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PRODUCT ANOSOV DIFFEOMORPHISMS AND THE TWO-SIDED LIMIT SHADOWING PROPERTY

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ABSTRACT. We characterize product Anosov diffeomorphisms in terms of the two-sided limit shadowing property. It is proved that an Anosov diffeomorphism is a product Anosov diffeomorphism if and only if any lift to the universal covering has the unique two-sided limit shadowing property. Then we introduce two maps in a suitable Banach space such that fixed points of these maps are related with shadowing orbits on the universal covering.

1. PRODUCT ANOSOV DIFFEOMORPHISMS

An *Anosov diffeomorphism* defined in a smooth manifold M is a smooth diffeomorphism $f\colon M \to M$ satisfying:

(1) for every $x \in M$ there is a splitting $T_xM = E^s(x) \oplus E^u(x)$ which is invariant under the derivative map $Df(x)\colon T_xM \to T_{f(x)}M$, that is,

$$Df(x)(E^s(x)) = E^s(f(x)) \text{ and } Df(x)(E^u(x)) = E^u(f(x)).$$

We call $E^s(x)$ the *stable space* of x and $E^u(x)$ is called the *unstable space* of x.

(2) There exist a Riemannian metric in M and a constant $0 < \lambda < 1$ such that

$$|Df^k(x)(v)|_{f^k(x)} \leq \lambda^k |v|_x \text{ and } |Df^{-k}(x)(w)|_{f^{-k}(x)} \leq \lambda^k |w|_x$$

for all $v \in E^s(x)$, $w \in E^u(x)$, $k \in \mathbb{Z}$ and $x \in M$, where $|.|_x$ denotes the norm in T_xM induced by the Riemannian metric. This metric is said to be *adapted* to f.

Such systems have been intensely studied since the works of Anosov [1] and Smale [13] in the sixties. They introduced several examples of Anosov diffeomorphisms and stated some questions about them that have been not answered yet (to our best knowledge). The central problem of this theory is to understand all examples of Anosov diffeomorphisms (up to topological conjugacy). Smale conjectured that every Anosov diffeomorphism must be topologically conjugated to an Anosov automorphism on an infra-nilmanifold. The following properties are expected to be true for an Anosov diffeomorphism $f\colon M \to M$:

(1) M is an infra-nilmanifold and the universal covering is the Euclidean space,
(2) the lift of f to $\mathbb{R}^n$ is topologically conjugated to a hyperbolic matrix,
(3) the stable and unstable foliations are on global product structure,

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(4) there exists a fixed point of f,
(5) if M is compact and connected, then f is transitive.

We recall some definitions. We denote by M a closed and connected smooth n-dimensional manifold, by $\widetilde{M}$ its universal covering and by $\pi\colon \widetilde{M} \to M$ the covering projection. Consider a Riemannian metric $\langle .,. \rangle$ in M and the distance d induced by this metric. We can lift this metric to a Riemannian metric in $\widetilde{M}$ (which we also denote by $\langle .,. \rangle$) as follows: for $x \in \pi^{-1}(x_0)$ and $v, w \in T_x\widetilde{M}$ we define

$$\langle v, w \rangle_x = \langle D\pi(x)(v), D\pi(x)(w) \rangle_{x_0},$$

where $\langle .,. \rangle_{x_0}$ is the inner product in $T_{x_0}M$. We also denote by d the distance in $\widetilde{M}$ induced by the lifted metric. By definition, the covering map π is a local isometry. Hence, there exists $\varepsilon_0 > 0$ such that for each $x \in \widetilde{M}$, π maps the ε_0-neighborhood of x isometrically onto the ε_0-neighborhood of $\pi(x)$.

For an Anosov diffeomorphism $f_0\colon M \to M$ we consider $f\colon \widetilde{M} \to \widetilde{M}$ any lift of f_0 to the universal covering. Since π is a local diffeomorphism, the derivative map $D\pi(x)\colon T_x\widetilde{M} \to T_{x_0}M$ is a linear isomorphism and the splitting $T_{x_0}M = E^s(x_0) \oplus E^u(x_0)$ can be lifted to a splitting $T_x\widetilde{M} = E^s(x) \oplus E^u(x)$ that is invariant by $Df(x)$. If the adapted metric is lifted, then it is easy to check that f is an Anosov diffeomorphism and that the lifted metric is adapted to f.

The sets

$$W^s(x_0) = \{y \in M;\ d(f^k(y), f^k(x_0)) \to 0,\ k \to \infty\} \quad \text{and}$$
$$W^u(x_0) = \{y \in M;\ d(f^k(y), f^k(x_0)) \to 0,\ k \to -\infty\}$$

are called the *stable set* and the *unstable set* of the point x_0, respectively. For an Anosov diffeomorphism these sets are leafs of two respective foliations which we call *stable foliation* and *unstable foliation*. We denote by $\widetilde{W}^s(x_0)$ and by $\widetilde{W}^u(x_0)$ the lift of the stable and unstable leaves, respectively, to the universal covering. Actually, $\widetilde{W}^s(x_0)$ ($\widetilde{W}^u(x_0)$) is the stable (unstable) set of x with respect to the lifted Anosov diffeomorphism.

