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COMMUTATIVITY OF NORMAL COMPACT OPERATORS VIA PROJECTIVE SPECTRUM

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Abstract. In this note we obtain commutativity criteria for normal compact operators using the projective spectrum. We thus improve a corresponding result obtained by Chagouel, Stessin and Zhu in Trans. Amer. Math. Soc. 368 (2016), 1559–1582.

1. Introduction

In [4], R. Yang introduced the concept of projective spectrum. For an n-tuple $\mathbb{A} = (A_1, \dots, A_n)$ of operators acting on a Hilbert space H, the *projective spectrum* of $\mathbb{A}$ is defined by

$$\Sigma(\mathbb{A}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1A_1 + \cdots + z_nA_n \text{ is noninvertible}\}.$$

Obviously, if H is infinite-dimensional and all A_i's are compact, then $\Sigma(\mathbb{A}) = \mathbb{C}^n$. To study the commutativity of normal compact operators, in [2] the authors gave the following modified definition of projective spectrum:

$$\sigma(\mathbb{A}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : I + z_1A_1 + \cdots + z_nA_n \text{ is noninvertible}\},$$

and the point projective spectrum:

$$\sigma_p(\mathbb{A}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \ker(I + z_1A_1 + \cdots + z_nA_n) \neq 0\}.$$

By using the modified projective spectrum, I. Chagouel, M. Stessin and K. Zhu obtained the following theorem.

Theorem 1.1 (Chagouel, Stessin and Zhu, 2016)**.** *Let $\mathbb{A} = (A_1, A_2, \dots, A_n)$ be an n-tuple of compact operators on a Hilbert space H. Suppose that*

(1) each A_i is self-adjoint and $\dim H = \infty$,
(2) each A_i is normal and $\dim H < \infty$.

Then the operators $A_1, \dots, A_n$ pairwise commute if and only if their projective spectrum $\sigma_p(\mathbb{A})$ consists of countably many, locally finite, complex hyperplanes in $\mathbb{C}^n$, where, "locally finite" means that for each $z_0 \in \mathbb{C}^n$, there is a neighborhood U_0 of z_0 such that $U_0 \cap \sigma_p(\mathbb{A})$ has finite branches.

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The paper [2] also pointed out that the theorem does not hold without a normality condition on the tuple. In the present paper, we will show that such a result is true for normal tuples under some mild conditions. As a particular case, we recover the cited result of Chagouel, Stessin and Zhu. In the following we shall use the notation from [2]. To state our result, we recall that an operator A satisfies Agmon's condition [1] if there is a ray $\{\text{Arg}\lambda = \theta\}$ such that A has no eigenvalues on the ray. With Agmon's condition, S. Seeley studied the complex powers of elliptic operators. Inspired by Agmon's condition, we introduce the following strengthening of Agmon's condition.

Definition 1.2. A normal compact operator A is said to satisfy the strong Agmon condition if there is an $\epsilon > 0$ and $\theta \in (0, 2\pi)$ such that A has no nonzero eigenvalues in $\{z : \theta - \epsilon < \text{Arg}z < \theta + \epsilon\}$.

The following theorem is the main result in the present note.

Theorem 1.3. *Let $\mathbb{A} = (A_1, A_2, \ldots, A_n)$ be a tuple of normal compact operators satisfying the strong Agmon condition. Then the following conditions are equivalent:*

1) *$\mathbb{A}$ is commutative.*
2) *$\sigma_p(\mathbb{A})$ consists of countably many, locally finite, complex hyperplanes in $\mathbb{C}^n$.*

Since self-adjoint compact operators and normal matrices satisfy the strong Agmon condition, Theorem 1.1 is a consequence of Theorem 1.3. The result is proved as follows. At first we will need the following technical condition.

Condition A. A normal compact operator A is said to satisfy Condition A if there is an $\epsilon > 0$ such that the set $\bigcap_{\lambda\in\sigma_p(A)}\{z \in \mathbb{C} : |1 + \lambda z| \geq \epsilon\}$ is unbounded.

It will be shown that the strong Agmon condition implies Condition A. As in [2], to get our main result, the key point is to consider the case $n = 2$. We will prove that if A satisfies Condition A and B is a normal compact operator, then $[A, B] = 0$ if and only if $\sigma_p(A, B)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$.

