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BALANCED SUBDIVISIONS AND FLIPS ON SURFACES

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Abstract. In this paper, we show that two balanced triangulations of a closed surface are not necessarily connected by a sequence of balanced stellar subdivisions and welds. This answers a question posed by Izmestiev, Klee and Novik. We also show that two balanced triangulations of a closed surface are connected by a sequence of three local operations, which we call the pentagon contraction, the balanced edge subdivision and the balanced edge weld. In addition, we prove that two balanced triangulations of the 2-sphere are connected by a sequence of pentagon contractions and their inverses if none of them are the octahedral sphere.

1. Introduction

A classical result in combinatorial topology [Al] shows that two PL-homeomorphic simplicial complexes are connected by a sequence of stellar subdivisions and their inverses. A closely related result is Pachner's result [Pa1, Pa2] which shows that two PL-homeomorphic combinatorial manifolds are connected by a sequence of bistellar flips (see also [Li] for the proofs of both results). A combinatorial d-manifold is a triangulation of a d-manifold all of whose vertex links are PL $(d-1)$-spheres. A combinatorial d-manifold is said to be *balanced* if its graph is $(d+1)$-colorable. Recently, Izmestiev, Klee and Novik [IKN] proved an analogue of Pachner's result for balanced combinatorial manifolds. They introduced a version of bistellar flips that preserves the balanced property, which they call *cross-flips*, and proved that two PL-homeomorphic balanced combinatorial manifolds are connected by a sequence of cross-flips. In this paper, we study the following questions related to their result in the special case of triangulated surfaces:

- There is an analogue of stellar subdivisions for balanced simplicial complexes, called *balanced stellar subdivisions* (see [IKN, §2.5]). Are two PL-homeomorphic balanced combinatorial manifolds connected by a sequence of balanced stellar subdivisions and their inverses?
- It is known that not all cross-flips are necessary to connect any two PL-homeomorphic balanced combinatorial manifolds. How many different types of cross-flips are indeed necessary?

A *triangulation* G of a closed surface F^2 is a simple graph embedded on the surface such that each face of G is bounded by a 3-cycle and any two faces share at

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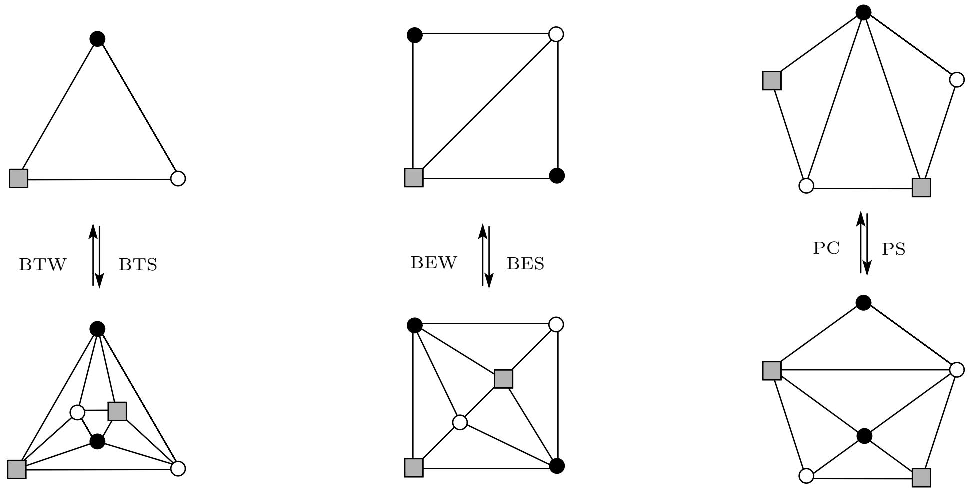

FIGURE 1. Six necessary cross-flips.

most one edge. (Combinatorially, considering such an embedded graph is equivalent to considering a simplicial complex which is homeomorphic to F^2.) By a result of Izmestiev, Klee and Novik [IKN, Theorem 1.1], two different balanced triangulations of a fixed closed surface are connected by a sequence of cross-flips. A cross-flip in dimension d is an operation that exchanges a shellable and co-shellable d-ball in the boundary of the $(d+1)$-dimensional cross-polytope with its complement (see [IKN] for the precise definition). In dimension 2, there are 9 different types of cross-flips, but it is known that only 6 flips, described in Figure 1, are necessary (see [IKN, Remark 3.9]). Note that, in Figure 1, it is not allowed to make a double edge by the operations and each triangle must be a face. In this paper, we call these six operations, a balanced triangle subdivision (BT-subdivision or BTS), a balanced triangle weld (BT-weld or BTW), a balanced edge subdivision (BE-subdivision or BES), a balanced edge weld (BE-weld or BEW), a pentagon splitting (P-splitting or PS) and a pentagon contraction (P-contraction or PC). A BT-subdivision (resp., -weld) and a BE-subdivision (resp., -weld) are collectively referred to as *balanced subdivisions* (resp., *-welds*). Izmestiev, Klee and Novik [IKN, Problem 3] asked if balanced subdivisions and balanced welds suffice to transform any balanced triangulation of a closed surface into any other balanced triangulation of the same surface. We answer this question.

