreover, $0 = -4f(0)$ and $y_p = -4p^2(p-1)^2/(2p-1)^2 = -4f(p)$. Thus, for any $y \in (y_p, 0)$, there exists a unique $t_y \in (0, p)$ such that $y = -4f(t_y)$. Note that the functions $T(y, t)$ and $T_1(y, t)$ in (2.24)-(2.25) have a branch cut for $y \in (-4f(t), 0)$. We introduce the one-side limit functions for $-4f(t) < y < 0$ (or equivalently, $t_y < t < p$):

$$(2.31) \qquad T^\pm(y; t) := \lim_{\varepsilon \to 0^+} T(y \pm i\varepsilon; t) = y + 2f(t) \pm i\sqrt{-y[y + 4f(t)]},$$

and

$$(2.32) \qquad T_1^\pm(y; t) := \lim_{\varepsilon \to 0^+} T_1(y \pm i\varepsilon; t) = f(t) h_1(t) \mp i \frac{y f(t) h_1(t) + 2 f(t)^2 h_0(t)}{\sqrt{-y[y + 4f(t)]}}.$$

We remark that $T(y; t)$ and $T_1(y; t)$, treated as functions in y, can be analytically continued to the interval $y_p < y < -4f(t)$. For convenience, we still define the one-side limit functions $T^\pm(y; t) = T(y; t)$ and $T_1^\pm(y; t) = T_1(y; t)$ for $y_p < y < -4f(t)$

(i.e., $0 < t < t_y$). Also, we define the functions

$$S^{\pm}(y;t) := \lim_{\varepsilon \to 0^+} \sqrt{(y \pm i\varepsilon)[y \pm i\varepsilon + 4f(t)]}$$

(2.33)
$$= \begin{cases} \pm i\sqrt{-y[y + 4f(t)]}, & -4f(t) < y < 0, \\ -\sqrt{(-y)[-y - 4f(t)]}, & y_p < y < -4f(t). \end{cases}$$

It can be shown from (2.29)-(2.33) that

(2.34) $$g^{\pm}(y) := \lim_{\varepsilon \to 0^+} g(y \pm i\varepsilon; t) = \int_0^p \ln T^{\pm}(y;t) dt,$$

and

$$r^{\pm}(y) := \lim_{\varepsilon \to 0^+} r(y \pm i\varepsilon; t)$$

(2.35)
$$= \frac{1}{2}\ln \frac{T^{\pm}(y;p)}{T^{\pm}(y;0)} + \int_0^p \frac{T_1^{\pm}(y;t)}{T^{\pm}(y;t)} dt - \int_0^p \frac{f(t)h_0(t)}{S^{\pm}(y;t)} dt - \Omega^{\pm}(y),$$

where

(2.36) $$\Omega^{\pm}(y) = \text{P.V.} \int_0^p \frac{f(t)h_0(t)}{y + 4f(t)} dt \pm i\pi \frac{f(t_y)h_0(t_y)}{4f'(t_y)}.$$

In (2.36), "P.V." denotes the Cauchy principal value and f' denotes the derivative of f. In (2.35), the first three integrals are obtained by taking one-side limits, while the last term $\Omega^{\pm}(y)$ is derived from one-side limits of Stieltjes transform and Cauchy's theorem. Note that the last integral of $r(y)$ in (2.30) has a logarithmic singularity at $t = t_y$. We rewrite it as a Stieltjes integral

$$\Omega(y) := \int_0^p \frac{f(t)h_0(t)}{y + 4f(t)} dt = \int_0^p \frac{F(y;t)}{t - t_y} dt, \ y \in \mathbf{C} \setminus [y_p, 0],$$

where $F(y;t) := f(t)h_0(t)(t - t_y)/[y + 4f(t)]$. As a function of t, $F(y;t)$ can be continually defined at $t = t_y$. An application of Cauchy's theorem yields

$$\Omega^{\pm}(y) := \lim_{\varepsilon \to 0^+} \Omega(y \pm i\varepsilon) = \text{P.V.} \int_0^p \frac{F(y;t)}{t - t_y} dt \pm i\pi \lim_{t \to t_y} F(y;t)$$

for $y \in (y_p, 0)$. By l'Hôpital's rule,

$$\lim_{t \to t_y} F(y;t) = \lim_{t \to t_y} \frac{f(t)h_0(t)(t - t_y)}{y + 4f(t)} = f(t_y)h_0(t_y) \lim_{t \to t_y} \frac{t - t_y}{y + 4f(t)} = \frac{f(t_y)h_0(t_y)}{4f'(t_y)}.$$

This gives (2.36).

