PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, Number 3, March 2018, Pages 1113–1121
http://dx.doi.org/10.1090/proc/13796
Article electronically published on September 28, 2017

HARMONIC MAPPINGS OF BOUNDED BOUNDARY ROTATION

D. BSHOUTY, A. LYZZAIK, AND F. M. SAKAR

(Communicated by Jeremy Tyson)

ABSTRACT. The purpose of this paper is to investigate the valency of planar harmonic mappings of bounded boundary rotation of the open unit disc $\mathbb{D}$. The paper is motivated by the earlier work of the first two authors [Complex Analysis Oper. Theory **5** (2011), 767–774] and the recent work of T. Hayami [Complex Var. Elliptic Equ. **59** (2014), 1214–1222].

First, the authors give a counterexample showing that both the main result of Hayami, Theorem 2.1, and the related conjecture, Conjecture 4.1, are false. Second, the authors give a valency criterion for planar harmonic mappings of bounded boundary rotation of $\mathbb{D}$, proving an ameliorated statement of Theorem 2.1 and settling a modified version of Conjecture 4.1.

1. INTRODUCTION AND PRELIMINARY RESULTS

A *planar harmonic mapping* of a domain Ω in the complex plane $\mathbb{C}$ is a complex-valued function of the form

$$f(z) = u(z) + iv(z),$$

where $z = x + iy$ and u and v are real harmonic functions. If Ω is simply connected and $z_0 \in \Omega$, then f admits the *canonical representation*

$$f = h + \overline{g},$$

where h and g are analytic functions in Ω and $g(z_0) = 0$. The mapping f is both sense-preserving and univalent in some open neighborhood of z_0 if, and only if, its Jacobian $|h'|^2 - |g'|^2$ is positive at z_0. Also, f is sense-preserving in Ω if, and only if, its Jacobian is nonnegative in Ω or, equivalently, if its *second dilatation*

$$\omega = \frac{g'}{h'}$$

is analytic and satisfies $|\omega(z)| < 1$ in Ω.

A simply connected subdomain Ω of $\mathbb{C}$ is called *close-to-convex* if its complement $\mathbb{C} \setminus \Omega$ is the union of closed half lines with pairwise disjoint interiors. A univalent analytic or harmonic mapping of the open unit disc $\mathbb{D}$ is called *close-to-convex* if its image set $f(\mathbb{D})$ is close-to-convex.

In 2011, the first two authors proved the following result [2], which settled a conjecture of Mocanu [8]:

Received by the editors August 3, 2016 and, in revised form, August 3, 2016, April 4, 2017 and April 13, 2017.

2010 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. Planar harmonic mapping, harmonic mapping of bounded boundary rotation, close-to-convex function, multivalent function.

Theorem A. *Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$, with $h'(0) \neq 0$, that satisfies $g'(z) = zh'(z)$ and*

$$\Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{1}{2}, \quad z \in \mathbb{D}. \tag{1}$$

Then f is a univalent close-to-convex mapping.

Suppose that $f = h + \overline{g}$ is a harmonic mapping of the closed unit disc $\overline{\mathbb{D}}$ with $h' \neq 0$ on $\partial\mathbb{D}$ and $g'(z) = z^m h'(z)$ for some positive integer m. To study the behavior of f on $\partial\mathbb{D}$, we write

$$\begin{aligned}\frac{d}{dt}f(e^{it}) &= ie^{it}h'(e^{it}) + \overline{ie^{it}g'(e^{it})} \\ &= ie^{it}h'(e^{it}) + \overline{ie^{i(m+1)t}h'(e^{it})} \\ &= ie^{-imt/2}\left\{e^{i(m+2)t/2}h'(e^{it}) - \overline{e^{i(m+2)t/2}h'(e^{it})}\right\} \\ &= -2e^{-imt/2}\Im\left\{e^{i(m+2)t/2}h'(e^{it})\right\}.\end{aligned}$$

Observe that $\Im\left\{e^{i(m+2)t/2}h'(e^{it})\right\}$ changes sign at finitely many values $t_k \in [0, 2\pi)$, if any, and that elsewhere every single-valued continuous branch of arg $df(e^{it})/dt$ decreases steadily on every complementary component (interval). It follows that $f(\partial\mathbb{D})$ admits a cusp, called a *harmonic cusp*, at every point $f(e^{it_k})$ and is locally concave with respect to $f(\mathbb{D})$ at every other point [7].

