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DERIVED EQUIVALENCE, ALBANESE VARIETIES, AND THE ZETA FUNCTIONS OF 3–DIMENSIONAL VARIETIES

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Abstract. We show that any derived equivalent smooth, projective varieties of dimension 3 over a finite field $\mathbb{F}_q$ have equal zeta functions. This result is an application of the extension to smooth, projective varieties over any field of Popa and Schnell's proof that derived equivalent smooth, projective varieties over $\mathbb{C}$ have isogenous Albanese torsors; this result is proven in an appendix by Achter, Casalaina-Martin, Honigs and Vial.

The problem of characterizing the bounded derived category of coherent sheaves of a variety has connections to birational geometry, the minimal model program, mirror symmetry (in particular, the conjecture of Kontsevich [12]), and motivic questions.

Orlov has conjectured that derived equivalent smooth, projective varieties have isomorphic motives [13]. This conjecture predicts that smooth, projective varieties over a finite field that are derived equivalent have equal zeta functions. The prediction holds in the case of curves since derived equivalent smooth, projective curves over a finite field are isomorphic: proof in the genus 1 case is given by Antieau, Krashen and Ward [2, Example 2.8], and proof in all other cases is a consequence of Bondal and Orlov [3, Theorem 2.5], which shows that derived equivalent varieties with ample or anti-ample canonical bundle must be isomorphic. In [8], it was verified that derived equivalent smooth, projective varieties over a finite field that are abelian or of dimension 2 have equal zeta functions.

In this paper, we prove the following extension of these results:

Theorem A. *Let $X, Y/\mathbb{F}_q$ be derived equivalent smooth, projective varieties of dimension 3, where $\mathbb{F}_q$ is a finite field with q elements. Then $\zeta(X) = \zeta(Y)$.*

The proof of Theorem A is similar to the argument in [8] proving that derived equivalent smooth, projective surfaces over any finite field have equal zeta functions: it is accomplished by comparing the eigenvalues of the geometric Frobenius morphism acting on the ℓ-adic étale cohomology groups of the varieties in question.

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The crucial ingredient for making this comparison between the point-counts of 3-dimensional varieties is the following theorem, proven in Appendix A, which has as a corollary that if X and Y are derived equivalent smooth, projective varieties over a finite field $\mathbb{F}_q$ and $\overline{X}$, $\overline{Y}$ are their base changes to $\overline{\mathbb{F}_q}$, then there is an isomorphism

$$H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \cong H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)$$

that is compatible with the action of the q-th power geometric Frobenius morphism:

Theorem B. *Derived equivalent smooth, projective varieties X and Y over an arbitrary field k have isogenous Albanese varieties.*

The strategy of the proof of Theorem B is similar to Popa and Schnell's proof for the case $k = \mathbb{C}$ [15]. However, to work over an arbitrary base field, we account for some pathologies concerning non-reduced group schemes and use the Albanese torsor to circumvent the selection of a rational point.

An alternate proof of Theorem B over $\mathbb{C}$ has been obtained by R. Abuaf in his Theorem 3.0.14 of [1]. It is conceivable that similar methods to those in loc. cit. can be used over algebraically closed fields of arbitrary characteristic.

1. Background

We take a *variety* to be a separated, integral scheme of finite type over a field. In this section, X and Y denote smooth, projective varieties.

Definition 1.1. An exact functor F between derived categories $D^b(X)$ and $D^b(Y)$ is a *Fourier–Mukai transform* if there exists an object $P \in D^b(X \times Y)$, called a *Fourier–Mukai kernel*, such that

$$F \cong p_{Y*}(p_X^*(-) \otimes P) =: \Phi_P, \tag{1}$$

where p_X and p_Y are the projections $X \times Y \to X$ and $X \times Y \to Y$. A Fourier–Mukai transform that is an equivalence of categories is called a *Fourier–Mukai equivalence*. The pushforward, pullback, and tensor in (1) are all in their derived versions, but the notation is suppressed.