Compared to [2], firstly, our proof is shorter and more elementary. Secondly, we do not need a stronger hypothesis for the case of normal operators. We conjecture that the result is true for normal compact operators without any extra condition.

2. Proof of the main result

In this section, we will prove our main theorem. At first, we will show that the strong Agmon condition implies Condition A.

Lemma 2.1. *If a compact operator A satisfies the strong Agmon condition, then there exists $0 < \epsilon < 1$ and a complex sequence $\{z_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} z_n = \infty$$

and for every $\lambda \in \sigma_p(A)$ and $n \in \mathbb{N}$,

$$|1 + \lambda z_n| \geq \epsilon.$$

Proof. By Definition 1.2, there exist $0 \leq \theta < 2\pi$ and $0 < \delta < \pi$ such that

$$\sigma_p(e^{i\theta}A)\backslash\{0\} \subseteq \{z \in \mathbb{C} : 0 \leq \text{Arg}(z) < \pi - \delta \quad \text{or} \quad \pi + \delta < \text{Arg}(z) < 2\pi\}.$$

Take $0 < \epsilon < \sin\delta$, $z_n = e^{i\theta}n$; then $\lim_{n\to\infty} z_n = \infty$. Now, for any $\lambda \in \sigma_p(A)$,

$$e^{i\theta}\lambda \in \sigma_p(e^{i\theta}A) \subseteq \{z \in \mathbb{C} : 0 \leq \text{Arg}(z) < \pi - \delta \quad \text{or} \quad \pi + \delta < \text{Arg}(z) < 2\pi\} \cup \{0\}.$$

Obviously, if $\lambda = 0$, then

$$|1 + \lambda z_n| = 1 \geq \epsilon.$$

If $\lambda \neq 0$, then $-\frac{1}{e^{i\theta}\lambda} \in \{z \in \mathbb{C} : \delta < \text{Arg}z < 2\pi - \delta\}$, since $\text{Arg}(e^{i\theta}\lambda) = \pi - \text{Arg}(-\frac{1}{e^{i\theta}\lambda})$. The distance between $-\frac{1}{e^{i\theta}\lambda}$ and the positive x-axis is

$$\inf_{x>0} \left|x - (-\frac{1}{e^{i\theta}\lambda})\right| \geq \frac{\sin\delta}{|\lambda|};$$

then

$$|1 + \lambda z_n| = |\lambda| \cdot \left|z_n - (-\frac{1}{\lambda})\right| = |\lambda| \cdot \left|n - (-\frac{1}{e^{i\theta}\lambda})\right| \geq \sin\delta \geq \epsilon.$$

□

Lemma 2.2. *For compact operators A and B, suppose A is normal and satisfies Condition* A. *If $\mu \neq 0$ is a complex number such that the complex line $\{(z,w) \in \mathbb{C}^2 : \mu w + 1 = 0\}$ is contained in $\sigma_p(A,B)$ and $|\mu| = \|B\|$, then there exists a unit vector x such that*

$$Ax = 0 \quad \text{and} \quad Bx = \mu x. \tag{2.1}$$

Proof. Write

$$A = \sum_j \lambda_j e_j \otimes e_j,$$

where $\{e_j\}$ is an orthonormal sequence of eigenvectors of A with corresponding eigenvalues λ_j. Since A satisfies Condition A, there exists $0 < \epsilon < 1$ and a complex sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} z_n = \infty,$$

and for every $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|1 + \lambda_j z_n| \geq \epsilon.$$

Because the complex line $\mu w + 1 = 0$ is contained in $\sigma_p(A,B)$, for every $z \in \mathbb{C}$, $I + zA - \frac{1}{\mu}B$ has nontrivial kernel. There exists a unit vector v_n such that

$$\Big(I + z_n A - \frac{1}{\mu}B\Big)v_n = 0. \tag{2.2}$$

Since the unit ball of a Hilbert space is weakly compact, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ which converges weakly to some vector $v_0 \in H$. Since A, B are compact, we have

$$\lim_{k\to\infty} Av_{n_k} = Av_0 \quad \text{and} \quad \lim_{k\to\infty} Bv_{n_k} = Bv_0. \tag{2.3}$$

Let P_0 be the orthogonal projection onto $\ker A$. Now, we claim that $v_0 \neq 0$. To see this, we argue by contradiction. Assume $v_0 = 0$; then

$$\lim_{k\to\infty} (I + z_{n_k}A)v_{n_k} = \lim_{k\to\infty} \frac{1}{\mu}Bv_{n_k} = \frac{1}{\mu}Bv_0 = 0. \tag{2.4}$$