The notion of a harmonic cusp for a more general setting may be found in [7].

Definition 1. A sense-preserving harmonic mapping $f : \mathbb{D} \to \mathbb{C}$ is called *p-valent* if it takes every value at most p times, counting multiplicity.

In an attempt to extend this theorem, Hayami [4] proved the following:

Theorem B. *Let $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be an analytic function of $\overline{\mathbb{D}}$, with $H(z) = h'(z)/z^{p-1} \neq 0$ on $\overline{\mathbb{D}}$, and let*

$$F(t) = (2p + m - 1)t + 2\arg\left\{H(e^{it})\right\}, \quad -\pi \leq t < \pi,$$

for some $m = 2, 3, 4, \ldots$. If for each $k \in K = \{0, \pm 1, \pm 2, \ldots, \pm\lfloor(2p + m + 1)/2\rfloor\}$ the equation $F(t) = 2k\pi$ has at most a single root in $[-\pi, \pi)$, and for all $k \in K$ there exist exactly $2p + m - 1$ such roots in $[-\pi, \pi)$, then the harmonic function $f(z) = h(z) + \overline{g(z)}$, with $g'(z) = z^{m-1}h'(z)$, is p-valent in $\mathbb{D}$ and maps $\partial\mathbb{D}$ onto a curve comprising $2p + m - 1$ harmonic cusps whose vertices subdivide $f(\partial\mathbb{D})$ into $2p + m - 1$ concave curves.

Following the proof of Theorem B, Hayami posed the following conjecture [4, Conjecture 4.1]:

Conjecture C. *Let $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z)$ be an analytic function of $\mathbb{D}$ satisfying $g'(z) = z^{m-1}h'(z)$ for some $m = 2, 3, \ldots$, and*

$$\Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{m-1}{2}, \quad z \in \mathbb{D}. \tag{2}$$

Then the harmonic mapping $f(z) = h(z) + \overline{g(z)}$ is p-valent in $\mathbb{D}$.

The purpose of this paper is two-fold: first, to give an example showing that both Theorem B and Conjecture C are false, and second, to give a valency criterion for planar harmonic mappings of bounded boundary rotation of $\mathbb{D}$ proving an ameliorated version of Theorem B and settling a stronger version of Conjecture C. This criterion has interesting applications, of which one (Corollary 1) yields a new and more direct and informative proof of Theorem A.

Definition 2. A sense-preserving harmonic mapping f of $\mathbb{D}$ is said to belong to the class $VH_k(p)$, where p is a positive integer and k is a real number at least 2, if f has $p-1$ critical points in $\mathbb{D}$, counting multiplicity, and

$$\limsup_{r\to 1^-} \int_0^{2\pi} \left| \frac{d}{dt} \arg \frac{\partial}{\partial t} f(re^{it}) \right| dt \le pk\pi. \tag{3}$$

The classes $VH_k(p)$, for all p and k, constitute all *multivalent harmonic mappings of bounded boundary rotation.*

The subclass of $VH_k(p)$ consisting only of analytic functions of $\mathbb{D}$ is denoted by $V_k(p)$; particularly, $V_k(p)$ is a proper subclass of $VH_k(p)$. The classes $V_k(p)$ were first introduced and investigated by Leach [5] and further studied by Lyzzaik [6]. The class of functions $f \in V_k(1)$ normalized by $f(0) = 0$ and $f'(0) = 1$, denoted simply by V_k, was introduced by Paatero [9], who showed that V_4 consists only of univalent functions. Much later, Brannan [1] showed that V_4 is properly contained in the class of univalent close-to-convex functions. However, every class V_k with $k > 4$ contains nonunivalent functions.

The geometries of the classes $V_k(p)$ were investigated by the second author [6]. By deploying a cutting and pasting method that applies essentially in the same manner to the classes $VH_k(p)$, questions related to their decomposition and valency, among others, were settled.

