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GROUPS WITH LOCALLY MODULAR HOMOGENEOUS PREGEOMETRIES ARE COMMUTATIVE

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ABSTRACT. It is well known that strongly minimal groups are commutative. Whether this is true for various generalisations of strong minimality has been asked in several different settings. In this note we show that the answer is positive for groups with locally modular homogeneous pregeometries.

1. Introduction

The basic setting of geometric stability theory is a strongly minimal theory where algebraic closure induces a pregeometry. In the literature there are several approaches in generalising this basic setting.

One approach is to ask for every definable set to be countable or cocountable (instead of finite or cofinite in strongly minimal structures). An uncountable structure satisfying this condition is called *quasiminimal*.

Another option is to substitute the finite/cofinite dichotomy by large/small dichotomy with respect to some ultrafilter on the boolean algebra of definable sets (i.e. a type). So given a type $p \in \mathrm{S}(M)$ on a structure M, one can look at the operator associating to $A \subseteq M$ the union of all small A-definable sets (i.e. those that are not in p). The type p is called *strongly regular* if this operator is a closure operator (i.e. a pregeometry possibly without exchange).

The third possibility is to simply require the existence of a homogeneous pregeometry on a structure M. The pregeometry $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ is called *homogeneous* if M is infinite dimensional with respect to cl and for every finite $A \subset M$ and $b, c \in M \setminus \mathrm{cl}(A)$ there is an automorphism f of M (and of cl) fixing A pointwise with $f(b) = c$.

By a well-known result of [Rei75] all strongly minimal groups are commutative. It has been asked whether the same is true in each of the above three settings (in [Mae07] for quasiminimality, in [PT11] for regular types and in [Hyt02, HLS05] for groups with homogeneous pregeometries). Reineke's argument has been generalised to show that if a noncommutative group has one of the above properties, then it must have the strict order property, so in particular is unstable. (See also [GK14] which, as a step towards a potential counterexample, constructs an uncountable group where all nontrivial elements are conjugate and have countable centralisers.)

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However not much is known beyond this. The main result of this paper is the following:

Theorem 1. *A group with a locally modular homogeneous pregeometry is commutative.*

Using the connections between quasiminimality, strong regularity and homogeneous pregeometries (see [PT11]), we have an analogous statement for quasiminimal groups of cardinality at least $\aleph_2$ and groups with symmetric strongly regular types, whose respective pregeometries are locally modular.

The initial idea of the proof was suggested to the author by Alfonso Ruiz Guido. It used the representation theorem of projective geometries and required a stronger homogeneity hypothesis. Ivan Tomasic suggested to eliminate the use of the representation theorem by finding the contradiction explicitly. Doing that helped relax the homogeneity hypothesis. The author expresses his gratitude to them and to Rahim Moosa for encouraging the author to write this.

2. Background on pregeometries

Although the motivation for Theorem 1 comes from model theory, its statement and proof involve none. In order to make the presentation accessible to a wider audience, we provide a brief introduction to pregeometries.

Definition 2. A *pregeometry* (also called *matroid*) on a set X is an operator $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions for all $A \subseteq X$ and $a, b \in X$:

- Reflexivity: $A \subseteq \mathrm{cl}(A)$.
- Transitivity: $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
- Finite character: $a \in \mathrm{cl}(A)$ if and only if $a \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$.
- Exchange: $a \in \mathrm{cl}(A, b) \setminus \mathrm{cl}(A)$ implies $b \in \mathrm{cl}(A, a)$.

(Here and throughout we may use comma for union and omit the brackets for singletons, so that e.g. $\mathrm{cl}(A, b)$ means $\mathrm{cl}(A \cup \{b\})$.)

On any set X the operator cl defined by $\mathrm{cl}(A) = A$ is a pregeometry. Other examples of pregeometries include a vector space V where $\mathrm{cl}(A)$ is the linear span of A and an algebraically closed field F where $\mathrm{cl}(A)$ is the algebraic closure of the field generated by A.

A subset $A \subseteq X$ is called *independent* if $a \notin \mathrm{cl}(A \setminus a)$ for every $a \in A$. It is called *generating* if $\mathrm{cl}(A) = X$ and a *basis* if it is both independent and generating. The exchange axiom can be used to show that all bases have the same cardinality which is then called the *dimension* of X and denoted by $\dim(X)$.

Given $Y \subseteq X$ we can define the *restriction* cl^Y of cl onto Y. This is a pregeometry on Y defined by $\mathrm{cl}^Y(A) = \mathrm{cl}(A) \cap Y$. We also define the *localisation* cl_Y of cl on Y. The localisation is a pregeometry on X defined by $\mathrm{cl}_Y(A) = \mathrm{cl}(A \cup Y)$. By $\dim(Y)$ we mean the dimension of cl^Y and by $\dim(Y/Z)$ we mean the dimension of Y in cl_Z.

A pregeometry cl is called a *geometry* if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(a) = \{a\}$ for every $a \in A$. The idea is that closed subsets of dimension 1, 2, 3, etc., correspond to points, lines, planes, etc. (Note that what we call the dimension is 1 more than the geometric dimension.) In such a geometry one can only talk about incidence however it turns out to be surprisingly rich.

