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A NEW LOOK AT BERNOULLI'S INEQUALITY

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ABSTRACT. In this work, a generalization of the well-known Bernoulli inequality is obtained by using the theory of discrete fractional calculus. As far as we know our approach is novel.

1. Introduction

In classical analysis the following inequality is attributed to Bernoulli: for a real number $x > -1$ and a nonnegative integer n, it holds that

$$(1+x)^n \geq 1 + nx. \tag{1.1}$$

One can find in the literature several (elementary) different proofs of inequality (1.1) (see e.g. [1, 10]). Moreover, various generalizations were also obtained throughout the years (cf. [9]) as well as different kinds of applications (see e.g. [7]).

In this work we obtain an inequality that generalizes (1.1) in a completely different direction than the ones mentioned before. The reasoning is that we use the theory of discrete fractional calculus [4], in particular, the (delta) Riemann–Liouville fractional operators which were introduced by Miller and Ross in 1988 [8] and for which real developments happened only in the past eight years. Therefore, we actually believe our main results to be new and obtained following a novel procedure.

In order to accomplish our desires we need to further develop the theory of linear fractional difference equations, which was initially started in the work [2] and generalized afterwards in [3]. More specifically, we solve explicitly the IVP[1]

$$\begin{aligned}&(\Delta^{\nu}_{a+\nu-1}x)(t) = y(t+\nu-1)x(t+\nu-1) + z(t+\nu-1),\ t \in \{a, a+1, a+2, \ldots\},\\ &x(a+\nu-1) = x_{a+\nu-1},\end{aligned}$$

and, after deducing some of its important consequences, we use a recent (comparison) result of [6] to deduce our Bernoulli-type inequality.

This paper is organized as follows: In section 2 we provide the reader some background on the discrete fractional calculus theory. In section 3 we present our achievements.

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[1]This result alone might obviously be used by researchers in other contexts.

2. Preliminaries on discrete fractional calculus

In this section we introduce the reader to basic concepts and results about discrete fractional calculus (the monograph [4], particularly Chapter 2, could be useful to the reader).

Throughout this work and as usual we assume that empty sums and products equal 0 and 1, respectively.

The power function is defined by

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}, \text{ for } x, x-y \in \mathbb{R}\backslash\{\dots,-2,-1\},$$
$$x^{(y)} = 0, \text{ for } x \notin \mathbb{Z}^- \text{ and } x-y \in \mathbb{Z}^-.$$

For $a \in \mathbb{R}$ and $0 < \nu \leq 1$ we define the set $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$. Also, we use the notation $\sigma(s) = s+1$ for the shift operator and $(\Delta f)(t) = f(t+1) - f(t)$ for the forward difference operator.

For a function $f : \mathbb{N}_a \to \mathbb{R}$, the *discrete fractional sum of order* $\nu \geq 0$ is defined as

$$(\Delta_a^0 f)(t) = f(t), \quad t \in \mathbb{N}_a,$$
$$(\Delta_a^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s), \quad t \in \mathbb{N}_{a+\nu-1},\ \nu > 0. \tag{2.1}$$

Remark 2.1. Note that the operator $\Delta_a^{-\nu}$ with $\nu > 0$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu-1}$. Also observe that if $\nu = 1$, then we get the summation operator

$$(\Delta_a^{-1} f)(t) = \sum_{s=a}^{t-1} f(s).$$

The *discrete fractional derivative of order* $\nu \in (0,1]$ is defined by

$$(\Delta_a^{\nu} f)(t) = (\Delta\Delta_a^{-(1-\nu)} f)(t), \quad t \in \mathbb{N}_{a+\nu-1}.$$

Remark 2.2. Note that if $\nu = 1$, then the fractional derivative is just the forward difference operator.

3. Main results

This section is devoted in great part to deducing our generalized Bernoulli inequality.