Definition 3. Let P be a complex polynomial, and let γ be a ray in $\mathbb{C}$. A simple unbounded curve l is called a P-ray if $P : l \to \gamma$ is a homeomorphism.

In this paper, we need both Corollary 6.3 and Theorem 6.3 of [6] formulated for the more general classes $VH_k(p)$ as follows.

Proposition 1. *Let $f \in VH_k(p)$. Then $f = P \circ \phi$, where P is a polynomial of degree at most*

(a) *the largest integer less than $k/2$ if $k > 2$ and $p = 1$ and*
(b) *the smallest integer larger than $pk/2 - 1$ if $p > 1$,*

and ϕ is a homeomorphism of $\mathbb{D}$ into $\mathbb{C}$ such that the zeros of P' lie in $\overline{\phi(\mathbb{D})}$, and $\mathbb{C} \setminus \phi(\mathbb{D})$ is either empty or is a union of P-rays of disjoint interiors starting from $\partial\phi(\mathbb{D})$.

Note that the notation $\{pk/2 - 1\}$ was used in [6] to indicate the quantities appearing above in items (a) and (b) of Proposition 1 and that if $f \in V_k(p)$, then ϕ is an analytic univalent function and $\mathbb{C} \setminus \phi(\mathbb{D})$ is nonempty.

2. Main results

We begin this section by giving an example showing that both Theorem B and Conjecture C are, in fact, false [4, Theorem 2.1 and Conjecture C], at least in the case $p = 1, m = 4$. In doing so, we shall avoid the tedious calculations by adhering to the associated figures from *Mathematica*®.

Example 1. Let

$$h'(z) = (1 - 0.7z)^{-5} \quad \text{and} \quad g'(z) = z^3 h'(z).$$

We show that $f(z) = h(z) + \overline{g(z)}$, with $h(0) = g(0) = 0$, satisfies the assumptions of Theorem B and Conjecture C for the case $p = 1$ and $m = 4$; nonetheless f is 2-valent.

Indeed, $h'(0) = 1$ and

$$\Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} = \Re\left\{\frac{10 + 28z}{10 - 7z}\right\} > -\frac{3}{2}, \quad z \in \mathbb{D}.$$

Hence, inequality (2) of Conjecture C holds. On the other hand,

$$\begin{aligned} F(t) &= 5t + 2\arg\left\{(1 - 0.7e^{it})^{-5}\right\} \\ &= 5\arg\left\{\frac{e^{it}}{(1 - 0.7e^{it})^2}\right\}, \end{aligned}$$

where arg is the single-valued continuous branch of the argument satisfying $\arg 1 = 0$. It is immediate that F is a strictly increasing function on $[0, 2\pi]$ with range $[0, 10\pi]$; hence it satisfies the assumption of Theorem B for the respective case.

Note that the circle shown in Figure 1 does not relate in any manner to $f(\partial\mathbb{D})$, but it is sketched only to distinguish between two parts of a subdivision of $f(\partial\mathbb{D})$ which are sketched for the sake of clarity with two different scalings: the part depicted in Figure 1 lying outside the circle, and the part depicted in Figure 2 lying inside the circle. It is evident from both figures that $f(\partial\mathbb{D})$ has exactly 5 harmonic cusps and that f takes on exactly twice every value in the connected component of $\mathbb{C} \setminus f(\partial\mathbb{D})$ containing the interval $(-1.4, -1.3)$, and takes on at most once every other value of $\mathbb{C} \setminus f(\partial\mathbb{D})$. Therefore, f is 2-valent and both Theorem B and Conjecture C are false.

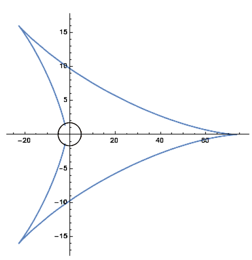

FIGURE 1. Graph of $f(\partial\mathbb{D})$ outside the disc

Remark 1. Example 1 is based on a self-intersecting concave subarc of the subdivision of $f(\partial\mathbb{D})$ induced by the set of vertices of the harmonic cusps. This shows that Hayami's apparent assumption that all the concave subarcs of the subdivision are simple is generally not true, and it is the reason why his Theorem 2.1 [4] falls short of giving the better valency result of Theorem 1 of this paper.