Definition 1.1. The stable and the unstable foliations are on *global product structure* if for every $x, y \in M$ the leaves $\widetilde{W}^s(x)$ and $\widetilde{W}^u(y)$ intersect in exactly one point in the universal covering. If this is the case, we say that f_0 is a *product Anosov diffeomorphism.*

When M is compact and connected, the following three classes of diffeomorphisms are expected to be the same:

- Anosov diffeomorphisms,
- transitive Anosov diffeomorphisms,
- product Anosov diffeomorphisms.

Product Anosov diffeomorphisms are transitive but the converse of this statement is not known (see J. Franks' thesis [8]). In [4] transitive Anosov diffeomorphisms are characterized in terms of the *two-sided limit shadowing property.* Recall that a homeomorphism f defined in a metric space (X, d) is *transitive* if for every pair (U, V) of non-empty open subsets of X there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. We say that f has the two-sided limit shadowing property if for every sequence $(x_k)_{k\in\mathbb{Z}} \subset X$ satisfying

$$d(f(x_k), x_{k+1}) \to 0, \quad |k| \to \infty,$$

there exists $z \in X$ satisfying

$$d(f^k(z), x_k) \to 0, \quad |k| \to \infty.$$

The sequence $(x_k)_{k\in\mathbb{Z}}$ is called a *two-sided limit pseudo-orbit* and we say that it is *two-sided limit shadowed* (see [2], [4], [5] and [6] for more information on this property). The following is proved in [4] but we state it here since it is not stated as a theorem in [4]:

Theorem 1.2. *An Anosov diffeomorphism, defined in a compact and connected manifold, is transitive if and only if it has the two-sided limit shadowing property.*

Our first result characterizes product Anosov diffeomorphisms in terms of the two-sided limit shadowing property. We say that f has the *unique two-sided limit shadowing property* when every two-sided limit pseudo-orbit is two-sided limit shadowed by a *single point*.

Theorem A. *An Anosov diffeomorphism is a product Anosov diffeomorphism if and only if any lift to the universal covering has the unique two-sided limit shadowing property.*

So the following three classes of diffeomorphisms are expected to be the same:

- Anosov diffeomorphisms,
- Anosov diffeomorphisms with the two-sided limit shadowing property,
- Anosov diffeomorphisms with the unique two-sided limit shadowing property on the universal covering.

We note that the uniqueness of the shadowing point and also the uniqueness of the intersection between the lifted stable and unstable leaves is a problem apart from the existence of these points. Though it is an interesting problem, on this note we will focus on the existence of these points.

2. Lifting shadowing properties

In [9] (and also in [10]) it is proved that a homeomorphism $f\colon M \to M$ has the *shadowing property* if and only if any lift of f to the universal covering also has it. We prove this also holds for the *limit shadowing property*. First, we state some definitions. For any number $\delta > 0$ we say that a sequence $(x_k)_{k\in\mathbb{Z}} \subset M$ is a *δ-pseudo-orbit* if

$$d(f(x_k), x_{k+1}) < \delta, \quad k \in \mathbb{Z}.$$

This sequence is *ε-shadowed* by a point $z \in M$ if

$$d(f^k(z), x_k) < \varepsilon, \quad k \in \mathbb{Z}.$$

We say that f has the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed. A sequence $(x_k)_{k\in\mathbb{N}} \subset M$ is a *limit pseudo-orbit* if

$$d(f(x_k), x_{k+1}) \to 0, \quad k \to \infty.$$

This sequence is *limit shadowed* if there exists $z \in M$ such that

$$d(f^k(z), x_k) \to 0, \quad k \to \infty.$$

We say that f has the limit shadowing property if every limit pseudo-orbit is limit shadowed. Information about these properties can be found in [12].

Lemma 2.1. *If f_0 is a homeomorphism defined in a compact manifold M and f is any lift of f_0 to the universal covering $\widetilde{M}$, then f has the limit shadowing property if and only if f_0 also has it.*

Proof. Suppose that f has the limit shadowing property and consider $(x_k)_{k\in\mathbb{N}} \subset M$ a limit pseudo-orbit for f_0. Choose $N \in \mathbb{N}$ such that

$$d(f_0(x_k), x_{k+1}) < \varepsilon_0, \quad k \geq N.$$

The number ε_0 was chosen in the first section. Note that from the definition of ε_0 we have

$$\varepsilon_0 < \min\{d(\tilde{x}, \tilde{y});\ x \in M,\ \tilde{x}, \tilde{y} \in \pi^{-1}(x),\ \tilde{x} \neq \tilde{y}\}.$$

Thus, for each choice of $y_N \in \pi^{-1}(x_N)$ there exists a unique limit pseudo-orbit $(y_k)_{k\geq N}$ of f such that $y_k \in \pi^{-1}(x_k)$ and $d(f(y_k), y_{k+1}) < \varepsilon_0$ for every $k \geq N$. Since f has the limit shadowing property, there exists $z \in \widetilde{M}$ that limit shadows $(y_k)_{k\geq N}$. Therefore $\pi(f^{-N}(z))$ limit shadows $(x_k)_{k\in\mathbb{N}}$. Since this holds for every limit pseudo-orbit, it follows that f_0 has the limit shadowing property.