In [3] it was shown that, for $t \in \mathbb{N}_{a+\nu-1}$,

$$x(t) = \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} x_{a+\nu-1} + \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s+\nu-1, x(s+\nu-1)) \tag{3.1}$$

is the solution of the following nonlinear fractional difference initial value problem:

$$(\Delta_{a+\nu-1}^{\nu} x)(t) = f(t+\nu-1, x(t+\nu-1)), \quad t \in \{a, a+1, a+2, \dots\},$$
$$x(a+\nu-1) = x_{a+\nu-1}.$$

For our purposes we need the solution of (3.1) in the particular case when $f(t,x) = y(t)x + z(t)$. The next result is obtained following the same procedure as for the case $f(t,x) = y(t)x$ done in [3], and, therefore, we leave the details of its proof to the reader.

Theorem 3.1. *Let $a \in \mathbb{R}$ and $\nu \in (0,1]$. Suppose that $y : \mathbb{N}_{a+\nu-1} \to \mathbb{R}$ is a function. Define an operator T by*

$$\begin{aligned}(T_y^0 f)(t) &= f(t),\\ (T_y^1 f)(t) &= (T_y f)(t) = \left(\Delta_a^{-\nu} y(s+\nu-1) f(s+\nu-1)\right)(t),\\ (T_y^{k+1} f)(t) &= (T_y T_y^k)(t), \quad k \in \mathbb{N}^1,\end{aligned}$$

for $t \in \mathbb{N}_{a+\nu-1}$. Then, the function
(3.2)

$$x(t) = \sum_{k=a}^{t-(\nu-1)} \left[\frac{x_{a+\nu-1}}{\Gamma(\nu)} \left(T_y^{k-a} (s-a)^{(\nu-1)} \right)(t) + \left(T_y^{k-a} \Delta_a^{-\nu} z(s+\nu-1) \right)(t) \right]$$

is the solution of the summation equation

$$x(t) = \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} x_{a+\nu-1} + \left(\Delta_a^{-\nu}[y(s+\nu-1)x(s+\nu-1) + z(s+\nu-1)]\right)(t),$$

for all $t \in \mathbb{N}_{a+\nu-1}$.

Remark 3.2. In Theorem 3.1, the notation

$$\left(\Delta_a^{-\nu} y(s+\nu-1) f(s+\nu-1)\right)(t)$$

stands for $\left(\Delta_a^{-\nu} g\right)(t)$ where $g(s) = y(s+\nu-1) f(s+\nu-1)$.

Let us now introduce some notation. We define a function $E : \mathbb{N}_{a+\nu-1} \times \mathbb{R}^5 \to \mathbb{R}$ (it can be thought of as a discrete Mittag–Leffler function) by

$$E(t,a,\nu,\beta,c,\lambda) = \sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\beta)} (t-\lambda+(k-a)(\nu-1))^{((k-a)\nu+\beta-1)}$$

whenever the right hand side makes sense.

Corollary 3.3. *If $y(t) = c$ for some $c \in \mathbb{R}$ and all $t \in \mathbb{N}_{a+\nu-1}$ in Theorem 3.1, then the solution given by (3.2) is*

$$x(t) = x_{a+\nu-1} E(t,a,\nu,\nu,c,a) + \sum_{r=a}^{t-\nu} E(t,a,\nu,\nu,c,\sigma(r)) z(r+\nu-1), \quad t \in \mathbb{N}_{a+\nu-1}.$$

It is pertinent to formulate the following consequence of Corollary 3.3.