In the basis $\{e_j\}_j$,

$$(I - P_0)(I + z_{n_k}A)v_{n_k} = \sum_j (1 + \lambda_j z_{n_k})\langle v_{n_k}, e_j\rangle e_j,$$

which tends to 0, that is,

$$\lim_{k\to\infty}\sum_j |1+\lambda_j z_{n_k}|^2 |\langle v_{n_k}, e_j\rangle|^2 = 0. \tag{2.5}$$

Recall that $|1+\lambda_j z_n| \geq \epsilon$. Combining with (2.4), we get

$$\begin{aligned}\sum_j |1+\lambda_j z_{n_k}|^2 |\langle v_{n_k}, e_j\rangle|^2 &\geq \epsilon^2 \sum_j |\langle v_{n_k}, e_j\rangle|^2 \\ &= \epsilon^2(\|v_{n_k}\|^2 - \|P_0 v_{n_k}\|^2) \\ &= \epsilon^2(1-\|P_0(I+z_{n_k}A)v_{n_k}\|^2) \\ &\to \epsilon^2,\end{aligned}$$

which contradicts (2.5). By (2.2) and (2.3)

$$Av_0 = \lim_{k\to\infty} Av_{n_k} = \lim_{k\to\infty}\frac{1}{z_{n_k}}(-I+\frac{1}{\mu}B)v_{n_k} = 0,$$

by which $v_0 \in \ker A$. Recall that P_0 is the orthogonal projection onto $\ker A$; then

$$\begin{aligned} v_0 &= P_0 v_0 \\ &= w-\lim_{k\to\infty} P_0 v_{n_k} \\ &= w-\lim_{k\to\infty} P_0(-z_{n_k}A+\frac{1}{\mu}B)v_{n_k} \\ &= w-\lim_{k\to\infty}\frac{1}{\mu}P_0 B v_{n_k} \\ &= \frac{1}{\mu}P_0 B v_0.\end{aligned}$$

Since $|\mu| = \|B\|$, by the Pythagorean theorem

$$\begin{aligned}\|(I-P_0)Bv_0\|^2 &= \|Bv_0\|^2 - \|P_0 B v_0\|^2 \\ &\leq \|B\|^2\|v_0\|^2 - \|\mu v_0\|^2 = 0,\end{aligned}$$

that is,

$$Bv_0 = P_0 B v_0 = \mu v_0, \tag{2.6}$$

which shows that v_0 is a common eigenvector of A and B. By normalizing v_0, we get the unit vector x satisfying (2.1). □

We also need the following lemma.

Lemma 2.3. *For compact operators A and B, suppose A is normal and $(\lambda,\mu) \neq (0,0)$ are complex numbers such that the complex line $\{(z,w)\in\mathbb{C}^2 : \lambda z+\mu w+1=0\}$ is contained in $\sigma_p(A,B)$ and λ is an isolated eigenvalue of A. Then there exists a unit vector x such that*

$$Ax = \lambda x \quad \text{and} \quad \mu = \langle Bx, x\rangle. \tag{2.7}$$

Proof. We can choose a disc $D = D(\lambda,\delta)$ containing λ for a small $\delta > 0$ such that:

(1) $0 \notin D$ if $\lambda \neq 0$,
(2) $D \cap \sigma_p(A) = \{\lambda\}$,
(3) $uI - A$ is invertible for $u \in \partial D$.

Define

$$A_\epsilon := A + \epsilon B, \qquad \lambda_\epsilon := \lambda + \epsilon\mu.$$