Now suppose that f_0 has the limit shadowing property and consider $(x_k)_{k\in\mathbb{N}} \subset \widetilde{M}$ a limit pseudo-orbit for f. Since π is a local isometry, the sequence $(\pi(x_k))_{k\in\mathbb{N}} \subset M$ is a limit pseudo-orbit of f_0 and, hence, it is limit shadowed by some point $z \in M$. Let $K \in \mathbb{N}$ be such that

$$d(f_0^k(z), \pi(x_k)) < \varepsilon_0, \quad k \geq K.$$

There is a unique point $\tilde{z} \in \pi^{-1}(f_0^K(z))$ satisfying $d(\tilde{z}, x_K) < \varepsilon_0$. We will prove that $w = f^{-K}(\tilde{z})$ limit shadows $(x_k)_{k\in\mathbb{N}}$. Indeed, $f^k(w) \in \pi^{-1}(f_0^k(z))$ for every $k \in \mathbb{N}$, and, moreover, $d(f^k(w), x_k) < \varepsilon_0$ if $k \geq K$. It follows that

$$d(f^k(w), x_k) = d(f_0^k(z), \pi(x_k)), \quad k \geq K,$$

since π is an isometry in every ball $B(x_k, \varepsilon_0)$. This equality proves that w limit shadows $(x_k)_{k\in\mathbb{N}}$, because z limit shadows $(\pi(x_k))_{k\in\mathbb{N}}$. Since this holds for every limit pseudo-orbit, it follows that f has the limit shadowing property. □

Proof of Theorem A. We first prove that if the lift f of an Anosov diffeomorphism f_0 to the universal covering has the unique two-sided limit shadowing property, then f_0 is a product Anosov diffeomorphism. For each pair $(x, y) \in \widetilde{M} \times \widetilde{M}$ consider the two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_k = \begin{cases} f^k(y), & k \geq 0, \\ f^k(x), & k < 0. \end{cases}$$

Since f has the unique two-sided limit shadowing property, there exists a unique point $z \in \widetilde{M}$ that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. In particular, $z \in W^u(x) \cap W^s(y)$. Moreover, $W^u(x) \cap W^s(y) = \{z\}$, since $w \in W^u(x) \cap W^s(y)$ implies that w two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$, which, in turn, implies that $w = z$. This proves that f has the global product structure.

Now we prove the converse statement. Let f be any lift of f_0 to the universal covering and $(x_k)_{k\in\mathbb{Z}}$ be any two-sided limit pseudo-orbit of f. Since f_0 is an Anosov diffeomorphism, it has the limit shadowing property (see [12] for a proof) and the previous lemma assures that f also has it. Then there exist two points $z_1, z_2 \in \widetilde{M}$ satisfying

$$d(f^k(z_1), x_k) \to 0, \quad k \to -\infty \quad \text{and} \quad d(f^k(z_2), x_k) \to 0, \quad k \to \infty.$$

Since f_0 is a product Anosov diffeomorphism there exists a unique point $z \in W^u(z_1) \cap W^s(z_2)$. It is easy to see that z two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Moreover, this point is unique because if a point two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$, then it belongs to $W^u(z_1) \cap W^s(z_2)$, which is a singleton. $\square$

3. Fixed point problem in the universal covering

Theorem A says to us that the shadowing theory might play an important role in solving Smale's conjectures. Then we use some ideas contained in the proof of the Shadowing Lemma of [12] and introduce some new techniques: for any two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ in the universal covering, we discuss two maps F and G, from a suitable Banach space to itself, such that fixed points of these maps are related with points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. In this section, we define these maps, the Banach space and clarify this relation.

Let f_0 be a homeomorphism defined in a compact manifold M, f be any lift of f_0 to the universal covering $\widetilde{M}$ and $(x_k)_{k\in\mathbb{Z}} \subset \widetilde{M}$ be a two-sided limit pseudo-orbit of f. We denote by C the set of all bilateral sequences $\bar{v} = (v_k)_{k\in\mathbb{Z}}$ where $v_k \in T_{x_k}\widetilde{M}$ for every $k \in \mathbb{Z}$. Let B denote the subset of C consisting of bounded sequences, i.e., sequences $(v_k)_{k\in\mathbb{Z}} \in C$ that satisfy

$$\sup_{k\in\mathbb{Z}} |v_k|_{x_k} < \infty,$$

where $|.|_{x_k}$ is the norm in $T_{x_k}\widetilde{M}$ induced by the lifted metric. The map $||.||\colon B \to \mathbb{R}^+$ defined by

$$||\bar{v}|| = \sup_{k\in\mathbb{Z}} |v_k|_{x_k}$$

is a norm in B that makes $(B; ||.||)$ a Banach space. We consider the subspace C_0 of B as the space of sequences $(v_k)_{k\in\mathbb{Z}} \in B$ that satisfy

$$|v_k|_{x_k} \to 0, \;\; |k| \to \infty.$$

The set C_0 is a closed subspace of B with respect to the norm defined above, so it is also a Banach space. We define a map $F : C_0 \to C_0$ as follows: for each sequence $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$ we define $F(\bar{v})$ as the sequence

$$F(\bar{v})_k = \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}), \quad k \in \mathbb{Z},$$

where $\exp_x\colon T_x\widetilde{M} \to \widetilde{M}$ is the exponential map at x in the universal covering. The map F is already considered in the proof of the Shadowing Lemma. However, the pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ is considered in M, which is a compact manifold, so the exponential maps are just defined locally. If we suppose that the ambient manifold has non-positive sectional curvature, then the Hadamard Theorem assures that the exponential map $\exp_x$ is a global diffeomorphism for each $x \in \widetilde{M}$, and this assures that F is defined in the whole space C_0. This puts a restriction on the ambient manifold but it seems to be no problem, since it is expected that the universal covering of a manifold supporting an Anosov diffeomorphism is the Euclidean space. For F to be well defined we just have to prove the following.