Our second result is stated below. Its proof is based on the observation that the real part of an asymptotic approximant for orthogonal polynomials in the complex plane is half of the corresponding asymptotic approximant in the interval of orthogonality on the real line.

Theorem 2.3. *Let $\pi_n(z)$ be the monic Racah polynomials with $\alpha + 1 = -N$. Assume $n/N = p$ is fixed in $(0, 1/2)$. For $y \in (y_p, 0)$ with $y_p = -4p^2(p - 1)^2/(2p - 1)^2$, we have*

(2.37)
$$\pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n \left\{ e^{Ng^+(y) + r^+(y)}\left[1 + O\left(\frac{1}{n}\right)\right] + e^{Ng^-(y) + r^-(y)}\left[1 + O\left(\frac{1}{n}\right)\right] \right\},$$

as $N \to \infty$, where $g^{\pm}(y)$ and $r^{\pm}(y)$ are defined as in (2.34) and (2.35), respectively.

Proof. First, we note that the functions $g^{\pm}(y)$ and $r^{\pm}(y)$ can be analytically continued to a small neighborhood of any compact subset of $(y_p, 0)$ in the complex plane. In such a neighborhood, we further observe that the difference

$$g^+(y) - g^-(y) = \int_{t_y}^{p} \ln \frac{T^+(y;t)}{T^-(y;t)} dt$$

has positive real part if y is in the upper-half plane and negative real part if y is in the lower-half plane. Thus, the function $e^{N[g^+(y)-g^-(y)]}$ is exponentially large on the upper-half plane and exponentially small on the lower-half plane, which together with (2.28) implies that the asymptotic formula (2.37) is valid on both upper- and lower-half planes. By taking limits, this formula is also valid for $y \in (y_p, 0)$.   □

3. CASE II: $\gamma + 1 = -N$

In this section, we consider the case $\gamma + 1 = -N$ and α, β, δ are fixed. It follows from (1.3) and (1.5) that

$$a_k = -\frac{(k+\alpha+1)(k+\alpha+\beta+1)(k+\beta+\delta+1)(k-N)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}$$
$$-\frac{k(k+N+\alpha+\beta+1)(k+\alpha-\delta)(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)};$$
$$b_k = \frac{k(k+N+\alpha+\beta+1)(k+\alpha-\delta)(k+\beta)(k+\alpha)(k+\alpha+\beta)(k+\beta+\delta)(k-N-1)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)^2(2k+\alpha+\beta+1)}.$$

Put $t := k/N$. We obtain for $0 < t < 1$,

(3.1)
$$\frac{a_k}{N^2} = -\frac{t^2}{2} - \frac{t(2\alpha+2\beta+2)-(\alpha+\beta+2\delta+2)}{4N} + O\left(\frac{1}{N^2}\right);$$
$$\frac{b_k}{N^4} = \frac{t^2(t^2-1)}{16} + \frac{(\alpha+\beta)t(t-1)(2t+1)-2t^2}{16N} + O\left(\frac{1}{N^2}\right),$$

and

(3.2)
$$\frac{a_{k+1}-a_k}{N} = -t + O\left(\frac{1}{N}\right); \quad \frac{b_{k+1}-b_k}{N^3} = \frac{t(2t^2-1)}{8} + O\left(\frac{1}{N}\right).$$

It is noted that $b_k < 0$ for large k and N such that $k < N$. Hence, Favard's theorem [3, §§1.4 & 1.5] is no longer applicable and it is expected that the zeros of $\pi_n(z)$ are not necessarily all real; see Figure 1.