Theorem 1.1. *For every closed surface F^2, there are balanced triangulations G and G' of F^2 such that G' cannot be obtained from G by a sequence of balanced subdivisions and welds.*

Next, we consider how many different types of cross-flips are necessary. The above result shows that at least a P-contraction or a P-splitting is necessary. Then since we can apply neither a P-contraction nor a P-splitting to the octahedral sphere (the boundary of the 3-dimensional cross-polytope), we at least need three different cross-flips to transform any balanced triangulation of the 2-sphere to any other balanced triangulation of the 2-sphere. We show that a result of Kawarabayashi, Nakamoto and Suzuki [KNS] implies the following result which guarantees that three flips are indeed enough.

Theorem 1.2. *Any two balanced triangulations of a closed surface F^2 can be transformed into each other by a sequence of BE-subdivisions, BE-welds and P-contractions.*

As we mentioned, the set of three moves in the theorem is minimal possible. However, somewhat surprisingly, we show in Theorem 4.3 that most balanced triangulations of a fixed closed surface are actually connected by only P-splittings and P-contractions. In particular, we prove the following stronger statement for the 2-sphere.

Theorem 1.3. *Any two balanced triangulations of the 2-sphere except the octahedral sphere can be transformed into each other by a sequence of P-splittings and P-contractions.*

This paper is organized as follows. In the next section, we introduce some operations defined for bipartite graphs, and show a key lemma to prove our main theorem. Section 3 is devoted to proving our first main result in the paper. In Section 4, we discuss how many different types of cross-flips are sufficient to connect given two balanced triangulations of a closed surface.

2. Operations for bipartite graphs

In this section, we consider bipartite graphs which are not necessarily embedded on surfaces, and prove the key lemma to prove our first main theorem.

We first introduce some notation. In the paper, we consider simple graphs. Let G be a simple graph. We denote by $V(G)$ the vertex set of G. The *degree* of the vertex v in G is the number of edges of G that contains v. The *minimal degree* of G is the minimum of degrees of vertices of G. An edge on vertices a and b will be denoted by ab and a face on vertices a, b and c will be denoted by abc. A graph G is *d-colorable* if there is a map $c : V(G) \to \{1, 2, \dots, d\}$ such that $c(v) \neq c(u)$ for any edge uv of G. A 2-colorable graph is called a *bipartite graph*. For bipartite graphs, we define the following three operations: Let H be a bipartite graph.

- (I) Add a pendant edge vw with $v \in V(H)$ and $w \notin V(H)$. (A pendant edge is an edge such that one of its vertex has degree 1.)
- (II) Replace an edge $e = uv$ of H with three edges up, pq, qv, where p and q are new vertices.
- (III) Add a vertex $w \notin V(H)$ and two incident edges xw, wy, where x and y have distance 2 in H (i.e., xy is not an edge of H and there is a vertex z such that xz and yz are edges of H).

The inverse operations of the above (I), (II) and (III) are represented by (I'), (II') and (III'), respectively (see Figure 2). In particular, we call (II) the *subdivision* of uv and call (II') the *smoothing* of the edges up, pq, qv. Note that each of these six operations preserves the bipartiteness of the graph.

A set of two adjacent vertices $\{p, q\}$ of degree 2 in a bipartite graph H is said to be *smoothable* if it is possible to apply (II') to H, which removes vertices p and q; that is, there exists no cycle of length 4 containing p and q. Furthermore, a vertex w of degree 2 in a bipartite graph H is said to be *removable* if we can remove the vertex w by applying (III'); that is, there exists a 4-cycle in H containing w.

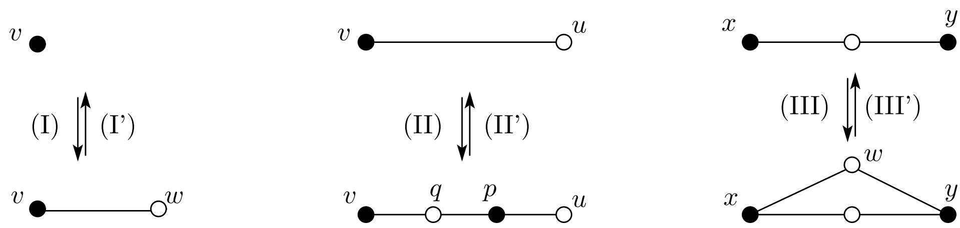

FIGURE 2. Six operations defined for bipartite graphs.

The following lemma plays an important role when we prove our main theorem in the next section.