Take $\sigma > 0$ small enough such that for $0 < |\epsilon| < \sigma$, $uI - A_\epsilon$ is invertible for $u \in \partial D$, $\lambda_\epsilon \in D$ and $\lambda_\epsilon \neq 0$. Since $(-\frac{1}{\lambda_\epsilon}, -\frac{\epsilon}{\lambda_\epsilon}) \in \sigma_p(A, B)$ and

$$\lambda_\epsilon I - A_\epsilon = \lambda_\epsilon(I - \frac{1}{\lambda_\epsilon}A - \frac{\epsilon}{\lambda_\epsilon}B),$$

we have λ_ϵ is an eigenvalue of A_ϵ. For any fixed $\epsilon > 0$ small enough, take a unit v_ϵ such that

$$(A_\epsilon - \lambda_\epsilon I)v_\epsilon = 0.$$

Consider the Riesz projections [3]

$$P_\epsilon = \frac{1}{2\pi i}\int_{\partial D}(uI - A_\epsilon)^{-1}\mathrm{d}u$$

and

$$P_0 = \frac{1}{2\pi i}\int_{\partial D}(uI - A)^{-1}\mathrm{d}u; \tag{2.8}$$

then $P_\epsilon \to P_0$ as $\epsilon \to 0$. Obviously $P_\epsilon v_\epsilon = v_\epsilon$. Rewrite P_ϵ as

$$\begin{aligned} P_\epsilon &= \frac{1}{2\pi i}\int_{\partial D}(uI - A_\epsilon)^{-1}\mathrm{d}u \\ &= \sum_{r=0}^{\infty}\frac{1}{2\pi i}\int_{\partial D}\epsilon^r((uI - A)^{-1}B)^r(uI - A)^{-1}\mathrm{d}u \\ &= \frac{1}{2\pi i}\int_{\partial D}(uI - A)^{-1}\mathrm{d}u \\ &\quad + \frac{1}{2\pi i}\epsilon\int_{\partial D}(uI - A)^{-1}B(uI - A)^{-1}\mathrm{d}u + O(\epsilon^2) \\ &= P_0 + \epsilon\tilde{P} + O(\epsilon^2), \end{aligned}$$

where $\tilde{P} = \frac{1}{2\pi i}\int_{\partial D}(uI - A)^{-1}B(uI - A)^{-1}\mathrm{d}u$. Accordingly $(A_\epsilon - \lambda_\epsilon I)P_\epsilon$ can be written as

$$\begin{aligned} (A_\epsilon - \lambda_\epsilon)P_\epsilon &= \big(A - \lambda I + \epsilon(B - \mu I)\big)\big(P_0 + \epsilon\tilde{P} + O(\epsilon^2)\big) \\ &= (A - \lambda I)P_0 + \epsilon\big((A - \lambda I)\tilde{P} + (B - \mu I)P_0\big) + O(\epsilon^2). \end{aligned} \tag{2.9}$$

Please note that

$$(A - \lambda I)P_0 = P_0(A - \lambda I) = 0.$$

Multiplying P_0 to the left of (2.9), we get

$$P_0(A_\epsilon - \lambda_\epsilon I)P_\epsilon = \epsilon P_0(B - \mu I)P_0 + O(\epsilon^2). \tag{2.10}$$

Recall that v_ϵ is a unit eigenvector, together with (2.10):

$$P_0(B - \mu I)P_0 v_\epsilon = O(\epsilon). \tag{2.11}$$

If $\lambda \neq 0$, then since A is compact, the range $\mathrm{Ran}P_0$ is of finite dimension. Thus we can choose a converging subsequence of $\{P_0 v_\epsilon\}$ with the limit v_0. In (2.11), let $\epsilon \to 0$ in the subsequence

$$P_0(B - \mu I)v_0 = 0. \tag{2.12}$$

We have $\|v_0\| = 1$ because

$$1 \geq \|v_0\| \geq \|P_\epsilon v_\epsilon\| - \|P_\epsilon v_\epsilon - P_0 v_\epsilon\| - \|v_0 - P_0 v_\epsilon\|.$$

If $\lambda = 0$, then $\mu \neq 0$. Consider $\tilde{B} = P_0(B - \mu I)P_0$ and an operator on $\mathrm{Ran}P_0$. Then $\tilde{B}$ has nontrivial kernel. Otherwise suppose it were injective. Since P_0BP_0 is compact, by Riesz-Schaulder theory, $\tilde{B}$ is invertible. Therefore there exists $d > 0$ such that

$$\|P_0(B - \mu I)P_0 v\| \geq d\|P_0 v\|, \quad \text{for all } v \in H,$$

which contradicts (2.11).