A *derived equivalence* is an exact equivalence between derived categories; varieties are said to be derived equivalent if their associated bounded derived categories are exact equivalent. By the following theorem, in the context of this paper, derived equivalence and Fourier–Mukai equivalence are synonymous.

Theorem 1.2 (Orlov [14, Theorem 3.2.1])**.** *Let X and Y be smooth projective varieties and $F : D^b(X) \to D^b(Y)$ an exact equivalence. Then there is an object $\mathcal{E} \in D^b(X \times Y)$ such that F is isomorphic to the functor $\Phi_{\mathcal{E}}$, and the object $\mathcal{E}$ is determined uniquely up to isomorphism.*

The full statement of [14, Theorem 3.2.1] is stronger than what is given here, but the statement in Theorem 1.2 is sufficient for the purposes of this paper.

1.3. Let X and Y be smooth, projective varieties over a perfect field. Any Fourier–Mukai transform gives a map on Chow groups: The functor $\Phi_{\mathcal{E}}$ induces a map

$$\Phi_{\mathcal{E}}^{\mathrm{CH}} = p_{Y*}(v(\mathcal{E}) \cup p_X^*(-)) : \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(Y)_{\mathbb{Q}}$$

where $v(\mathcal{E}) := \mathrm{ch}(\mathcal{E}).\sqrt{\mathrm{td}(X \times Y)}$ is the *Mukai vector* of $\mathcal{E}$ (see for instance [10, Definition 5.28]). Since $\Phi_{\mathcal{E}}$ is an equivalence, $\Phi_{\mathcal{E}}$ is a bijection (cf. [10, Remark 5.25, Proposition 5.33]).

Similarly, the cycle class of $v(\mathcal{E})$ inside any Weil cohomology theory H (i.e., de Rham, singular, crystalline or ℓ-adic étale) induces a map $\Phi_{\mathcal{E}}^H = p_{Y*}(\mathrm{cl}(v(\mathcal{E})) \cup p_X^*(-))$ that factors through the above map on Chow rings with rational coefficients. This map on cohomology does not necessarily preserve degree, and, in the case of ℓ-adic étale cohomology of varieties over finite fields, Tate twists must be accounted for the map to be compatible with the action of geometric Frobenius, so care must be taken with the domain and codomain of $\Phi_{\mathcal{E}}^H$. The map $\Phi_{\mathcal{E}}^H$ gives the following isomorphisms compatible with the action of geometric Frobenius φ between the *even* and *odd Mukai–Hodge structures* [8,9], of X and Y, where d denotes $\dim(X)(=\dim(Y))$:

$$\bigoplus_{i=0}^{d} H^{2i}(X)(i) \cong \bigoplus_{i=0}^{d} H^{2i}(Y)(i), \tag{2}$$

$$\bigoplus_{i=1}^{d} H^{2i-1}(X)(i) \cong \bigoplus_{i=1}^{d} H^{2i-1}(Y)(i). \tag{3}$$

2. Zeta functions

Theorem A. *Let $X, Y/\mathbb{F}_q$ be derived equivalent smooth, projective varieties of dimension 3, where $\mathbb{F}_q$ is a finite field with q elements. Then $\zeta(X) = \zeta(Y)$.*

Proof. Let $\overline{X}$, $\overline{Y}$ be the base changes of X and Y to the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Fix $\ell \in \mathbb{Z}^+$ prime such that $(q, \ell) = 1$.

By the Lefschetz fixed-point formula for Weil cohomologies (see Proposition 1.3.6 and Section 4 of Kleiman [11]), to prove this theorem it is sufficient to show that for any $n \in \mathbb{N}$, the traces of the geometric q^n-th power Frobenius map φ^n acting on $H^i(\overline{X}, \mathbb{Q}_\ell)$ and $H^i(\overline{Y}, \mathbb{Q}_\ell)$ are the same for each $0 \le i \le 6$.