Lemma 2.1. *Let H be a bipartite graph with minimum degree at least* 3*. If H' can be obtained from H by a sequence of operations (I), (II), (III), (I'), (II') and (III'), then H' can be obtained from H by a sequence of operations (I), (II) and (III).*

Proof. In the following argument, we say that a bipartite graph is *configurable* from H (by at most t steps) if it can be obtained from H by applying operations (I), (II) and (III) (at most t times).

Let H' be a graph obtained from H by a sequence of operations (I), (II), (III), (I'), (II') and (III'). Then, there is a sequence of bipartite graphs $H = H_0, H_1, \dots, H_t = H'$ such that H_{i+1} is obtained from H_i by one of the six operations for $i = 0, \dots, t-1$, as shown in the following diagram:

$$H = H_0 \xrightarrow{o_1} H_1 \xrightarrow{o_2} H_2 \longrightarrow \cdots \xrightarrow{o_{t-2}} H_{t-2} \xrightarrow{o_{t-1}} H_{t-1} \xrightarrow{o_t} H_t = H'.$$

We claim that $H_t = H'$ is configurable from H by at most t steps. We proceed by induction on t. Since any vertex of H has degree at least 3, o_1 must be (I), (II) or (III). Thus the assertion is obvious when $t = 1$. Suppose $t \geq 2$. To prove the desired assertion, it only suffices to show the case when each of $o_1, \dots, o_{t-1}$ is one of (I), (II) and (III), and o_t is one of (I'), (II') and (III').

Case 1. Suppose that o_t is (I'), which removes a vertex w and an edge vw from H_{t-1}. If w is not a vertex of H_{t-2}, then o_{t-1} is (I), which adds w and vw. In this case, it is clear that $H_{t-2} = H_t$ and hence H_t is configurable. Thus, we assume that w is a vertex of H_{t-2}. Since none of (I), (II) and (III) decrease the degrees of vertices, w has degree 1 in H_{t-2}. Let v' denote the unique neighbor of w in H_{t-2}.

First, suppose that $v \neq v'$. In this case, o_{t-1} is (II), which subdivides $v'w$, and hence a graph isomorphic to H_t can be obtained from H_{t-2} by adding a pendant edge to w by applying (I) (see Figure 3). Next, we suppose that $v = v'$. We delete a vertex w from H_{t-2} and denote the resulting graph by H'_{t-2}. Since $H_{t-2} \to H'_{t-2}$ is an operation (I'), by the induction hypothesis, H'_{t-2} is configurable from H by at most $t-1$ steps. Furthermore, since o_{t-1} is not (II), which subdivides wv, we can apply the same operation as o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t. Therefore, H_t is configurable from H also in this case.

Case 2. Suppose that o_t is (II'), which replaces edges up, pq, qv with uv. First, suppose that both p and q are vertices of H_{t-2} and $\{p, q\}$ is smoothable in H_{t-2}. Let u' and v' denote the vertices such that $u'p, pq, qv'$ are edges of H_{t-2}. We apply

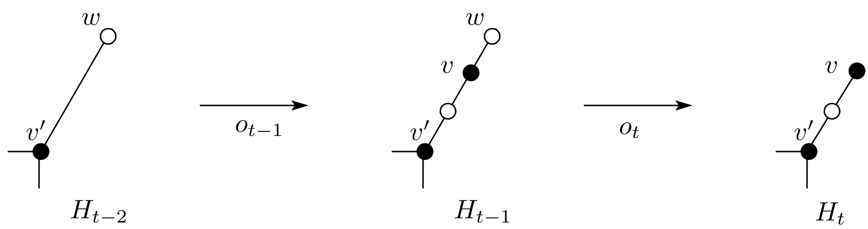

FIGURE 3. Configurations in Case 1 in the proof of Lemma 2.1.

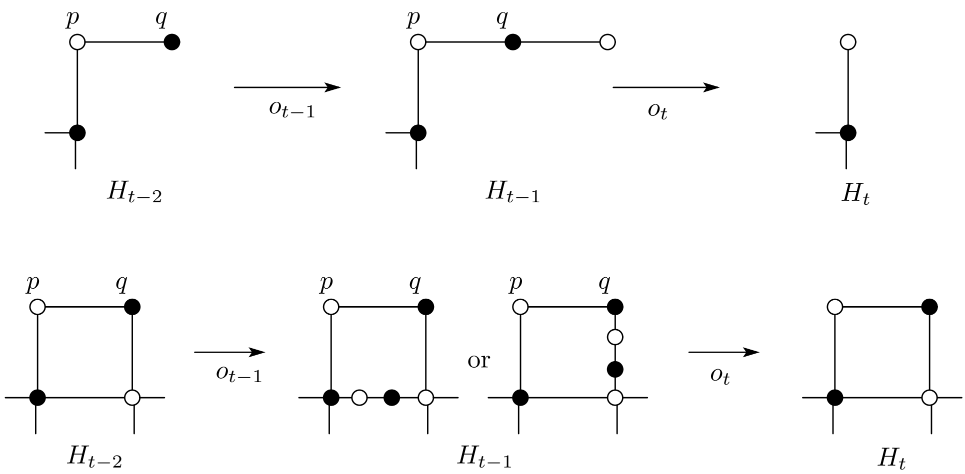

FIGURE 4. Configurations of (C) in Case 2 in the proof of Lemma 2.1.