Lemma 3.1. *If $\bar{v} \in C_0$, then $F(\bar{v}) \in C_0$.*

Proof. A standard compactness argument (which we omit here) proves that f is uniformly continuous on $\widetilde{M}$. Thus for each $\varepsilon > 0$ we can choose $0 < \delta < \frac{\varepsilon}{2}$ such that $d(x,y)<\delta$ implies $d(f(x),f(y))<\frac{\varepsilon}{2}$. Since $\bar{v}\in C_0$ and $(x_k)_{k\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of f we can choose $K\in\mathbb{N}$ such that for $|k|\geq K$ we have

$$|v_k|_{x_k}<\delta \quad \text{and} \quad d(f(x_k),x_{k+1})<\delta.$$

Thus for $|k|>K$ we have

$$d(\exp_{x_{k-1}}(v_{k-1}),x_{k-1})=|v_{k-1}|_{x_{k-1}}<\delta,$$

which imply

$$\begin{aligned}
|F(\bar{v})_k|_{x_k} &= |\exp_{x_k}^{-1}\circ f\circ\exp_{x_{k-1}}(v_{k-1})|_{x_k}\\
&= d(f(\exp_{x_{k-1}}(v_{k-1})),x_k)\\
&\leq d(f(\exp_{x_{k-1}}(v_{k-1})),f(x_{k-1}))+d(f(x_{k-1}),x_k)\\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}$$

This is enough to prove that $F(\bar{v})\in C_0$. □

We prove that fixed points of F are in a bijective relation with the set of points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$.

Theorem B. *There exists a bijection between the set of fixed points of F and the set of points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$.*

Proof. For a two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ of f, consider the space C_0 and the map F defined above. We suppose that $\bar{v}$ is a fixed point of F and we prove that the sequence $(\exp_{x_k}(v_k))_{k\in\mathbb{Z}}$ is an orbit that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Indeed, for each $k\in\mathbb{Z}$ we have

$$v_k=F(\bar{v})_k=\exp_{x_k}^{-1}\circ f\circ\exp_{x_{k-1}}(v_{k-1}),$$

which implies

$$\exp_{x_k}(v_k)=f\circ\exp_{x_{k-1}}(v_{k-1}).$$

By induction, we obtain

$$\exp_{x_k}(v_k)=f^k(\exp_{x_0}(v_0)), \quad k\in\mathbb{Z}.$$

Therefore,

$$d(f^k(\exp_{x_0}(v_0)),x_k)=d(\exp_{x_k}(v_k),x_k)=|v_k|_{x_k}\to 0, \quad |k|\to\infty.$$

Now, suppose that z two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. For each $k\in\mathbb{Z}$ let

$$v_k=\exp_{x_k}^{-1}(f^k(z)).$$

We have $\bar{v}=(v_k)_{k\in\mathbb{Z}}\in C_0$ since

$$|v_k|_{x_k}=d(\exp_{x_k}(v_k),x_k)=d(f^k(z),x_k)\to 0, \quad |k|\to\infty.$$

Moreover, for each $k\in\mathbb{Z}$

$$F(\bar{v})_k=\exp_{x_k}^{-1}\circ f\circ\exp_{x_{k-1}}(v_{k-1})=\exp_{x_k}^{-1}(f^k(z))=v_k,$$

which proves $\bar{v}$ is a fixed point of F. These arguments construct the desired bijection. □

The map F can be defined for any homeomorphism f_0 defined in a compact manifold, any lift f of f_0 to the universal covering and any two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ of f. It is not expected that F admits fixed points in all cases though, but it is when f_0 is an Anosov diffeomorphism. In this case, we use the hyperbolic structure of f to define a new map G in C_0, with more structure than F, but with the same fixed points. We can assume, with no restriction, that $(x_k)_{k\in\mathbb{Z}}$ has the form

$$x_k = \begin{cases} f^k(y), & k \geq 0, \\ f^k(x), & k < 0, \end{cases}$$

for some points $x, y \in \widetilde{M}$ (see the proof of Theorem A). Consider a linear isomorphism $I : T_x\widetilde{M} \to T_y\widetilde{M}$ satisfying

(1) $I(E^s(x)) = E^s(y)$,
(2) $I(E^u(x)) = E^u(y)$,

and define a map $T : C_0 \to C_0$ by

$$T(\bar{v})_k = \begin{cases} Df(x_{k-1})(v_{k-1}), & k \neq 0, \\ I \circ Df(x_{-1})(v_{-1}), & k = 0. \end{cases}$$

Note that $T(\bar{v}) \in C_0$ if $\bar{v} \in C_0$. Let Id denote the identity map in C_0. The following theorem allows us to define the map G.