Let φ be the (q-th power) geometric Frobenius morphism. By Theorem 1.2, the derived equivalence $D^b(X) \cong D^b(Y)$ is isomorphic to a Fourier–Mukai functor $\Phi_{\mathcal{E}} := p_{Y*}(p_X^*(-) \otimes \mathcal{E})$ for some $\mathcal{E} \in D^b(X \times Y)$. Taking the traces of the action of φ^* on the equations (2), (3), and using the fact that the presence of a Tate twist (j) has the effect of multiplying the eigenvalues of the action of φ^* on cohomology by $\frac{1}{q^j}$, we have:

$$\sum_{i=0}^{3} \frac{1}{q^i} \operatorname{Tr}(\varphi^* | H^{2i}(\overline{X}, \mathbb{Q}_\ell)) = \sum_{i=0}^{3} \frac{1}{q^i} \operatorname{Tr}(\varphi^* | H^{2i}(\overline{Y}, \mathbb{Q}_\ell)), \tag{4}$$

$$\sum_{i=1}^{3} \frac{1}{q^i} \operatorname{Tr}(\varphi^* | H^{2i-1}(\overline{X}, \mathbb{Q}_\ell)) = \sum_{i=1}^{3} \frac{1}{q^i} \operatorname{Tr}(\varphi^* | H^{2i-1}(\overline{Y}, \mathbb{Q}_\ell)). \tag{5}$$

The values $\operatorname{Tr}(\varphi^* | H^i(\overline{X}, \mathbb{Q}_\ell))$ and $\operatorname{Tr}(\varphi^* | H^i(\overline{Y}, \mathbb{Q}_\ell))$ are trivially equal for $i = 0, 6$, so (4) reduces to

$$\begin{aligned} &\tfrac{1}{q} \operatorname{Tr}(\varphi^* | H^2(\overline{X}, \mathbb{Q}_\ell)) + \tfrac{1}{q^2} \operatorname{Tr}(\varphi^* | H^4(\overline{X}, \mathbb{Q}_\ell)) \\ &\qquad = \tfrac{1}{q} \operatorname{Tr}(\varphi^* | H^2(\overline{Y}, \mathbb{Q}_\ell)) + \tfrac{1}{q^2} \operatorname{Tr}(\varphi^* | H^4(\overline{Y}, \mathbb{Q}_\ell)). \end{aligned} \tag{6}$$

By Deligne's Hard Lefschetz Theorem for ℓ-adic étale cohomology [6, Théorème 4.1.1], or by Poincaré duality, we have the following lemma:

Lemma 2.1 ([8, Lemma 4.2])**.** *Let $V/\mathbb{F}_q$ be a smooth, projective variety of dimension d. If the multiset of eigenvalues of φ^* acting on $H^i_{\text{ét}}(\overline{V}, \mathbb{Q}_\ell)$, $0 \le i < d$,* *are* $\{\alpha_1, \dots, \alpha_n\}$*, then the set of eigenvalues of* φ^* *acting on* $H^{2d-i}_{\text{ét}}(\overline{V}, \mathbb{Q}_\ell)$ *are* $\{q^{d-i}\alpha_1, \dots, q^{d-i}\alpha_n\}$.

By Lemma 2.1, (6) implies that $\operatorname{Tr}(\varphi^*|H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^*|H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell))$ and $\operatorname{Tr}(\varphi^*|H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^*|H^4_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell))$.

By Lemma 2.1, (5) implies that

$$\begin{aligned} &\tfrac{2}{q}\operatorname{Tr}(\varphi^*|H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) + \tfrac{1}{q^2}\operatorname{Tr}(\varphi^*|H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) \\ &\qquad = \tfrac{2}{q}\operatorname{Tr}(\varphi^*|H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)) + \tfrac{1}{q^2}\operatorname{Tr}(\varphi^*|H^3_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)). \end{aligned} \tag{7}$$

By Corollary A.4,

$$\operatorname{Tr}(\varphi^*|H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^*|H^1_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)).$$

So, by Lemma 2.1, we have $\operatorname{Tr}(\varphi^*|H^5_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^*|H^5_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell))$.