(II'), which replaces $u'p, pq, qv'$ with $u'v'$ to H_{t-2} and denote the resulting graph by H'_{t-2}. By the induction hypothesis, H'_{t-2} is configurable from H by at most $t-1$ steps. If o_{t-1} is not (II), which subdivides either $u'p$ or qv', then we can apply o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t. On the other hand, if o_{t-1} is (II), which subdivides either $u'p$ or qv', then H_t and H_{t-2} are clearly isomorphic. In either case, H_t is configurable from H by at most t steps.

By the above argument, we only need to discuss the case when at least one of p and q is not a vertex of H_{t-2} or $\{p, q\}$ is not smoothable in H_{t-2}. We divide the argument into three cases (2A), (2B) and (2C) depending on the situation.

(2A) Neither p nor q is a vertex of H_{t-2}: In this case, o_{t-1} is clearly an operation adding p and q, that is, o_{t-1} is (II), which subdivides uv in H_{t-2}. It is easy to see that $H_{t-2} = H_t$.

(2B) p is a vertex of H_{t-2} but q is not of H_{t-2}: Note that there exists no cycle of length 4 containing p and q in H_{t-1} since $\{p, q\}$ is smoothable in H_{t-1}. Under the condition, q must be added by o_{t-1}, and we can conclude that o_{t-1} is (II), which subdivides an edge incident to p. (If o_{t-1} is (III), then p and q would lie on a 4-cycle in H_{t-1}.) As a result, H_{t-2} is isomorphic to H_t and hence H_t is configurable from H.

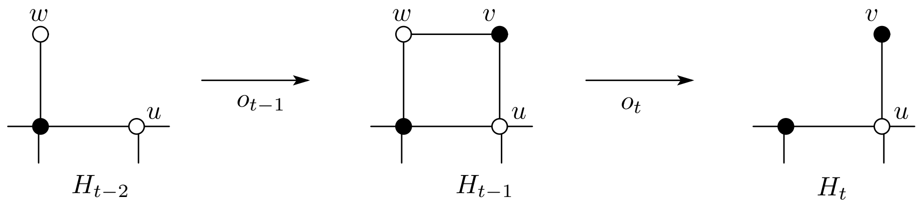

FIGURE 5. Configurations of (3B) in Case 3 in the proof of Lemma 2.1.

(2C) Both of p and q are the vertices of H_{t-2}: Here note that p and q are adjacent and have degree at most 2 in H_{t-2} since each of (I), (II) and (III) does not decrease the degrees of vertices and does not join two non-adjacent vertices. If one of p and q, say q, has degree 1, then o_{t-1} is (I), which adds an edge incident to q, since (III) would generate a 4-cycle containing p and q. In this case, a graph isomorphic to H_t can be obtained from H_{t-2} by deleting q using operation (I'), and hence H_t is configurable from H by the induction hypothesis (see the upper diagram in Figure 4). On the other hand, if each of p and q has degree 2, then there exists a 4-cycle containing p and q in H_{t-2} under our assumption. Since $\{p, q\}$ is smoothable in H_{t-1}, o_{t-1} is (II), which subdivides an edge on the 4-cycle. In any case, H_{t-2} and H_t is isomorphic to each other (see the bottom diagram in Figure 4).

Case 3. Suppose that o_t is (III') deleting a vertex w of degree 2 and two edges xw and yw. Note that H_{t-1} must have a 4-cycle that contains w. First assume that w is a vertex of H_{t-2} and is removable in H_{t-2}. Let x' and y' denote the vertices adjacent to w in H_{t-2}. Now, since there exists a 4-cycle containing w in H_{t-1}, o_{t-1} is not (II), which subdivides $x'w$ or wy'. Thus, we have $\{x', y'\} = \{x, y\}$. We delete w from H_{t-2} by applying (III') and denote the resulting graph by H'_{t-2}. Since o_{t-1} is not (II) subdividing xw or wy, we can apply the same operation as o_{t-1} to H'_{t-2} and obtain a graph isomorphic to H_t. By the induction hypothesis, H_t is configurable from H by at most t steps.

By the above argument, we may assume that w is not a vertex of H_{t-2} or w is not removable in H_{t-2}. It suffices to discuss the following three cases (3A), (3B) and (3C).