There is a canonical way of associating a geometry to the pregeometry cl on X. We define an equivalence relation $\sim$ on $\hat{X} = X \setminus \mathrm{cl}(\emptyset)$ by $a \sim b$ iff $\mathrm{cl}(a) = \mathrm{cl}(b)$. Then the operator $\hat{\mathrm{cl}}$ defined as $\hat{\mathrm{cl}}(A/\sim) = \{b/\sim : b \in \mathrm{cl}(A)\}$ is a geometry on $\hat{X}$.

Let cl be a pregeometry on X. We distinguish the following types of pregeometries:

- The pregeometry is called *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$. In the associated geometry this means that every set is closed.
- It is called *modular* if $\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B)$ for all finite dimensional closed A and B. The pregeometry associated with the linear closure in a vector space is modular.
- The pregeometry is called *locally modular* if the above modularity equation holds provided $\dim(A \cap B) > 0$. This is equivalent to cl_a being modular for every $a \in X \setminus \mathrm{cl}(\emptyset)$. An example of locally modular pregeometry that is not modular is the affine closure in a vector space. (The closed sets are translates of linear subspaces.)
- A geometry that is modular but not trivial is called *projective*. A classical result in projective geometry is that every projective geometry of dimension at least 4 comes from a vector space over a division ring.

A pregeometry on a structure M is a pregeometry cl on the base set of M such that every automorphism of M is an automorphism of cl (i.e. preserves closed sets). The pregeometry cl on M is called *homogeneous* if $\dim(M)$ is infinite and for every finite $A \subset M$ and $b, c \in M \setminus \mathrm{cl}(A)$ there is an automorphism of M fixing A pointwise and taking b to c.

3. Main result

From now on let G be a group (possibly with extra structure) and $\mathrm{cl} : \mathcal{P}(G) \to \mathcal{P}(G)$ be a homogeneous pregeometry in the above sense. We call a tuple $\bar{a} \in G^n$ *generic* over $A \subset G$ if $\dim(\bar{a}/A) = n$. For $A, B \subseteq G$ we say that B is A-*invariant* if B is fixed setwise by all automorphisms of G that fix A pointwise.

We start with some simple observations.

Proposition 3. *If a is A-invariant for some finite $A \subset G$, then $a \in \mathrm{cl}(A)$.*

Proof. Indeed, pick $c \in G$ generic over $A \cup a$. This is possible since G is infinite dimensional. If $a \notin \mathrm{cl}(A)$, then there is an automorphism fixing A and taking a to c. □

In particular if a is first-order definable from A, then $a \in \mathrm{cl}(A)$. We will mostly use the proposition in this form.

Corollary 4. *Let $A \subset G$ be a finite subset. If a is generic over $A \cup b$, then $a \cdot b$ is also generic over $A \cup b$.*

Proof. Indeed, a is definable from b and $a \cdot b$. Hence $a \in \mathrm{cl}(A, b, a \cdot b)$. So in particular $a \cdot b \notin \mathrm{cl}(A, b)$. □

As another corollary we can exclude one type of pregeometry.

Corollary 5. *The pregeometry (G, cl) is not trivial.*

Proof. Pick a generic pair a, b (i.e. $\dim(a, b) = 2$) and consider $a \cdot b$. By the above $a \cdot b \notin \mathrm{cl}(a) \cup \mathrm{cl}(b)$. However $a \cdot b \in \mathrm{cl}(a, b)$ demonstrates that cl is not trivial. □

Another consequence is that proper invariant (in particular definable) subgroups are small.

Proposition 6. *Let A be a finite subset and $H \leq G$ an A-invariant subgroup. If H contains an element generic over A, then $H = G$.*

Proof. Let $b \in G$ be an arbitrary element. Pick a generic over $A \cup b$. Since H is A-invariant and contains a generic element, we have that $a \in H$. (Recall that there is an automorphism over A mapping one generic element to another.) By Proposition 3, $a \cdot b$ is also generic over A. Hence as above $a \cdot b \in H$. Finally since H is a subgroup, we have that $b \in H$. □

As a corollary we get the following criterion for a group with homogeneous pregeometry to be commutative.

Proposition 7. *If a generic pair commutes in a group with a homogeneous pregeometry, then the group is commutative.*

Proof. Let (a, b) be a generic pair and assume that a and b commute. Consider the centraliser $C_G(a)$ of a. It is definable over a and contains a generic element b. So by the previous proposition, we have $C_G(a) = G$. By homogeneity the same is true for every generic element. Now given $c \in \mathrm{cl}(\emptyset)$, by the above it commutes with a. Hence $C_G(c)$ is c-definable and contains a generic element and therefore must be the whole group G. □

We will apply the following improved criterion.