Since (2) and (3) are compatible with the action of φ^*, they are compatible with the action of φ^{n*}, and hence the above statements comparing the traces of the action of φ^* also hold true if φ^* is replaced by φ^{n*}. In particular, by (7), we have

$$\operatorname{Tr}(\varphi^{n*}|H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^{n*}|H^3_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell)),$$

and now we have demonstrated that $\operatorname{Tr}(\varphi^{n*}|H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\varphi^{n*}|H^i_{\text{ét}}(\overline{Y}, \mathbb{Q}_\ell))$ for all $0 \le i \le 6$, as required. □

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Appendix A. Derived equivalent varieties have isogenous Picard varieties

Although Popa and Schnell only claim [15, Theorem A] that derived equivalent complex varieties have isogenous Picard varieties, their result (and its proof) is valid, with minimal changes, over an arbitrary field. Our goal in this appendix is to explain:

Theorem A.1. *Let* X *and* Y *be smooth projective varieties over a field* K*. If* X *and* Y *are derived equivalent, then* $\operatorname{Pic}^0(X)_{\text{red}}$ *and* $\operatorname{Pic}^0(Y)_{\text{red}}$ *are isogenous (over* K*).*

In outline, the proof of Theorem A.1 in [15] for varieties over an algebraically closed field k proceeds as follows. A theorem of Rouquier implies that there is an isomorphism of group schemes

$$F : (\operatorname{Aut}^0_{X/k})_{\text{red}} \times (\operatorname{Pic}^0_{X/k})_{\text{red}} \xrightarrow{\sim} (\operatorname{Aut}^0_{Y/k})_{\text{red}} \times (\operatorname{Pic}^0_{Y/k})_{\text{red}}. \tag{8}$$

Unfortunately, F need not preserve the given decompositions of the source and target schemes. Using F, Popa and Schnell identify distinguished subgroups (actually, abelian varieties) $A_X \subseteq (\operatorname{Aut}^0_{X/k})_{\text{red}}$ and $A_Y \subseteq (\operatorname{Aut}^0_{Y/k})_{\text{red}}$, and show that F induces an isomorphism

$$A_X \times (\operatorname{Pic}^0_{X/k})_{\text{red}} \xrightarrow{\sim} A_Y \times (\operatorname{Pic}^0_{Y/k})_{\text{red}}.$$

By Poincaré reducibility, it now suffices to show that A_X and A_Y are isogenous. For this, they construct a homomorphism

$$\pi : A_X \times (\mathrm{Pic}^0_{X/k})_{\mathrm{red}} \longrightarrow A_X \times A_Y \times \widehat{A}_X \times \widehat{A}_Y$$

and show that $\mathrm{Im}(\pi)$ is isogenous via the projections p_{13} and p_{24} to both $A_X \times_k \widehat{A}_X$ and $A_Y \times_k \widehat{A}_Y$.

If we now consider varieties over an arbitrary (perfect) field K, since the formation of automorphism and Picard schemes commutes with base extension, it makes sense to descend F (and subsequent constructions) from $\overline{K}$ to K. This goes through without incident, except that the construction of π detailed in [15] involves a choice of point in the support of the kernel of the Mukai transform. We circumvent this appeal to the existence of rational points on X and Y by invoking the Albanese torsor.

A.1. **Preliminaries.** For an arbitrary group scheme G over a field, the maximal reduced subscheme G_{red} need not be a group scheme; but this does not happen for the Picard scheme [7, VI.3]. For a smooth projective variety X/K, let $\mathrm{P}(X) = \mathrm{Pic}^0(X)_{\mathrm{red}}$ and $\mathrm{G}(X) = \mathrm{Aut}^0(X)_{\mathrm{red}}$. Then $\mathrm{P}(X)$ is an abelian variety; and we will only work with $\mathrm{G}(X)$ when the base field is perfect, in which case $\mathrm{G}(X)$ is an irreducible group scheme.