(3A) w is not a vertex of H_{t-2}: Clearly, w must be added by o_{t-1}. Since H_{t-1} has a 4-cycle containing w, o_{t-1} cannot be (II); that is, o_{t-1} is (III). Then, it is easy to see that $H_{t-2} = H_t$.

(3B) w has degree 1 in H_{t-2}: In this case, o_{t-1} is clearly (III). We assume that o_{t-1} adds a vertex v and edges wv and vu (see Figure 5). We remove w from H_{t-2} by applying (I'), and denote the resulting graph by H'_{t-2}. By the induction hypothesis, H'_{t-2} is configurable from H by at most $t-1$ steps. Furthermore, H_t is obtained from H'_{t-2} by (I), which adds an edge incident to u. Thus, H_t is also configurable from H by at most t steps.

(3C) w is a vertex of degree 2 in H_{t-2}: Denote two vertices adjacent to w in H_{t-2} by x' and y'. By our assumption, w is not removable in H_{t-2}, that is, there exists no cycle of length 4 containing w. On the other hand, w is removable and there exists such a 4-cycle in H_{t-1}. To satisfy these conditions, o_{t-1} must be (III),

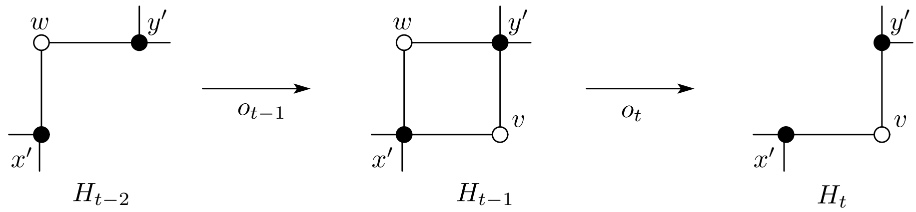

FIGURE 6. Configurations of (3C) in Case 3 in the proof of Lemma 2.1.

which adds a vertex v and two edges $x'v$ and vy'. However, it is easy to see that H_{t-2} is isomorphic to H_t (see Figure 6).

Now, we have considered all cases and hence the lemma follows. □

3. Proof of Theorem 1.1

An *even embedding* $\mathcal{H}$ of a graph H on a closed surface F^2 is an embedding of H on F^2 such that each face is bounded by a cycle of even length. We call H the *underlying graph* of $\mathcal{H}$. For an even embedding $\mathcal{H}$ of F^2, its *face subdivision*, denoted by $S(\mathcal{H})$, is the triangulation of F^2 obtained from $\mathcal{H}$ by adding a new vertex into each face of $\mathcal{H}$ and joining all vertices on the corresponding boundary cycle. If H is 2-colorable, then, since no vertices of $S(\mathcal{H})$ which are not vertices of H are adjacent, $S(\mathcal{H})$ is a balanced triangulation. Conversely, for any balanced triangulation G of F^2, we can obtain a 2-colorable even embedding $\mathcal{H}$ of F^2 such that $G = S(\mathcal{H})$ by removing vertices of one color from G. We denote by $e(G)$ the number of edges of a graph G. Let $\mathcal{K}$ and $\mathcal{K}'$ be even embeddings of a closed surface F^2 and let K and K' be the underlying graphs of $\mathcal{K}$ and $\mathcal{K}'$ respectively. Since $|V(S(\mathcal{H}))|$ equals the sum of the number of vertices of $\mathcal{H}$ and the number of faces of $\mathcal{H}$, by Euler's formula, one has $|V(S(\mathcal{K}))| > |V(S(\mathcal{K}'))|$ if and only if $e(K) > e(K')$.

For each closed surface F^2, there are infinitely many graphs of minimal degree at least 3 that admit 2-colorable even embeddings on F^2. Hence the next result proves Theorem 1.1.

Theorem 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be 2-colorable even embeddings of a closed surface F^2 and let H and K be the underlying graphs of $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that H and K have minimal degree at least 3. If $e(H) \neq e(K)$, then $S(\mathcal{K})$ cannot be obtained from $S(\mathcal{H})$ by a sequence of balanced subdivisions and welds.*

Proof. We may assume $e(H) > e(K)$, and in particular $|V(S(\mathcal{H}))| > |V(S(\mathcal{K}))|$. Let $G = S(\mathcal{H})$ and let $G' \neq G$ be a balanced triangulation of F^2 which can be obtained from G by a sequence of balanced subdivisions and welds. To prove the desired statement, it is enough to prove that $|V(G)| \leq |V(G')|$.