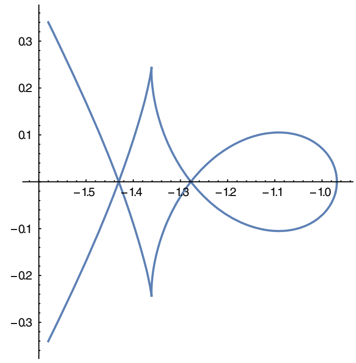

FIGURE 2. Graph of $f(\partial\mathbb{D})$ inside the disc of Figure 1

A modified version of Conjecture C will be settled by proving the following theorem:

Theorem 1. *Let* $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 1$, *and* $g(z)$ *be an analytic function of* $\mathbb{D}$ *satisfying* $h'(z)/z^{p-1} \neq 0$, $g'(z) = z^m h'(z)$, $m = 1, 2, \ldots,$ *and the inequality*

$$\Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{m}{2}, \quad z \in \mathbb{D}. \tag{4}$$

Then the harmonic mapping $f = h + \overline{g}$ *satisfies the following properties:*

(a) $f \in VH_{2(1+m/p)}(p)$;
(b) $f = P \circ \phi$, *where* P *is a polynomial of degree at most* m *if* $p = 1$ *and* $p+m$ *otherwise, and* ϕ *is a homeomorphism of* $\mathbb{D}$ *into* $\mathbb{C}$ *such that the zeros of* P' *lie in* $\overline{\phi(\mathbb{D})}$ *and* $\mathbb{C} \setminus \phi(\mathbb{D})$ *is nonempty and is a union of* P*-rays of disjoint interiors starting from* $\partial\phi(\mathbb{D})$;
(c) f *is* m*-valent if* $p = 1$ *and* $(p+m)$*-valent otherwise.*

Furthermore, if h *is analytic on* $\overline{\mathbb{D}}$ *and* h' *is nonvanishing on* $\partial\mathbb{D}$, *then* $f(\partial\mathbb{D})$ *subdivides into* $2p+m$ *concave curves and comprises the same number of harmonic cusps whose vertices are the points of the subdivision.*

A requisite for the proof of this theorem is the following lemma:

Lemma 1. *Fix* r, $0 < r \leq 1$. *For* $z \in \mathbb{D} \setminus \{0\}$, *let* $\kappa(z) = z^{m/2} z h'(z)$ *and* $\eta(z) = \kappa(z) - |z|^m \overline{\kappa(z)}$. *Then, under the assumptions of Theorem 1, there exists a strictly increasing, single-valued, differentiable branch of* $\arg \eta(re^{it})$, $-\infty < t < \infty$, *whose net (total) variation over* $[0, 2\pi]$ *is exactly* $(2p+m)\pi$.

Proof. Since κ is nonvanishing in $\mathbb{D} \setminus \{0\}$, it is locally analytic there and there exists a single-valued differentiable branch of $\arg \kappa(re^{it})$. From inequality (4) we infer that

$$\begin{aligned} \frac{\partial}{\partial t} \arg \kappa(re^{it}) &= \Im\left\{\frac{\partial}{\partial t} \log \kappa(re^{it})\right\} \\ &= \frac{m}{2} + \Re\left\{1 + re^{it}\frac{h''(re^{it})}{h'(re^{it})}\right\} > 0. \end{aligned}$$

But since h' has $p-1$ zeros in $\mathbb{D}$ located at the origin,

$$\begin{aligned}\int_0^{2\pi} \frac{\partial}{\partial t} \arg \kappa(re^{it})\, dt &= \int_0^{2\pi} \left(\frac{m}{2} + \Re\left\{1 + re^{it}\frac{h''(re^{it})}{h'(re^{it})}\right\}\right) dt \\ &= m\pi + \int_0^{2\pi} \Re\left\{1 + re^{it}\frac{h''(re^{it})}{h'(re^{it})}\right\} dt \\ &= (2p+m)\pi.\end{aligned}$$

Hence, $\arg \kappa(re^{it})$ is a strictly increasing function that increases by exactly $(2p+m)\pi$ on $[0, 2\pi]$. With $\kappa(re^{it}) = \rho(t)e^{i\Theta(t)}$, it follows at once that there exist values t_k, $1 \le k \le 2p+m+1$, such that $t_1 < t_2 < \cdots < t_{2p+m} < t_{2p+m+1} = t_1 + 2\pi$ and $\Theta(t_k) = (k-1)\pi$.