Theorem 3.2. *The map $Id-T$ is a bounded linear isomorphism in C_0 with bounded inverse $(Id-T)^{-1}$.*

We prove this theorem in the next section and use it to define the map $G : C_0 \to C_0$ as follows. For each $\bar{v} \in C_0$ let

$$G(\bar{v}) = (Id-T)^{-1} \circ (F-T)(\bar{v}).$$

By definition, F and G have the same fixed points in C_0. This proves the following.

Theorem C. *There exists a bijection between the set of fixed points of G and the set of points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$.*

The map $(Id-T)^{-1}$ will be defined in the proof of Theorem 3.2 and the map $F-T$ is the following one:

$$(F-T)(\bar{v})_k = \begin{cases} \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}) - Df(x_{k-1})(v_{k-1}), & k \neq 0, \\ \exp_y^{-1} \circ f \circ \exp_x(v_{-1}) - I \circ Df(x_{-1})(v_{-1}), & k = 0. \end{cases}$$

We do not know how to obtain fixed points for the map G in the general case, but we do when f is linear or when the numbers $d(f(x_k), x_{k+1})$ are sufficiently small (which are known cases). In the first case, G is a constant map and in the second case, G is a contraction in an invariant small neighborhood of $\bar{0}$ in C_0 (see [12]). We hope some fixed point theorem applies in the general case.

If we consider a curve $\gamma\colon [0,1] \to \widetilde{M}$ such that $\gamma(0) = y$ and $\gamma(1) = x$, then for each $t \in [0,1]$ we can consider the two-sided limit pseudo-orbit $(x_k^t)_{k\in\mathbb{Z}}$ defined as the past orbit of $\gamma(t)$ and the future orbit of y. This induces one-parameter families of Banach spaces $(C_0^t)_{t\in[0,1]}$ and maps $(G_t)_{t\in[0,1]}$ where C_0^t and G_t were defined above with respect to $(x_k^t)_{k\in\mathbb{Z}}$.

When M has zero sectional curvature, the uniqueness of space forms says that the universal covering is the Euclidean space $\mathbb{R}^n$ and the lifted metric is the Euclidean metric. In this case, the Banach space C_0 does not depend on the two-sided limit pseudo-orbit. Thus, the family $(G_t)_{t\in[0,1]}$ is a one-parameter family of maps defined on the same space C_0. For $t=0$ the sequence $(x_k^0)_{k\in\mathbb{Z}}$ is the orbit of x_0 and $\bar{0}$ is a fixed point of G_0. We think some *fixed point continuation theorem* applies and the fixed point of G_0 can be carried through $(G_t)_{t\in[0,1]}$ for a fixed point of G_1. More precisely, we would like to obtain a curve $\Gamma\colon [0,1]\to C_0$ such that $\Gamma(0)=\bar{0}$ and $\Gamma(t)$ is a fixed point of G_t for each $t\in[0,1]$.

In J. Franks' Ph.D. thesis [8] there is a similar discussion about product Anosov diffeomorphisms: it might happen (though it is not expected) that there exists some $t_0\in(0,1)$ such that for $s<t_0$ there exists an intersection between $W^u(\gamma(s))$ and $W^s(x_0)$ but for $s\geq t_0$ there is not. In this case the intersections $W^u(\gamma(s))\cap W^s(x_0)$ go to infinity in $\mathbb{R}^n$ when s converges to t_0 to the left. In our setting, this means the set

$$K=\{\bar{v}\in C_0;\;\; \bar{v}=G_t(\bar{v}) \text{ for some } t\in[0,1]\}$$

is unbounded in C_0. The problem is to determine if this set is bounded or not. If we could prove that K is bounded, then it would follow that there exists a finite number of points in $W^u(x)\cap W^s(y)$, which is a weaker form of product structure.

Remark. The only fixed points continuations theorems we know are for completely continuous maps on Banach spaces. These maps satisfy that the image of every bounded set is compact. Moreover, there is some hypothesis on the set of solutions that is called *a priori bound hypothesis* and is usually the most difficult one to check (see [11]). This hypothesis is equivalent to the set K defined above to be bounded.

Question. Is the map $H\colon C_0\times[0,1]\to C_0$ defined by $H(\bar{v},t)=G_t(\bar{v})$ a completely continuous map? Is it continuous with respect to the product topology in $C_0\times[0,1]$?

Remark. An interesting problem is to understand how these techniques translate to the theory of Anosov flows. Does there exist a Banach space and a map on this space such that an analogous of Theorem C holds? There are some examples of Anosov flows that are not product Anosov flows (see [3], [7]) so if we are able to obtain this map it should not admit fixed points.