Since balanced subdivisions and welds preserve balancedness, there is an even embedding $\mathcal{H}'$ on F^2 with the underlying graph H' such that $G' = S(\mathcal{H}')$. This H' can be obtained from H by a sequence of operations shown in Figure 7, which come from balanced subdivisions and welds. Furthermore, each operation in Figure 7 can be realized by a combination of the operations (I), (II), (III), (I'), (II') and (III'). Indeed, the first operation is a combination of (I) and (III) (or their inverses), the second one is (III) or (III'), and the third one is (II) or (II'). Then, by Lemma 2.1, the bipartite graph H' can be obtained from H by a sequence of operations (I),

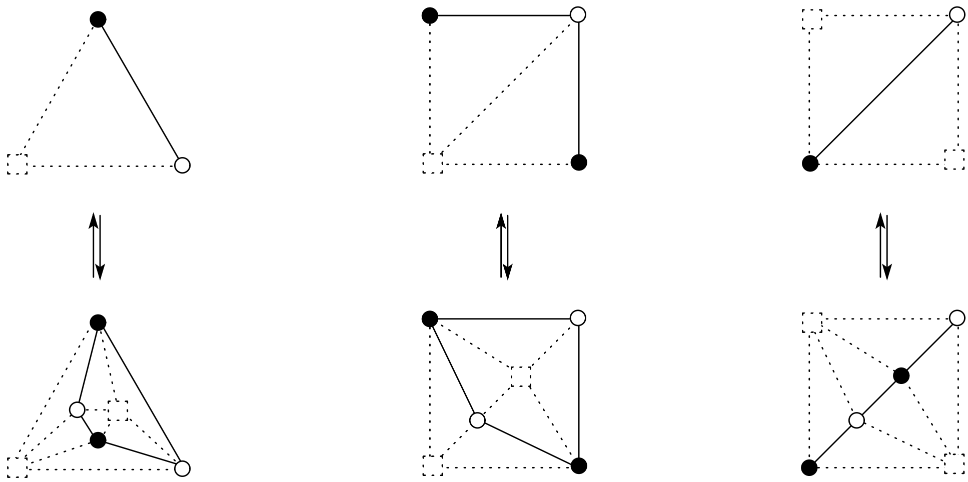

FIGURE 7. Corresponding operations in H.

(II) and (III). In particular, we have $e(H') \geq e(H)$. Then, since $G = S(\mathcal{H})$ and $G' = S(\mathcal{H}')$, we have $|V(G)| \leq |V(G')|$ as desired. □

Remark 3.2. The proof of Theorem 3.1 says that, in the theorem, if we assume $e(H) > e(K)$, then we do not need to assume that K has minimal degree ≥ 3. For example, if $S(\mathcal{H})$ is the face subdivision of the cube and $S(\mathcal{K})$ is the octahedral sphere, then $S(\mathcal{K})$ cannot be obtained from $S(\mathcal{H})$ by a sequence of balanced subdivisions and welds.

4. Necessary operations for balanced triangulations

In this section, we discuss how many different types of cross-flips are necessary. We first introduce operations called an *N-flip* and a *P_2-flip* originally defined in [NSS], as shown in Figure 8. (An N-flip is also found in cross-flips in [IKN, Figure 1].) Note that it is not allowed to make a double edge by the operations and each triangle in Figure 8 must be a face. Using those operations, Kawarabayashi et al. [KNS] proved the following theorem.

Theorem 4.1 (Kawarabayashi, Nakamoto and Suzuki [KNS])**.** *For any closed surface F^2, there exists an integer M such that any two balanced triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geq M$ can be transformed into each other by a sequence of N- and P_2-flips.*

We now prove Theorem 1.2 in the introduction, saying that BE-subdivisions, BE-welds and P-contractions are enough.

Proof of Theorem 1.2. Clearly, a P_2-flip can be replaced with a combination of a BE-subdivision and a BE-weld. Furthermore, an N-flip is replaced with a sequence of BE-subdivisions, P-contractions and a single BE-weld, as shown in Figure 9. Since a BE-subdivision increases the number of the vertices by two and a P-contraction decreases the number of the vertices by one, the desired assertion follows from Theorem 4.1. □

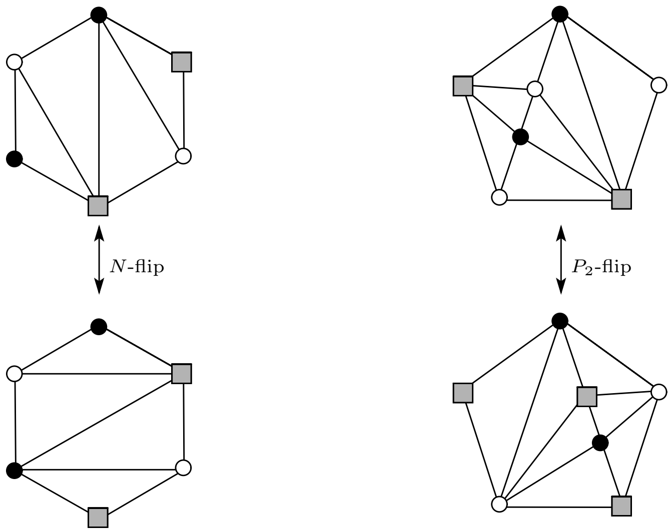

FIGURE 8. N-flip and P_2-flip.