In summary, there is a unit vector v_0 such that (2.12) holds whether $\lambda = 0$ or not. Letting $x = v_0$, we have (2.7). □

From the above technical lemma, we have:

Corollary 2.4. *Let A and B be normal compact operators such that A satisfies Condition* A. *If $\sigma_p(A, B)$ consists of complex lines, then A and B have a common eigenvector.*

Proof. Choose μ to be the eigenvalue of B with maximal norm, that is, $|\mu| = \|B\|$. The case $\mu = 0$ is trivial, so suppose $\mu \neq 0$. The point $(0, -\frac{1}{\mu})$ is contained in $\sigma_p(A, B)$. By the assumption on $\sigma_p(A, B)$, there is a complex line $\lambda z + \mu w + 1 = 0$ in $\sigma_p(A, B)$ containing $(0, -\frac{1}{\mu})$.

If $\lambda \neq 0$, then $(-\frac{1}{\lambda}, 0)$ is contained in $\sigma_p(A, B)$, which indicates that λ is a nonzero eigenvalue of A. By Lemma 2.3 we have the desired result.

If $\lambda = 0$, the corollary comes from Lemma 2.2. □

Suppose A and B satisfy the conditions in Corollary 2.4. Define two sets of subspaces of H:

$$\begin{aligned} \mathcal{V} &= \{V \subseteq H : A(V) \subseteq V,\, B(V) \subseteq V\}, \\ \mathcal{W} &= \{W \in \mathcal{V} : AB = BA \text{ on } W\}. \end{aligned}$$

We have $0 \in \mathcal{W}$, and A and B commute if and only if $H \in \mathcal{W}$.

By Zorn's lemma, $\mathcal{W}$ has a maximal element W with respect to inclusion, and we argue by contradiction to show that $W = H$. W is closed since $\overline{W} \in \mathcal{W}$. Assume that $W \subsetneq H$, that is, $W^\perp \neq 0$. If there exists a common eigenvector of A and B in $W^\perp$, let W' be the subspace generated by the vector. Then $0 \neq W' \subseteq W^\perp$ such that $W' \in \mathcal{W}$, then $W \oplus W' \in \mathcal{W}$, which contradicts the maximality of W.

Let $W^\perp \in \mathcal{V}$ because A and B are normal. Denote the restricted operators on the Hilbert space $W^\perp$ by A' and B'. We only need to show that A' and B' have a common eigenvector. This is done if the operators A' and B' on the Hilbert space $W^\perp$ satisfy the conditions in Corollary 2.4, which is assured by the following proposition.

Proposition 2.5. *Let A and B be normal compact operators such that $\sigma_p(A, B)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$. If W is a closed invariant subspace of both A and B, then the restricted operators on W have the same property as A and B; that is, $A|_W$ and $B|_W$ are normal compact operators over the Hilbert space W such that $\sigma_p(A|_W, B|_W)$ consists of countably many, locally finite, complex lines.*

Proof. This can be concluded from the proof of Theorem 11 of [2]. □

The reason that the commutativity of A and B implies that $\sigma_p(A, B)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$ is trivial, since A and B are diagonalized by an orthonormal basis; see the proof of Theorem 11 in [2] for details. We have our main result.

Theorem 2.6. *If A and B are normal and compact and A satisfies Condition* A*, then the following conditions are equivalent:*

(1) A, B are commutative.
(2) $\sigma_p(A,B)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$.

Because self-adjoint operators and finite rank operators satisfy the strong Agmon condition automatically, by Lemma 2.1 and Theorem 2.6, we have

Corollary 2.7. *Let A and B be normal compact operators. Suppose A is self-adjoint or of finite rank. Then the following are equivalent:*

(1) A, B are commutative.
(2) $\sigma_p(A,B)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$.

Obviously, if both A and B are of finite rank, then the commutativity of A and B is equivalent to the finite-dimensional case, and Corollary 2.7 recovers Theorem 1.1. Next, we give an example which shows that there is a normal compact operator that does not satisfy Condition A.

Example 2.8. Let H be a Hilbert space with an orthonormal basis

$$\{e_{n,i} : n \in \mathbb{N}; 1 \leq i \leq 2^n\}.$$

Set $\omega_{n,i}$ to be the ith root of $x^{2^n} = 1$. Let $\nu_n = \sum_{j=1}^{n} \frac{1}{j}$. Then

$$\lambda_{n,i} = \frac{1}{\nu_n \omega_{n,i}} \to 0.$$

It is easy to verify that the operator

$$A = \sum_{n,j} \lambda_{n,j} e_{n,j} \otimes e_{n,j}$$

does not satisfy Condition A.

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