Let X/K be a geometrically reduced variety over K. Then X admits an (abelian) Albanese variety $\mathrm{Alb}(X)/K$, a torsor $\mathrm{Alb}^1(X)$, and a morphism $X \to \mathrm{Alb}^1(X)$ which is universal for morphisms from X into torsors under abelian varieties (see, e.g., [17, §2]).

If $P \in X(K)$ is a point, then there is a pointed morphism $(X, P) \to (\mathrm{Alb}(X), \mathcal{O})$ which is universal for pointed morphisms from X to abelian varieties. We will sometimes denote $\mathrm{Alb}(X)$, together with this morphism, as $\mathrm{Alb}(X, P)$.

Let $\mathrm{G}(X) = \mathrm{Aut}^0(X)_{\mathrm{red}}$. If $P \in X(K)$ is a base point, Popa and Schnell compute [15, Lemma 2.2] a canonical morphism

$$\mathrm{Alb}(\mathrm{G}(X)) \xrightarrow{f_{(X,P)}} \mathrm{Alb}(X, P).$$

Lemma A.2. *Let X/K be a variety over a perfect field.*

(a) *There is a canonical action* $\mathrm{Alb}(\mathrm{G}(X)) \times \mathrm{Alb}^1(X) \to \mathrm{Alb}^1(X)$ *which induces a canonical morphism*

$$\mathrm{Alb}(\mathrm{G}(X)) \xrightarrow{g_X} \mathrm{Alb}(X).$$

(b) *If $P \in X(K)$ is a point, then the trivialization* $\mathrm{Alb}(X, P) \xrightarrow{\sim} \mathrm{Alb}^1(X)$ *makes the following diagram commute:*

$$\begin{tikzcd} \mathrm{Alb}(\mathrm{G}(X)) \arrow[r, "f_{(X,P)}"] \arrow[d, "="] & \mathrm{Alb}(X, P) \arrow[d, "\sim"] \\ \mathrm{Alb}(\mathrm{G}(X)) \arrow[r, "g_X"'] & \mathrm{Alb}^1(X). \end{tikzcd}$$

Proof. By the universal property of the Albanese, the composition $\mathrm{G}(X) \times X \to X \to \mathrm{Alb}^1(X)$ factors through $\mathrm{Alb}^1(\mathrm{G}(X) \times X) \cong \mathrm{Alb}^1(\mathrm{G}(X)) \times \mathrm{Alb}^1(X)$.

Since $\mathrm{G}(X)$ admits a K-rational point, its Albanese torsor coincides with its Albanese variety, and we obtain an action

$$\mathrm{Alb}(\mathrm{G}(X)) \times \mathrm{Alb}^1(X) \longrightarrow \mathrm{Alb}^1(X).$$

In particular, $\mathrm{Alb}(\mathrm{G}(X))$ acts as a connected group scheme of automorphisms of $\mathrm{Alb}^1(X)$; by Lemma A.3, we obtain a morphism $g_X : \mathrm{Alb}(\mathrm{G}(X)) \to \mathrm{Alb}(X)$.

Popa and Schnell construct $f_{(X,P)}$ in a similar way, except that they work with the Albanese varieties of the pointed varieties $(\mathrm{G}(X), \mathrm{id})$ and (X, P). Part (b) then follows from the universality of $X \to \mathrm{Alb}^1(X)$ into abelian torsors and of $X \to \mathrm{Alb}(X, P)$ into abelian varieties. □

Lemma A.3. *Let A/K be an abelian variety, and let T/K be a torsor under A. Then* $\mathrm{Aut}(T)^0 \cong A$.

Proof. Over an algebraic closure, we have $\mathrm{Aut}(T)^0_{\overline{K}} \cong \mathrm{Aut}(A)^0_{\overline{K}} \cong A_{\overline{K}}$. Therefore, the inclusion $A \hookrightarrow \mathrm{Aut}(T)^0$ induced by the faithful action of A on T is an isomorphism. □

A.2. **Proof of the Popa–Schnell theorem.** Having dispatched these preliminaries, we now explain how to adapt the proof of the complex version of Theorem A.1 in [15] to account for an arbitrary base field. At each stage, we will see that the morphisms used in [15], a priori defined over an algebraically closed field, actually descend to a field of definition.