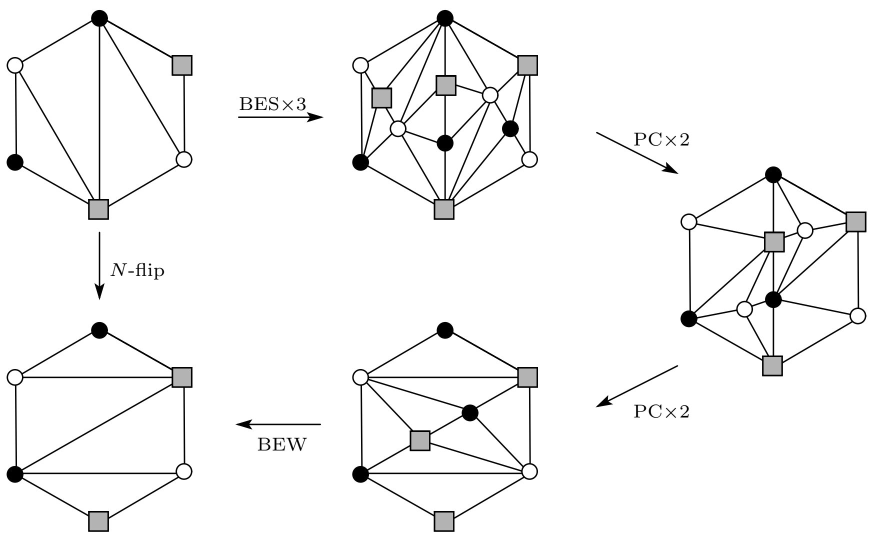

FIGURE 9. An N-flip realized by a sequence of other operations.

Next, we show that most balanced triangulations of a fixed closed surface F^2 are connected by a sequence of P-contractions and P-splittings. The following simple fact can be observed from Figure 10.

Lemma 4.2. *Let G and G' be balanced triangulations of a closed surface F^2 such that G' is obtained from G by applying the BE-subdivision to the edge v_0v_1 in G. Let xv_0v_1 and yv_0v_1 be the faces of G that contain v_0v_1 and let $u \neq v_0$ be the vertex such that xv_1u is a face of G. If uy is not an edge of G, then G' is obtained from G by a sequence of P-splittings.*

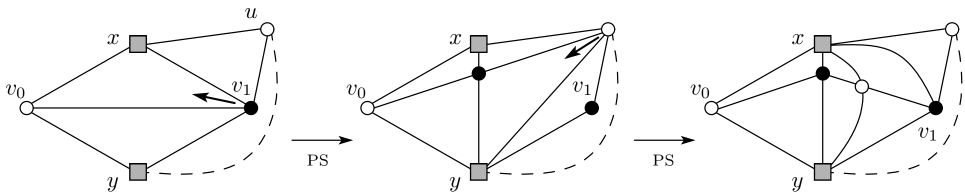

FIGURE 10. Two PSs corresponding to a BES.

Theorem 4.3. *For each closed surface F^2, with finitely many exceptions, any two balanced triangulations of F^2 can be transformed into each other by a sequence of P-splittings and P-contractions.*

Proof. First, observe that each of the BE-subdivisions and BE-welds applied in Figure 9 satisfies the assumption of Lemma 4.2. Hence any N-flip can be replaced by a sequence of P-splittings and P-contractions. Similarly, a P_2-flip can be replaced with a combination of P-splittings and P-contractions by Lemma 4.2. This observation also implies that if we can apply either an N-flip or a P_2-flip to a balanced triangulation, then we can apply a P-splitting.

Now, let M be an integer given by Theorem 4.1, and let G and G' be balanced triangulations of F^2 with $|V(G')| \geq |V(G)| \geq M$. By taking M sufficiently large, we may assume that there is a balanced triangulation $H \neq G$ of F^2 with $|V(H)| = |V(G)|$. By Theorem 4.1, G can be transformed into H by a sequence of N- and P_2-flips. This implies that we can apply a P-splitting to G. After applying a P-splitting to G, we obtain a balanced triangulation of F^2 with $|V(G)|+1$ vertices, and we can apply a P-splitting to this new triangulation again. Thus we can repeat applying P-splittings until the number of vertices becomes $|V(G')|$; denote the resulting graph by G_0. By Theorem 4.1 and the above argument G_0 and G' can be transformed into each other by a sequence of P-splittings and P-contractions. Therefore, we conclude that G and G' are connected by only P-splittings and P-contractions. Then the assertion follows since there exist only finitely many balanced triangulations of F^2 with the number of vertices less than M. □