Proposition 8. *Let (a, b) be a generic pair over some finite A. If $b \cdot a \in \mathrm{cl}(A, a \cdot b)$, then G is commutative.*

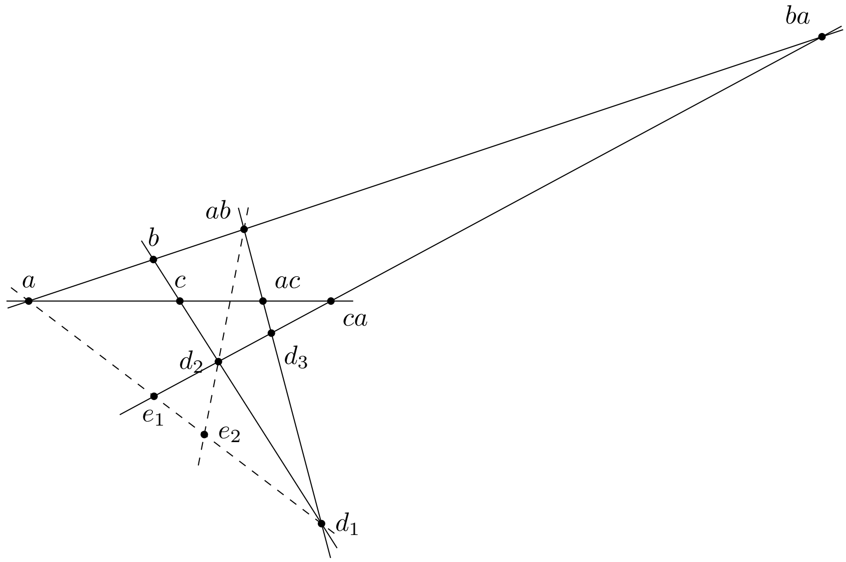
Proof. Assume that $b \cdot a \in \mathrm{cl}(A, a \cdot b)$. By homogeneity it is true for every generic pair over A. Now $(b, b^{-1} \cdot a)$ is a generic pair over A. And so $b^{-1}ab \in \mathrm{cl}(A, a)$. Now consider the set $\{x \in G : x^{-1}ax = b^{-1}ab\}$. It is definable over $A, a, b^{-1}ab$ and contains a generic element b. Hence it contains all generic elements over A, a. Pick c generic over A, a, b. Then $c^{-1}ac = b^{-1}ab$ and therefore $(bc^{-1})a = a(bc^{-1})$. Since (a, bc^{-1}) is a generic pair, by the above G is commutative. □

So far we have not used local modularity of the pregeometry. So everything said so far also applies to groups with arbitrary homogeneous pregeometry. We now use the full strength of our assumptions for the main result.

Theorem 1. *A group with locally modular homogeneous pregeometry is commutative.*

Proof. Assume that cl is a locally modular homogeneous pregeometry on a group G. Pick a finite $A \not\subseteq \mathrm{cl}(\emptyset)$. Then the localisation cl_A of cl at A is modular. (Recall that cl_A is defined as $\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X)$.) Consider a generic pair (a, b) over A. By the previous proposition $b \cdot a \in \mathrm{cl}_A(a \cdot b)$ implies that G is commutative. So assume that $b \cdot a \notin \mathrm{cl}_A(a \cdot b)$. Pick an element $c \notin \mathrm{cl}_A(a, b)$. Similarly we assume that $c \cdot a \notin \mathrm{cl}_A(a \cdot c)$. Consider the *geometry* of the operator cl_A on $\mathrm{cl}_A(a, b, c)$. Since the geometry is modular but not trivial (Corollary 5), it is a projective plane. (This essentially means that any two lines intersect.)

We claim that the lines (b, c), (ab, ac) and (ba, ca) meet in a single point. Assume the opposite. Then these three lines meet in three distinct points d_1, d_2 and d_3 as shown in the diagram below.



Now consider the line (a, d_1) and assume that it meets the lines (ba, ca) and (ab, d_2) in points e_1 and e_2 respectively. By the homogeneity assumption there is an automorphism f of G that fixes a, e_1, e_2, d_1 and takes b to c. (Note that e_1, e_2, d_1 are points in the geometry so are equivalence classes of elements of G. What we mean by f fixing them is f fixing a representative.) Then f induces a collineation of the projective plane (which we again denote by f). Since f is an automorphism of G it takes ab to ac and ba to ca respectively. Then f fixes the lines (b, c) and (ba, ca) (since it also fixes two points d_1 and e_1 on them). Thus f fixes their intersection point d_2. But this implies that f fixes the line (d_2, e_2). However ab is on that line whereas ac is not. This contradiction proves the claim.

But now note that $c^{-1}b \in \mathrm{cl}_A(b, c) \cap \mathrm{cl}_A(ab, ac)$ and $bc^{-1} \in \mathrm{cl}_A(b, c) \cap \mathrm{cl}_A(ba, ca)$. Thus $c^{-1}b \in \mathrm{cl}_A(bc^{-1})$. Since (b, c^{-1}) is a generic pair over A, by Proposition 8 the group G is commutative. □

As a final remark, note that under the assumption of local modularity the geometry of cl on G (after localisation) is isomorphic to the projective geometry of some vector space over a division ring. The group structure of G however need not come from this vector space. Examples of locally modular strongly minimal groups that are not vector spaces can be found in [Pil96].

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