Let $\overline{K}$ be an algebraic closure of K; for a variety Z/K, let $\overline{Z} = Z_{\overline{K}}$.

Proof of Theorem A.1. By Chow rigidity [5, Thm. 3.19], two abelian varieties over K are isogenous if and only if they are isogenous over the perfect closure of K. Consequently, to prove the theorem we may and do assume that K is perfect. Since K is perfect, we have that if G and H are group schemes over K, then G_{red} and H_{red} are group schemes, and $(G \times H)_{\mathrm{red}} \cong G_{\mathrm{red}} \times H_{\mathrm{red}}$.

Moreover, we have $\mathrm{G}(\overline{Z}) \cong \mathrm{G}(Z)_{\overline{K}}$ and $\mathrm{P}(\overline{Z}) = \mathrm{P}(Z)_{\overline{K}}$.

Let $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be an equivalence of categories. A fundamental theorem of Orlov (Theorem 1.2) asserts that there is an object $\mathcal{E} \in \mathrm{D}^b(X \times Y)$ such that Φ is given by

$$\Phi = \Phi_{\mathcal{E}} : \mathcal{M}^\bullet \longmapsto p_{Y*}(p_X^* \mathcal{M}^\bullet \otimes \mathcal{E}) \ .$$

Over $\overline{K}$, a theorem of Rouquier [16, Thm. 4.18] shows that Φ induces an isomorphism

$$\mathrm{Aut}^0(\overline{X}) \times \mathrm{Pic}^0(\overline{X}) \longrightarrow \mathrm{Aut}^0(\overline{Y}) \times \mathrm{Pic}^0(\overline{Y}).$$

This induces an isomorphism on reduced subschemes which we denote $\overline{F}$:

$$\mathrm{G}(X)_{\overline{K}} \times \mathrm{P}(X)_{\overline{K}} \xrightarrow{\overline{F}} \mathrm{G}(Y)_{\overline{K}} \times \mathrm{P}(Y)_{\overline{K}} \ . \tag{9}$$

(Note that $\mathrm{G}(X)(\overline{K}) = \mathrm{Aut}^0(X)(\overline{K})$, etc.) On points, $\overline{F}$ is characterized by the fact that

$$\overline{F}(\phi, \mathcal{L}) = (\psi, \mathcal{M}) \Longleftrightarrow p_X^*\mathcal{L} \otimes (\phi \times \mathrm{id})^*\mathcal{E} \cong p_Y^*\mathcal{M} \otimes (\psi \times \mathrm{id})_*\mathcal{E}.$$

Since $\mathcal{E}$ is defined over K, the graph of this relation in $\mathrm{G}(X)_{\overline{K}} \times \mathrm{P}(X)_{\overline{K}} \times \mathrm{G}(X)_{\overline{K}} \times \mathrm{P}(X)_{\overline{K}}$ is stable under $\mathrm{Aut}(\overline{K}/K)$, and so isomorphism (9) descends to an isomorphism

$$\mathrm{G}(X) \times \mathrm{P}(X) \xrightarrow{F} \mathrm{G}(Y) \times \mathrm{P}(Y)$$

of connected, reduced group schemes over K.

Using the projections $p_{\mathrm{G}(Y)}$ and $p_{\mathrm{G}(X)}$, we obtain K-rational morphisms

$$\mathrm{P}(X) \xrightarrow{\alpha_Y = p_{\mathrm{G}(Y)} \circ F} \mathrm{G}(Y)$$
$$\mathrm{P}(Y) \xrightarrow{\alpha_X = p_{\mathrm{G}(X)} \circ F^{-1}} \mathrm{G}(X);$$

let $A_X = \alpha_X(\mathrm{P}(Y)) \subseteq \mathrm{G}(X)$ and $A_Y = \alpha_Y(\mathrm{P}(X)) \subseteq \mathrm{G}(Y)$. (Note that, since the formation of kernels commutes with base change, $\overline{A}_X \cong A_{\overline{X}}$.) The pointwise argument of [15], combined with the fact that F admits an inverse, shows that F induces an isomorphism

$$A_X \times \mathrm{P}(X) \longrightarrow A_Y \times \mathrm{P}(Y)$$

of abelian varieties over K. By Poincaré reducibility, it now suffices to show that A_X and A_Y are isogenous.