It would be natural to ask what the exceptions are in Theorem 4.3. Let F^2 be a closed surface. The proof of Theorem 4.3 says that there is an integer M such that any two balanced triangulations having at least M vertices are connected by a sequence of P-splittings and P-contractions. We say that a balanced triangulation G of F^2 is *exceptional* if G cannot be connected to a balanced triangulation G' of F^2 with $|V(G')| \geq M$ by a sequence of P-splittings and P-contractions (this condition does not depend on a choice of M). If we can apply a P-splitting to G, that is, there is a graph G' such that G' is obtained from G by a P-splitting, then we can again apply a P-splitting to G'. Thus if we can apply a P-splitting to G, then G is not exceptional. Also, if it is possible to apply a P-contraction to G, then it is also possible to apply a P-splitting to G. Thus we have the following criterion.

Proposition 4.4. *A balanced triangulation G is not exceptional if and only if G has faces vwx, vxy, vyz such that wz is not an edge of G.*

We think that exceptional balanced triangulations are quite rare. Indeed, for the 2-sphere we have the following result, which proves Theorem 1.3.

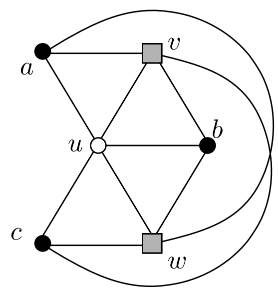

FIGURE 11. A configuration in the proof of Theorem 4.5.

Theorem 4.5. *The octahedral sphere is the only exceptional balanced triangulation of the* 2*-sphere.*

Proof. Let G be an exceptional balanced triangulation of the 2-sphere. Since the octahedral sphere is the only triangulation of the 2-sphere all whose vertices have degree 4, it suffices to show that every vertex of G has degree 4.

Suppose to the contrary that G has a vertex u whose degree is strictly greater than 4. Let a, b, c, v, w be vertices of G such that uav, uvb, ubw, uwc are faces of G (see Figure 11). By applying Proposition 4.4 to faces uav, uvb, ubw, we have that aw is an edge of G. Similarly, applying Proposition 4.4 to faces uvb, ubw, uwc implies that vc is an edge of G. Then G contains the complete bipartite graph $K_{3,3}$ as a subgraph with vertices partitioned as $\{a, b, c\}$ and $\{u, v, w\}$. This contradicts the planarity of G. □

We close the paper with a few remarks and one question.

Remark 4.6. In Theorem 4.1, it is also true that there is a sequence of N-flips and P_2-flips that transform G into G' and a given coloring of G into a given coloring of G' (this can be seen from the first paragraph of the proof of [KNS, Theorem 3]). Thus, like [IKN, Theorem 1.1], this stronger property is also true in Theorems 1.2 and 4.3.

Remark 4.7. There are balanced triangulations of surfaces whose underlying graph is the complete tripartite graph $K_{n,n,n}$ for any integer $n \geq 2$ [ESZ, RY, Wh]. By Proposition 4.4, these triangulations are exceptional. Steven Klee (personal communication) shows that the unique balanced triangulation of the torus whose graph is $K_{3,3,3}$ is the only exceptional balanced triangulation of the torus. This shows that a statement similar to Theorem 1.3 holds for the torus.

Remark 4.8. Any two balanced triangulations of a closed surface F^2 can be transformed into each other by a sequence of BT-subdivisions, BT-welds, P-contractions and P-splittings. Indeed, Figure 12 shows that one can replace BE-subdivisions and BE-welds with combinations of BT-subdivisions, BT-welds, P-splittings and P-contractions.

Remark 4.9. It was asked in [IKN, Problem 4] if two even triangulations of the same combinatorial manifold M with the same coloring monodoromy are connected by cross-flips. Since Theorem 4.1 also holds for even triangulations having the

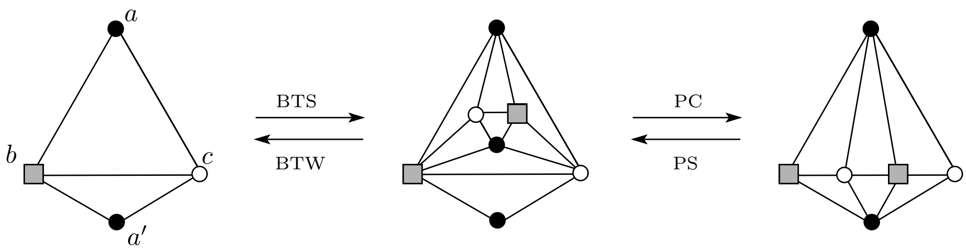

FIGURE 12. Replacement of a BEW and a BES with the other four operations.

same monodoromy, the answer to this problem is yes for closed surfaces. Also, Theorems 1.2 and 4.3 hold in this generality.

Question 4.10. Is there a generalization of Theorem 1.2 (or Theorem 4.3) in higher dimensions?

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