By the same argument as above, there exists a single-valued differentiable branch of $\arg \eta(re^{it})$ on $(-\infty, \infty)$. Since $\kappa(re^{it_1}) > 0$, we may choose $\arg \eta(re^{it})$ so that $\arg \eta(re^{it_k}) = (k-1)\pi$ for $k, 1 \le k \le 2p+m+1$. We write

$$\eta(re^{it}) = \rho(t)\left(e^{i\Theta(t)} - r^m e^{-i\Theta(t)}\right).$$

Then

$$\arg \eta(re^{it}) = \arg \left(e^{i\Theta(t)} - r^m e^{-i\Theta(t)}\right).$$

Consequently,

$$\begin{aligned}\frac{\partial}{\partial t} \arg \eta(re^{it}) &= \frac{\partial}{\partial t} \arg \left(e^{i\Theta(t)} - r^m e^{-i\Theta(t)}\right) \\ &= \Theta'(t) + \Im\left\{\frac{\partial}{\partial t} \log\left(1 - r^m e^{-2i\Theta(t)}\right)\right\} \\ &= \Theta'(t) + 2\Theta'(t)\Re\left\{\frac{r^m e^{-2i\Theta(t)}}{1 - r^m e^{-2i\Theta(t)}}\right\} \\ &> \Theta'(t) - \Theta'(t) = 0,\end{aligned}$$

since $\Theta' > 0$ for all real t and $\Re\{z/(1-z)\} > -1/2$ in $\mathbb{D}$. Hence, $\arg \eta(re^{it})$ is a strictly increasing function on $(-\infty, \infty)$.

Therefore, $\arg \eta(re^{it_k}) = (k-1)\pi$, $1 \le k \le 2p+m+1$, in $[t_1, t_1 + 2\pi]$ if, and only if, $t = t_k$, and the net (total) variation of $\arg \eta(re^{it})$ on the latter interval (or $[0, 2\pi]$) is exactly $(2p+m)\pi$.

A more elegant proof of the last conclusion goes as follows. Write

$$\arg \eta(re^{it}) = \arg\left\{\eta(re^{it})/|(1+r^m)\kappa(re^{it})|\right\} = \arg\left\{A \circ \left(\frac{\kappa(re^{it})}{|\kappa(re^{it})|}\right)\right\},$$

where A is the affine transformation $A(w) = (w - r^m\overline{w})/(1+r^m)$. Note that the curve $A(e^{i\varphi})$, $0 \le \varphi \le 2\pi$, is the positively directed ellipse whose minor axis is the closed interval $[-(1-r^m)/(1+r^m), (1-r^m)/(1+r^m)]$ and whose major axis is the closed interval extending from $-i$ to i. It follows that the point $\eta(re^{it})$ traverses the ellipse positively starting from $A(1) = (1-r^m)/(1+r^m)$ to $A(e^{i(2p+m)}) = (-1)^m(1-r^m)/(1+r^m)$ as t varies from zero to 2π. Therefore, $\arg \eta(re^{it})$ is a strictly increasing function whose net (total) variation on $[0, 2\pi]$ is $(2p+m)\pi$. This completes the proof of the lemma. □

Proof of Theorem 1. First, we show that $f \in VH_{2(1+m/p)}(p)$ and f is at most $(p+m)$-valent in $\mathbb{D}$. Assume that $\kappa(z)$ and $\eta(z)$ are defined as in Lemma 1.