As in the previous section, we introduce the ratios $w_k(z) = \pi_k(z)/\pi_{k-1}(z)$ for $k = 1, \cdots, n$. The corresponding difference equation remains the same:

$$w_{k+1}(z) = z - a_k - \frac{b_k}{w_k(z)}.$$

Let $z = N^2 y$. By using the same argument as in the proof of Lemma 2.1, we have

(3.3)
$$w_k = w_k^0[1 + w_k^1 + O(1/N^2)]$$

as $N \to \infty$, where the leading term is given by

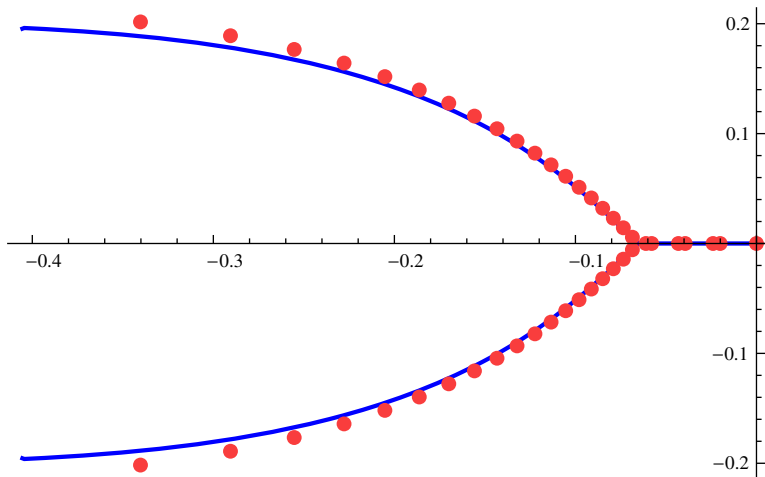$$w_k^0 = \frac{z - a_k + \sqrt{(z-a_k)^2 - 4b_k}}{2}$$

FIGURE 1. The zeros (red dots) of Racah polynomials $\pi_n(N^2 y)$ with $N = 50$, $n = 45$, $p = n/N = 0.9$, $\alpha = 0.3$, $\beta = 0.8$, $\delta = -1.2$, $\gamma = -N - 1 = -51$, and the branch cut Γ_p (blue curves) with $p = 0.9$ of the main function $g(y)$ defined in (3.8).

and the first-order term by

$$w_k^1 = \frac{a_{k+1} - a_k}{2\sqrt{(z - a_k)^2 - 4b_k}} + \frac{2(b_{k+1} - b_k) + (z - a_k)(a_{k+1} - a_k)}{2[(z - a_k)^2 - 4b_k]}.$$

On account of (3.1) and (3.2), we rewrite the above two equations as

$$\frac{2w_k^0}{N^2} = T(y; t) + \frac{T_1(y; t)}{N} + O\left(\frac{1}{N^2}\right),$$

and

$$Nw_k^1 = -\frac{t}{2\sqrt{y^2 + t^2(y + 1/4)}} - \frac{t(y + 1/4)}{2y^2 + 2t^2(y + 1/4)} + O\left(\frac{1}{N}\right),$$

where

(3.4)
$$T(y; t) := y + \frac{t^2}{2} + \sqrt{y^2 + yt^2 + \frac{t^2}{4}},$$

and

(3.5)

$$T_1(y; t)$$
$$:= \frac{[2t(\alpha + \beta + 1) - (\alpha + \beta + 2\delta + 2)]T(y; t) - (\alpha + \beta)t(t - 1)(t + 1/2) + t^2}{4\sqrt{y^2 + yt^2 + t^2/4}}.$$

Recall that $n/N = p$. It is easily seen from trapezoidal's rule that

(3.6)
$$\sum_{k=1}^n \ln w_k^0 = n \ln \frac{N^2}{2} + N \int_0^p \ln T(y; t) dt + \frac{1}{2} \ln \frac{T(y; p)}{T(y; 0)} + \int_0^p \frac{T_1(y; t)}{T(y; t)} dt + O\left(\frac{1}{N}\right),$$

and

$$(3.7) \qquad \sum_{k=1}^{n} w_k^1 = \frac{y - \sqrt{y^2 + p^2(y + 1/4)}}{2(y + 1/4)} - \frac{1}{4} \ln \frac{y^2 + p^2(y + 1/4)}{y^2} + O\left(\frac{1}{N}\right).$$

Finally, we introduce the main function

$$(3.8) \qquad g(y) := \int_0^p \ln T(y; t) dt,$$

and define the curve Γ_p to be the set of complex numbers y at which the real part of $g^+(y) - g^-(y)$ vanishes, where $g^{\pm}(y)$ are the one-sided limits of $g(y)$ on the curve Γ_p; see Figure 1.