4. Proof of Theorem 3.2

Now we turn our attention to the proof of Theorem 3.2. For each $k\in\mathbb{Z}$ we consider projections $\pi_k^s: T_{x_k}\widetilde{M}\to E^s(x_k)$ and $\pi_k^u: T_{x_k}\widetilde{M}\to E^u(x_k)$, parallel to $E^u(x_k)$ and $E^s(x_k)$ respectively. Since M is compact, we can choose $N\in\mathbb{N}$ such that for every $k\in\mathbb{Z}$ we have

$$|\pi_k^s(v)|_{x_k}\leq N|v|_{x_k} \quad\text{and}\quad |\pi_k^u(v)|_{x_k}\leq N|v|_{x_k}.$$

For each $k\in\mathbb{Z}$, consider the map $A_k\colon T_{x_k}\widetilde{M}\to T_{x_{k+1}}\widetilde{M}$ defined by

$$A_k(v)=\begin{cases} Df(x_k)(v), & k\neq -1,\\ I\circ Df(x_{-1})(v), & k=-1.\end{cases}$$

Since $A_k(E^s(x_k))=E^s(x_{k+1})$ for every $k\in\mathbb{Z}$, we can compose these maps to obtain

$$A_{k-1}\circ\cdots\circ A_n(E^s(x_n))=E^s(x_k),\quad n<k.$$

Since $A_k^{-1}(E^u(x_{k+1})) = E^u(x_k)$, we analogously have

$$A_k^{-1} \circ \cdots \circ A_n^{-1}(E^u(x_{n+1})) = E^u(x_k), \quad n \geq k.$$

Thus, we can define a map $\mathcal{G} : C_0 \to C_0$ as follows: for each $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$, the sequence $\mathcal{G}(\bar{v}) = (\mathcal{G}(\bar{v})_k)_{k\in\mathbb{Z}}$ is defined by

$$\mathcal{G}(\bar{v})_k = \pi_k^s(v_k) + \sum_{n=-\infty}^{k-1} A_{k-1} \circ \cdots \circ A_n \pi_n^s(v_n) - \sum_{n=k}^{+\infty} A_k^{-1} \circ \cdots \circ A_n^{-1} \circ \pi_{n+1}^u(v_{n+1}).$$

We will prove that the map $\mathcal{G}$ is exactly the inverse of the map $Id - T$. First, note that for every $v \in E^s(x_k)$ and $w \in E^u(x_{k+1})$ we have

$$|A_k(v)|_{x_{k+1}} \leq \begin{cases} \lambda |v|_{x_k}, & k \neq -1, \\ \|I\|\lambda|v|_{x_{-1}}, & k = -1, \end{cases}$$

and

$$|A_k^{-1}(w)|_{x_k} \leq \begin{cases} \lambda |w|_{x_{k+1}}, & k \neq -1, \\ \|I^{-1}\|\lambda|w|_{y}, & k = -1. \end{cases}$$

Thus, for every $v \in E^s(x_n)$ and $k > n$ we have

$$|A_{k-1} \circ \cdots \circ A_n(v)|_{x_k} \leq \begin{cases} \|I\|\lambda^{k-n}|v|_{x_n}, & \text{if} \quad n < -1 \leq k, \\ \lambda^{k-n}|v|_{x_n}, & \text{otherwise.} \end{cases}$$

Also, for every $w \in E^u(x_{n+1})$ and $k \leq n$ we have

$$|A_k^{-1} \circ \cdots \circ A_n^{-1}(w)|_{x_k} \leq \begin{cases} \|I^{-1}\|\lambda^{n-k+1}|w|_{x_{n+1}}, & \text{if} \quad k \leq -1 \leq n, \\ \lambda^{n-k+1}|w|_{x_{n+1}}, & \text{otherwise.} \end{cases}$$

Hence, if $k \geq 0$, then

$$\begin{aligned} \left| \sum_{n=-\infty}^{k-1} A_{k-1} \circ \cdots \circ A_n \circ \pi_n^s(v_n) \right|_{x_k} &\leq \sum_{n=-\infty}^{k-1} \|I\|\lambda^{k-n}|\pi_n^s(v_n)|_{x_n} \\ &\leq N\|I\| \sum_{n=-\infty}^{k-1} \lambda^{k-n}|v_n|_{x_n} \\ &\leq N\|I\|\|\bar{v}\| \sum_{n=-\infty}^{k-1} \lambda^{k-n} \\ &\leq N\|I\|\|\bar{v}\| \frac{\lambda}{1-\lambda} \end{aligned}$$

and

$$\left|\sum_{n=k}^{+\infty} A_k^{-1}\circ\cdots\circ A_n^{-1}\circ\pi_{n+1}^u(v_{n+1})\right|_{x_k} \le \sum_{n=k}^{+\infty}\lambda^{n-k+1}|\pi_{n+1}^u(v_{n+1})|_{x_{n+1}}$$
$$\le N\sum_{n=k}^{+\infty}\lambda^{n-k+1}|v_{n+1}|_{x_{n+1}}$$
$$\le N||\bar v||\sum_{n=k}^{+\infty}\lambda^{n-k+1}$$
$$\le N||\bar v||\frac{\lambda}{1-\lambda}.$$