Corollary 3.4. *Suppose that the function z is a constant equal to K in Corollary 3.3. Then,*

$$x(t) = x_{a+\nu-1} E(t,a,\nu,\nu,c,a) + K E(t,a,\nu,\nu+1,c,a), \quad t \in \mathbb{N}_{a+\nu-1}.$$

Proof. Let us first note that (cf. [4, Theorem 1.8])

$$\Delta_t (s-t)^{(r)} = -r(s-\sigma(t))^{(r-1)}, \; s,r \in \mathbb{R}.$$

Moreover,

$$\sum_{k=a}^{b-1} \Delta f(k) = f(b) - f(a).$$

Therefore,

$$
\begin{aligned}
&\sum_{r=a}^{t-\nu} E(t,a,\nu,\nu,c,\sigma(r))z(r+\nu-1)\\
&= K\sum_{r=a}^{t-\nu}\sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\nu)}(t-\sigma(r)+(k-a)(\nu-1))^{((k-a)\nu+\nu-1)}\\
&= K\sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\nu)} \sum_{r=a}^{t-(\nu-1)-1} (t-\sigma(r)+(k-a)(\nu-1))^{((k-a)\nu+\nu-1)}\\
&= K\sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\nu)((k-a)\nu+\nu)}\\
&\qquad\cdot \sum_{r=a}^{t-(\nu-1)-1} -((k-a)\nu+\nu)(t-\sigma(r)+(k-a)(\nu-1))^{((k-a)\nu+\nu-1)}\\
&= K\sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\nu+1)}[-(t-r+(k-a)(\nu-1))^{((k-a)\nu+\nu)}]_{r=a}^{r=t-(\nu-1)}\\
&= K\sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu+\nu+1)}(t-a+(k-a)(\nu-1))^{((k-a)\nu+\nu)}\\
&= KE(t,a,\nu,\nu+1,c,a),
\end{aligned}
$$

and the proof is done. □

We now need to address the question of the sign of the function $E(t,a,\nu,\nu,c,a)$. First we note that it is the solution of the fractional IVP:

$$
\begin{aligned}
&(\Delta^{\nu}_{a+\nu-1}x)(t) = cx(t+\nu-1), \quad t\in\{a,a+1,a+2,\ldots\},\ 0<\nu\le 1,\\
&x(a+\nu-1)=1.
\end{aligned}
$$

Let us now recall a recent result proved by Jia *et al.*:

Theorem 3.5 ([6, Theorem 4.2.]). *Assume $c_1(t)\ge c_2(t)\ge -\nu$, $0<\nu<1$ and $x(t)$, $y(t)$ are solutions of the equations*

$$(\Delta^{\nu}_{a+\nu-1}x)(t) = c_1(t)x(t+\nu-1), \quad t\in\{a,a+1,a+2,\ldots\}$$

and

$$(\Delta^{\nu}_{a+\nu-1}y)(t) = c_2(t)y(t+\nu-1), \quad t\in\{a,a+1,a+2,\ldots\},$$

respectively, satisfying $x(a+\nu-1)\ge y(a+\nu-1)>0$. Then,

$$x(t)\ge y(t), \quad t\in\mathbb{N}_{a+\nu-1}.$$

A closer look at the proof of the previous theorem permits us to conclude immediately that $y(a+\nu-1)$ might be equal to zero, and the result still remains. Moreover, the representation for the Riemann–Liouville fractional difference,

$$(\Delta^{\nu}_{a}f)(t) = \frac{1}{\Gamma(-\nu)}\sum_{k=a}^{t+\nu}(t-\sigma(k))^{(-\nu-1)}f(k),\ t\in\mathbb{N}_{a-\nu+1}, \tag{3.3}$$

is used to prove the result and, hence, the restriction $\nu\ne 1$. Nevertheless, M. Holm showed in [5, Theorem 2.2.] the continuity of the fractional difference operator (3.3) with respect to ν, and, therefore, one may also consider $\nu = 1$. In conclusion, for a real number a, $0 < \nu \leq 1$ and $c \geq -\nu$:

$$E(t, a, \nu, \nu, c, a) \geq 0, \quad t \in \mathbb{N}_{a+\nu-1}. \tag{3.4}$$

We are now ready to prove our main result:

Theorem 3.6 (Generalized Bernoulli inequality)**.** *Let $\nu \in (0, 1]$, $c \in [-\nu, \infty)$ and $a \in \mathbb{R}$. Then, the following inequality holds:*

$$cE(t, a, \nu, \nu + 1, c, a) \geq c\frac{(t-a)^{(\nu)}}{\Gamma(\nu + 1)}, \quad t \in \mathbb{N}_{a+\nu-1}. \tag{3.5}$$

Proof. Let $c \geq -\nu$ and $x : \mathbb{N}_{a+\nu-1} \to \mathbb{R}$ be the function defined by

$$x(t) = c\frac{(t-a)^{(\nu)}}{\Gamma(\nu + 1)}.$$

Then $x(a + \nu - 1) = 0$ and $(\Delta^{\nu}_{a+\nu-1}x)(t) = (\Delta^{\nu}_{a+\nu}x)(t) = c$ by [4, Theorem 2.40]. Therefore,

$$cx(t + \nu - 1) + c = c^2\frac{(t + \nu - 1 - a)^{(\nu)}}{\Gamma(\nu + 1)} + c \geq (\Delta^{\nu}_{a+\nu-1}x)(t).$$

Define the function m by

$$m(t + \nu - 1) = cx(t + \nu - 1) + c - (\Delta^{\nu}_{a+\nu-1}x)(t),$$

which is nonnegative. By Corollary 3.3 we get (note that $x_a = 0$)

$$x(t) = \sum_{r=a}^{t-\nu} E(t, a, \nu, \nu, c, \sigma(r))(c - m(r + \nu - 1)).$$

Hence,

$$\begin{aligned}
x(t) &= cE(t, a, \nu, \nu + 1, c, a) - \sum_{r=a}^{t-\nu} m(r + \nu - 1) \\
&\quad \cdot \sum_{k=a}^{t-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu + \nu)}(t - \sigma(r) + (k-a)(\nu - 1))^{((k-a)\nu+\nu-1)} \\
&= cE(t, a, \nu, \nu + 1, c, a) - \sum_{r=a}^{t-\nu} m(r + \nu - 1) \\
&\quad \cdot \sum_{k=a}^{t-\sigma(r)+a-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu + \nu)}(t - \sigma(r) + (k-a)(\nu - 1))^{((k-a)\nu+\nu-1)} \\
&= cE(t, a, \nu, \nu + 1, c, a) - \sum_{r=0}^{t-a-\nu} m(r + a + \nu - 1) \\
&\quad \cdot \sum_{k=a}^{t-\sigma(r)-(\nu-1)} \frac{c^{k-a}}{\Gamma((k-a)\nu + \nu)}(t - \sigma(r) - a + (k-a)(\nu - 1))^{((k-a)\nu+\nu-1)} \\
&= cE(t, a, \nu, \nu + 1, c, a) - \sum_{r=0}^{t-a-\nu} m(r + a + \nu - 1)E(t - \sigma(r), a, \nu, \nu, c, a).
\end{aligned}$$

Finally, by (3.4) we conclude that

$$x(t) \leq cE(t, a, \nu, \nu + 1, c, a),$$

which is equivalent to

$$c\frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)} \leq cE(t, a, \nu, \nu + 1, c, a).$$

The proof is done. □

We finish this work by showing that inequality (3.5) truly generalizes the Bernoulli inequality; i.e. when we let $\nu = 1$ (and $a = 0$) in (3.5), we get (1.1):

$$\begin{aligned} cE(t, 0, 1, 2, c, 0) &\geq c\frac{t^{(1)}}{\Gamma(2)} \\ \Leftrightarrow \sum_{k=0}^{t} \frac{c^{k+1}}{\Gamma(k+2)} t^{(k+1)} &\geq ct \\ \Leftrightarrow -1 + \sum_{k=0}^{t} \frac{c^{k}}{\Gamma(k+1)} t^{(k)} &\geq ct \\ \Leftrightarrow (1+c)^t &\geq 1 + ct, \quad t \in \mathbb{N}_0, \end{aligned}$$

where the last equivalency follows from [3, Remark 3.7].

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