With $0 < r < 1$, it can be easily computed that

$$\begin{aligned}\frac{\partial}{\partial t} f(re^{it}) &= ire^{it}h'(re^{it}) + \overline{i(re^{it})^m re^{it}h'(re^{it})}\\ &= ir^{-m/2}e^{-imt/2}\left\{r^{m/2}e^{imt/2}re^{it}h'(re^{it}) - r^m\overline{r^{m/2}e^{imt/2}re^{it}h'(re^{it})}\right\}\\ &= ir^{-m/2}e^{-imt/2}\eta(re^{it}).\end{aligned}$$

By invoking Lemma 1 we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial t}\arg\frac{\partial}{\partial t}f(re^{it})\right| &= \left|-\frac{m}{2} + \frac{\partial}{\partial t}\arg\eta(re^{it})\right|\\ &\leq \frac{m}{2} + \left|\frac{\partial}{\partial t}\arg\eta(re^{it})\right|\\ &\leq \frac{m}{2} + \frac{\partial}{\partial t}\arg\eta(re^{it}).\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{2\pi}\left|\frac{\partial}{\partial t}\arg\frac{\partial}{\partial t}f(re^{it})\right| dt &\leq \int_0^{2\pi}\frac{m}{2}\,dt + \int_0^{2\pi}\left\{\frac{\partial}{\partial t}\arg\eta(re^{it})\right\}dt\\ &= 2(p+m)\pi.\end{aligned}$$

Therefore, $f \in HV_{2(1+m/p)}(p)$ and (a) holds.

By invoking Proposition 1, we conclude immediately that (c) also holds. As for (b), it also holds except for the property that $\phi(\mathbb{D}) \neq \mathbb{C}$, which we now establish. The fact that $h'(z)/z^{p-1}$ is nonvanishing in $\mathbb{D}$ yields a single-valued analytic branch of $\log\{h'(z)/z^{p-1}\}$ there. Since

$$\frac{d}{dz}\log\left\{\frac{h'}{z^{p-1}}\right\} = \frac{1}{z}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} - \frac{p}{z},$$

inequality (4) implies that $(d/dz)\log\{h'(z)/z^{p-1}\}$ has finite radial (angular) limits a.e. on $\partial\mathbb{D}$ [3, p. 3 (Exercise 2) and p. 17 (Theorem 2.2)]. But the existence of the radial limit of the derivative of an analytic function of $\mathbb{D}$ at a point implies the existence of the radial limit of the function at the same point. Thus $h'(z)/z^{p-1}$ has finite radial limits a.e. on $\partial\mathbb{D}$. Evidently, the same holds for h' and likewise for g' since $g' = z^m h'$. Hence h and g, and consequently f, have finite radial limits a.e. on $\partial\mathbb{D}$. Suppose now that $\zeta \in \partial\mathbb{D}$ is a point where $\lim_{r\to 1^-} f(r\zeta)$ exists and is finite and that $\lim_{r\to 1^-}\phi(r\zeta)$ fails to exist. Then the radial cluster set of ϕ at ζ (the set of limits of all convergent sequences $\{f(r_n\zeta)\}$ where $r_n \to 1^-$) is a nondegenerate continuum, say C_ζ. But $f = P\circ\phi$, and the cluster set of f at ζ is also the nondegenerate continuum $f(C_\zeta)$. This gives a contradiction, and the radial limit of ϕ at ζ exists and is finite. Therefore $\phi(\mathbb{D}) \neq \mathbb{C}$.

Suppose now that h is analytic on $\overline{\mathbb{D}}$ and h' is nonzero on $\partial\mathbb{D}$. We contend that $f(\partial\mathbb{D})$ subdivides into $2p+m$ concave curves and comprises an equal number of harmonic cusps whose vertices are the points of the subdivision. We have

$$\frac{d}{dt}f(e^{it}) = -2e^{-imt/2}\,\Im\kappa(e^{it}). \tag{5}$$

Then by inequality (4) we obtain

$$\frac{d}{dt}\arg\kappa(e^{it}) = \frac{d}{dt}\left\{(m/2+1)t + \Im\{\log h'(e^{it})\}\right\} = \frac{m}{2} + \Re\left\{1 + e^{it}\frac{h''(e^{it})}{h'(e^{it})}\right\} > 0.$$