And if $k<0$, then

$$\left|\sum_{n=-\infty}^{k-1} A_{k-1}\circ\cdots\circ A_n\circ\pi_n^s(v_n)\right|_{x_k} \le \sum_{n=-\infty}^{k-1}\lambda^{k-n}|\pi_n^s(v_n)|_{x_n}$$
$$\le N\sum_{n=-\infty}^{k-1}\lambda^{k-n}|v_n|_{x_n}$$
$$\le N\|\bar v\|\sum_{n=-\infty}^{k-1}\lambda^{k-n}$$
$$\le N\|\bar v\|\frac{\lambda}{1-\lambda}$$

and

$$\left|\sum_{n=k}^{+\infty} A_k^{-1}\circ\cdots\circ A_n^{-1}\circ\pi_{n+1}^u(v_{n+1})\right|_{x_k} \le \sum_{n=k}^{+\infty}\|I^{-1}\|\lambda^{n-k+1}|\pi_{n+1}^u(v_{n+1})|_{x_{n+1}}$$
$$\le N\|I^{-1}\|\sum_{n=k}^{+\infty}\lambda^{n-k+1}|v_{n+1}|_{x_{n+1}}$$
$$\le N\|I^{-1}\|\|\bar v\|\sum_{n=k}^{+\infty}\lambda^{n-k+1}$$
$$\le N\|I^{-1}\|\|\bar v\|\frac{\lambda}{1-\lambda}.$$

Thus, if $k\ge 0$, then

$$|\mathcal{G}(\bar v)_k|_{x_k} \le N\|\bar v\|+N\|I\|\|\bar v\|\frac{\lambda}{1-\lambda}+N\|\bar v\|\frac{\lambda}{1-\lambda}$$
$$= N\left(\frac{1+\|I\|\lambda}{1-\lambda}\right)||\bar v||,$$

and if $k<0$, then

$$|\mathcal{G}(\bar v)_k|_{x_k} \le N\|\bar v\|+N\|\bar v\|\frac{\lambda}{1-\lambda}+N\|I^{-1}\|\|\bar v\|\frac{\lambda}{1-\lambda}$$
$$= N\left(\frac{1+\|I^{-1}\|\lambda}{1-\lambda}\right)||\bar v||.$$

This proves that $\mathcal{G}(\bar{v})_k \in T_{x_k}\widetilde{M}$ for every $k \in \mathbb{Z}$. Actually, the following proposition is proved.

Proposition 4.1. *If $\bar{v} \in C_0$, then $\mathcal{G}(\bar{v}) \in C_0$.*

Toward proving this proposition, we prove an auxiliary lemma:

Lemma 4.2. *If $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$, then $\sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n} \to 0$ when $k \to \infty$.*

Proof. For each $\varepsilon > 0$, choose $K \in \mathbb{N}$ such that

$$|v_n|_{x_n} < \frac{\varepsilon(1-\lambda)}{2}, \quad n \geq K.$$

If $k > K$, we can write

$$\sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n} = \sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} + \sum_{n=K+1}^{k-1} \lambda^{k-n}|v_n|_{x_n}.$$

Note that

$$\sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} = \lambda^k \sum_{n=1}^{K} \lambda^{-n}|v_n|_{x_n} \to 0, \quad k \to \infty.$$

Then we can choose $k \geq K$ such that

$$\sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} < \frac{\varepsilon}{2}.$$

Moreover,

$$\begin{aligned} \sum_{n=K+1}^{k-1} \lambda^{k-n}|v_n|_{x_n} &\leq \frac{\varepsilon(1-\lambda)}{2} \sum_{n=K+1}^{k-1} \lambda^{k-n} \\ &\leq \frac{\varepsilon(1-\lambda)}{2} \frac{1}{1-\lambda} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

Thus, for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\sum_{i=1}^{n-1} \lambda^{n-i}|v_i|_{x_i} < \varepsilon, \ \ n \geq k.$$

This proves that

$$\sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n} \to 0, \quad k \to \infty.$$

$\square$

Proof of Proposition 4.1. Let $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$. To prove that $\mathcal{G}(\bar{v}) \in C_0$ we need to show that $|\mathcal{G}(\bar{v})_k|_{x_k} \to 0$ when $|k| \to \infty$. We consider separately the three terms in the expression of $\mathcal{G}(\bar{v})_k$. The first one satisfies

$$|\pi_k^s(v_k)|_{x_k} \leq N|v_k|_{x_k} \to 0, \quad |k| \to \infty.$$

For the second term, it is enough to prove that

$$\sum_{n=-\infty}^{k-1} \lambda^{k-n}|v_n|_{x_n} \to 0, \quad |k| \to \infty.$$

If $k > 1$, we can write

$$\sum_{n=-\infty}^{k-1} \lambda^{k-n} |v_n|_{x_n} = \sum_{n=-\infty}^{0} \lambda^{k-n} |v_n|_{x_n} + \sum_{n=1}^{k-1} \lambda^{k-n} |v_n|_{x_n}.$$

We proved in Lemma 4.2 that the second sum goes to zero when $k \to \infty$. For the first sum, we argue as follows:

$$\begin{aligned} \sum_{n=-\infty}^{0} \lambda^{k-n} |v_n|_{x_n} &= \lambda^k \sum_{n=-\infty}^{0} \lambda^{-n} |v_n|_{x_n} \\ &\leq \lambda^k ||\bar{v}|| \sum_{n=-\infty}^{0} \lambda^{-n} \\ &= \lambda^k ||\bar{v}|| \frac{1}{1-\lambda} \to 0, \quad k \to \infty. \end{aligned}$$