Over an algebraically closed field, Popa and Schnell choose a point $(P, Q) \in (X \times Y)(\overline{K})$ in the support of $\mathcal{E}$, and use it to define morphisms of varieties over $\overline{K}$:

$$\overline{A}_X \times \overline{A}_Y \xrightarrow{\overline{f} = \overline{f}_X \times \overline{f}_Y} \overline{X} \times \overline{Y}$$
$$(\phi, \psi) \longmapsto (\phi(P), \psi(Q)).$$

The dual map $\overline{f}^* : \mathrm{P}(\overline{X}) \times \mathrm{P}(\overline{Y}) \to \widehat{\overline{A}_X} \times \widehat{\overline{A}_Y}$ is surjective.

Working now over a field which is only assumed to be perfect, Lemma A.2 supplies a canonical morphism $g_X : \mathrm{Alb}(\mathrm{G}(X)) \to \mathrm{Alb}^1(X)$ whose base change to $\overline{K}$ is $g_{X,\overline{K}} \cong \overline{f}_X$. In particular, by [15, Lemma 2.2], which depends only on [4] and is valid in any characteristic, $H_X := \ker g_X$ is a finite group scheme. Similarly, there is a canonical morphism $g_Y : \mathrm{Alb}(\mathrm{G}(Y)) \to \mathrm{Alb}(Y)$ with finite kernel H_Y, and $g_{Y,\overline{K}} \cong \overline{f}_Y$. We obtain a surjection $g^* : \mathrm{P}(X) \times \mathrm{P}(Y) \to \widehat{A}_X \times \widehat{A}_Y$ of abelian varieties over K which is an isogeny onto its image.

Consider the morphism

$$A_X \times A_Y \xrightarrow{\tau = (\tau_1, \tau_2 \pi_2)} (A_X \times A_Y) \times (\widehat{A}_X \times \widehat{A}_Y)$$

of abelian varieties over K, where

$$\tau_1 = (\mathrm{id}_{A_X}, p_{\mathrm{G}(Y)} \circ F),$$
$$\tau_2 = (g_X^* \circ \iota, g_Y^* \circ p_{\mathrm{P}(Y)} \circ F),$$

and ι denotes the inversion map on the abelian variety $\widehat{A_X}$. Let p_{13} (respectively, p_{24}) denote the projection of the codomain of τ onto the first and third (respectively, second and fourth) components.

After base change to $\overline{K}$, the morphisms $\overline{\tau}_1$, $\overline{\tau}_2$ and $\overline{\tau}$ coincide, respectively, with the morphisms π_1, π_2 and π constructed in [15, p. 533]. In particular $\overline{p_{13} \circ \tau}$ and $\overline{p_{24} \circ \tau}$, and therefore $p_{13} \circ \tau$ and $p_{24} \circ \tau$, are isogenies. By Poincaré reducibility, A_X and A_Y are isogenous. □

Corollary A.4. *Let X and Y be smooth projective varieties over a field K. If X and Y are derived equivalent, then $H^1(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)$ and $H^{2d-1}(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H^{2d-1}(Y_{\overline{K}}, \mathbb{Q}_\ell)$ as representations of $\mathrm{Gal}(\overline{K}/K)$, where $d = \dim X = \dim Y$ and ℓ is invertible in K.*

Proof. The claim for cohomology in degree one follows from Theorem A.1 and the canonical identifications $\mathrm{Pic}^0(X)[\ell^n](\overline{K}) \cong H^1(X_{\overline{K}}, \mu_{\ell^n})$ provided by the Kummer sequence. The second claim now follows from Poincaré duality. □

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