Since h' has p zeros at the origin and is nonvanishing on $\partial\mathbb{D}$,

$$\begin{aligned}\int_0^{2\pi} \frac{d}{dt}\left\{\arg\kappa(e^{it})\right\} dt &= \int_0^{2\pi}\left(\frac{m}{2}+1\right) dt + \int_0^{2\pi} \Re\left\{e^{it}\frac{h''(e^{it})}{h'(e^{it})}\right\} dt \\ &= (m+2)\pi + \Im\int_{\partial\mathbb{D}} \frac{h''(z)}{h'(z)}\, dz \\ &= (m+2)\pi + 2(p-1)\pi \\ &= (2p+m)\pi.\end{aligned}$$

Hence, $\arg\kappa(e^{it})$ is a strictly increasing function on $[0, 2\pi+\epsilon)$ so that its growth on the interval $[0, 2\pi]$ is exactly $(2p+m)\pi$. Consequently, $\Im\kappa(e^{it})$ changes sign exactly $2p+m$ times on $[0, 2\pi+\epsilon)$ for any arbitrarily small $\epsilon > 0$, and, by equation (5), our contention holds. This ends the proof. □

As immediate applications of Theorem 1 and Proposition 1, we obtain the following corollaries:

Corollary 1. *Under the assumptions of Theorem* A*,* $f \in VH_4(1)$ *and* f *is a univalent close-to-convex function.*

Corollary 2. *Under the assumptions of Theorem* 1*, and if* $m = 1$*, then* f *is* $(p+1)$*-valent.*

Remark 2. Corollary 1 is more informative than Theorem A, and its proof is different and more direct than that of the authors [2].

We have been unable to construct harmonic mapping satisfying the assumptions of Theorem 1 and having valency exactly $p+m$. In view of the corollaries and our negative attempts in this regard, we pose the following conjecture:

Conjecture 1. *The valency of harmonic mappings* f *satisfying the assumptions of Theorem* 1 *is at most* $m+p-1$*.*

Obviously, this conjecture is both true and sharp if $m = 1$ and $p = 1$.

Acknowledgement

The authors would like to thank the referee for a careful reading of the paper.

References

[1] D. A. Brannan, *On functions of bounded boundary rotation. I*, Proc. Edinburgh Math. Soc. (2) **16** (1968/1969), 339–347, DOI 10.1017/S001309150001302X. MR0264045

[2] D. Bshouty and A. Lyzzaik, *Close-to-convexity criteria for planar harmonic mappings*, Complex Anal. Oper. Theory **5** (2011), no. 3, 767–774, DOI 10.1007/s11785-010-0056-7. MR2836322

[3] P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970; reprinted with supplement by Dover Publications, Momeola, NY, 2000. MR0268655

[4] Toshio Hayami, *A sufficient condition for p-valently harmonic functions*, Complex Var. Elliptic Equ. **59** (2014), no. 9, 1214–1222, DOI 10.1080/17476933.2013.826656. MR3210296

[5] Ronald J. Leach, *Multivalent and meromorphic functions of bounded boundary rotation*, Canad. J. Math. **27** (1975), 186–199, DOI 10.4153/CJM-1975-024-9. MR0393458

[6] Abdallah Lyzzaik, *Multivalent functions of bounded boundary rotation and weakly close-to-convex functions*, Proc. London Math. Soc. (3) **51** (1985), no. 3, 478–500, DOI 10.1112/plms/s3-51.3.478. MR805718

[7] Abdallah Lyzzaik, *Local properties of light harmonic mappings*, Canad. J. Math. **44** (1992), no. 1, 135–153, DOI 10.4153/CJM-1992-008-0. MR1152671

[8] Petru T. Mocanu, *Three-cornered hat harmonic functions*, Complex Var. Elliptic Equ. **54** (2009), no. 12, 1079–1084, DOI 10.1080/17476930903275987. MR2589172

[9] V. Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Ann. Acad. Sci. Fenn. Ser. A **33** (1931), 79 pp.

Department of Mathematics, Technion, Haifa, Israel
E-mail address: daoud@technion.ac.il

Astra Executive Consultant, Fahad Bin Sultan University, Tabuk, Saudi Arabia
E-mail address: alyzzaik@gmail.com

Department of Business Administration, Batman University, Batman, Turkey
E-mail address: mugesakar@hotmail.com