If $k \leq 0$, then for each $\varepsilon > 0$ we choose $K \in \mathbb{N}$ such that

$$|v_n|_{x_n} < \frac{\varepsilon(1-\lambda)}{\lambda}, \quad n \leq -K.$$

Thus, if $k \leq -K$, then

$$\begin{aligned} \sum_{n=-\infty}^{k-1} \lambda^{k-n} |v_n|_{x_n} &\leq \frac{\varepsilon(1-\lambda)}{\lambda} \sum_{n=-\infty}^{k-1} \lambda^{k-n} \\ &= \frac{\varepsilon(1-\lambda)}{\lambda} \frac{\lambda}{1-\lambda} \\ &= \varepsilon. \end{aligned}$$

This proves that

$$\sum_{n=-\infty}^{k-1} \lambda^{k-n} |v_n|_{x_n} \to 0, \quad k \to -\infty.$$

The same arguments can be applied for the third term in $\mathcal{G}(\bar{v})_k$, so we leave the details to the reader. □

Lemma 4.3. *For each $\bar{v} \in C_0$ and each $k \in \mathbb{Z}$ the following holds:*

$$A_k(\mathcal{G}(\bar{v})_k) = \mathcal{G}(\bar{v})_{k+1} - v_{k+1}.$$

Proof. Note that for each $k \in \mathbb{Z}$

$$\begin{aligned} A_k(\mathcal{G}(\bar{v})_k) = A_k \circ \pi_k^s(v_k) &+ \sum_{n=-\infty}^{k-1} A_k \circ A_{k-1} \circ \cdots \circ A_n \pi_n^s(v_n) - \pi_{k+1}^u(v_{k+1}) \\ &- \sum_{n=k+1}^{+\infty} A_{k+1}^{-1} \circ \cdots \circ \pi_{n+1}^u(v_{n+1}). \end{aligned}$$

To obtain the desired equality, just put $-\pi_{k+1}^u(v_{k+1}) = \pi_{k+1}^s(v_{k+1}) - v_{k+1}$ in the last one. □

Proof of Theorem 3.2. We first prove that $Id - T$ is surjective. Indeed, $\mathcal{G}$ is a right inverse for $Id - T$. If $w \in C_0$ and $k \in \mathbb{Z}$, then

$$\begin{aligned} (Id - T)(\mathcal{G}(w))_k &= \mathcal{G}(w)_k - A_{k-1}(\mathcal{G}(w)_{k-1}) \\ &= w_k. \end{aligned}$$

The second equality is ensured by Lemma 4.3. To prove that $Id - T$ is injective, let $v \in C_0$ be such that $(Id - T)(v) = 0$, that is, $T(v) = v$. Then

$$A_{k-1}(v_{k-1}) = v_k, \quad k \in \mathbb{Z}.$$

By induction, we obtain

$$v_k = A_{k-1} \circ \cdots \circ A_0(v_0), \quad k > 0,$$

and

$$v_k = A_k^{-1} \circ \cdots \circ A_{-1}^{-1}(v_0), \quad k < 0.$$

This, and the fact that

$$|v_k|_{x_k} \to 0, \quad |k| \to \infty,$$

imply that

$$\pi_0^s(v_0) = \pi_0^u(v_0) = 0.$$

Indeed, if this is not the case, then $|v_k|_{x_k}$ would converge to ∞. This implies that $v_0 = 0$, and, hence, that $v_k = 0$ for every $k \in \mathbb{Z}$. This is enough to prove the injectivity of $Id - T$. Thus, the map $\mathcal{G}$ is a linear isomorphism in C_0, that is, the inverse of $Id - T$, and is bounded, since its norm satisfies

$$\begin{aligned} ||\mathcal{G}|| &= \sup_{\|\bar{v}\|=1} \|\mathcal{G}(\bar{v})\| \\ &= \sup_{\|\bar{v}\|=1} \sup_{k \in \mathbb{Z}} |\mathcal{G}(\bar{v})_k|_{x_k} \\ &\leq \max \left\{ N\left(\frac{1 + \|I\|\lambda}{1-\lambda}\right), N\left(\frac{1 + \|I^{-1}\|\lambda}{1-\lambda}\right) \right\}. \end{aligned}$$

□

Remark. If f is a partially hyperbolic diffeomorphism, then we can consider the map $\mathcal{G}$ as defined above. The difference is that Lemma 4.3 does not hold as it is written. Indeed, the following holds: for each $\bar{v} \in C_0$ and each $k \in \mathbb{Z}$ we have

$$A_k(\mathcal{G}(\bar{v})_k) = \mathcal{G}(\bar{v})_{k+1} - v_{k+1} + \pi_{k+1}^c(v_{k+1}),$$

where π_k^c is the projection in the central direction $E^c(x_k)$ parallel to $E^s(x_k) \oplus E^u(x_k)$. In this case, we have

$$(Id - T)(\mathcal{G}(w))_k = \mathcal{G}(w)_k - A_{k-1}(\mathcal{G}(w)_{k-1}) = w_k - \pi_k^c(w_k)$$

and $\mathcal{G}$ is not the inverse of $Id - T$ anymore.

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