Theorem 3.1. *Let $\pi_n(z)$ be the monic Racah polynomials with $\gamma = -N - 1$. Assume $n/N = p$ is fixed in $(0, 1)$. For any $y \in \mathbf{C} \setminus \bar{\Gamma}_p$, where $\bar{\Gamma}_p$ is the closure of Γ_p in the complex plane, we have as $N \to \infty$,*

$$(3.9) \qquad \pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n e^{Ng(y) + r(y)} \left[1 + O\left(\frac{1}{N}\right)\right],$$

where the main function $g(y)$ is given in (3.8) and the correction term is given by

$$(3.10) \qquad r(y) := \frac{1}{2} \ln \frac{T(y; p)}{T(y; 0)} + \int_0^p \frac{T_1(y; t)}{T(y; t)} dt$$

$$+ \frac{y - \sqrt{y^2 + p^2(y + 1/4)}}{2(y + 1/4)} - \frac{1}{4} \ln \frac{y^2 + p^2(y + 1/4)}{y^2}.$$

Here, the functions $T(y; t)$ and $T_1(y; t)$ are defined in (3.4) and (3.5), respectively. Let $r^{\pm}(y)$ denote the one-sided limits of $r(y)$ on Γ_p. For any $y \in \Gamma_p$, we have as $N \to \infty$,

$$(3.11)$$
$$\pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n \left\{e^{Ng^+(y) + r^+(y)} \left[1 + O\left(\frac{1}{n}\right)\right] + e^{Ng^-(y) + r^-(y)} \left[1 + O\left(\frac{1}{n}\right)\right]\right\}.$$

Proof. Just as in (2.23), we have

$$\ln \pi_n = \sum_{k=1}^{n} w_k = \sum_{k=1}^{n} \ln w_k^0 + \sum_{k=1}^{n} w_k^1 + O\left(\frac{1}{N}\right).$$

Substituting (3.6) and (3.7) in the above equation gives (3.9). Formula (3.11) follows from a similar argument as that in the proof of Theorem 2.3 and from the choice of the branch cut Γ_p. $\qquad \square$

4. Case III: $\beta + \delta + 1 = -N$

In this section, we consider the case $\beta + \delta + 1 = -N$ and α, δ are fixed. There are two subcases.

Subcase III.1: $\beta = -N - \delta - 1$ and α, δ, γ are fixed.
Subcase III.2: $\delta = -N - \beta - 1$ and α, β, γ are fixed.

For the subcase III.1, we introduce two parameters $\tilde{\alpha} := \beta + \delta = -N - 1$ and $\tilde{\beta} = \alpha - \delta$. Note that the $_4F_3$ hypergeometric function in the definition of Racah

polynomials (1.1) is symmetric with respect to the switch of parameters:

$$
{}_4F_3\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & \gamma+1 \end{matrix}\,\middle|\,1\right)
$$
$$
={}_4F_3\left(\begin{matrix} -n, & n+\tilde{\alpha}+\tilde{\beta}+1, & -x, & x+\gamma+\delta+1 \\ & \tilde{\alpha}+1, & \tilde{\beta}+\delta+1, & \gamma+1 \end{matrix}\,\middle|\,1\right).
$$

Thus, the subcase III.1 is the same as case I considered in Section 2.

For the subcase III.2, we introduce two parameters $\tilde{\gamma} := \beta+\delta = -N-1$ and $\tilde{\delta} = \gamma-\beta$. Note that the ${}_4F_3$ hypergeometric function in the definition of Racah polynomials (1.1) is symmetric with respect to the switch of parameters:

$$
{}_4F_3\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & \gamma+1 \end{matrix}\,\middle|\,1\right)
$$
$$
={}_4F_3\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\tilde{\gamma}+\tilde{\delta}+1 \\ & \alpha+1, & \beta+\tilde{\delta}+1, & \tilde{\gamma}+1 \end{matrix}\,\middle|\,1\right).
$$

That is, the subcase III.2 is the same as case II considered in Section 3.

Acknowledgment

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