

THE $(n + 1)$ -LIPSCHITZ HOMOTOPY GROUP OF THE HEISENBERG GROUP \mathbb{H}^n

PIOTR HAJLASZ

(Communicated by Jeremy Tyson)

ABSTRACT. We prove that for $n \geq 2$, the Lipschitz homotopy group $\pi_{n+1}^{\text{Lip}}(\mathbb{H}^n) \neq 0$ of the Heisenberg group \mathbb{H}^n is nontrivial.

1. INTRODUCTION

The Lipschitz homotopy groups of a metric space $\pi_n^{\text{Lip}}(X)$ are defined in the same way as the classical homotopy groups, with the exception that both the maps and homotopies are required to be Lipschitz. We emphasize that we make no restriction on the Lipschitz constants. In particular, we do not require the Lipschitz constant of a homotopy to be comparable to the Lipschitz constant of maps that are Lipschitz homotopic. The notion of Lipschitz homotopy groups was introduced in [4] with the main purpose of studying the Lipschitz homotopy groups of the Heisenberg group (equipped with the Carnot-Carathéodory metric). Note that the classical homotopy groups of the Heisenberg group are zero since the space is homeomorphic to a Euclidean space. The following results are known:

$$\pi_k^{\text{Lip}}(\mathbb{H}^n) \begin{cases} = 0 & \text{if } 1 \leq k < n & [11], \\ \neq 0 & \text{if } k = n & [2], \\ = 0 & \text{if } n = 1, k \geq 2 & [12], \\ \neq 0 & \text{if } n = 2\ell, k = 4\ell - 1 & [7]. \end{cases}$$

The main result of this paper adds one more case to this list.

Theorem 1.1. *For $n \geq 2$, $\pi_{n+1}^{\text{Lip}}(\mathbb{H}^n) \neq 0$.*

The case $n = 2$ is already contained in [7], but with a very different proof. As we will see, the result has a very short proof, and it is somewhat surprising that it has not been noticed before.

The proofs that $\pi_n^{\text{Lip}}(\mathbb{H}^n) \neq 0$ and $\pi_{4\ell-1}^{\text{Lip}}(\mathbb{H}^{2\ell}) \neq 0$ employ the fact that the groups $\pi_n(\mathbb{S}^n)$ and $\pi_{4\ell-1}(\mathbb{S}^{2\ell})$ are infinite, so one can use differential forms to detect non-trivial elements in these groups (using degree and the Hopf invariant respectively). To the best of my knowledge, Theorem 1.1 for $n \geq 3$ is a first result showing nontrivial Lipschitz homotopy groups of the Heisenberg groups when the corresponding homotopy groups of the spheres are finite, $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}_2$ for $n \geq 3$.

Received by the editors March 26, 2017 and, in revised form, May 2, 2017.

2010 *Mathematics Subject Classification.* Primary 53C17; Secondary 55Q40.

Key words and phrases. Heisenberg group, Lipschitz homotopy groups.

The author was supported by NSF grant DMS-1500647.

When $k > 1$, Lipschitz homotopy groups are abelian, but they are usually uncountably generated. For a detailed proof that $\pi_n^{\text{Lip}}(\mathbb{H}^n)$ is uncountably generated, see [4]. However, very little is known about the structure of these groups. For example it is not known if $\pi_n^{\text{Lip}}(\mathbb{H}^n)$ has elements of finite order. Nevertheless, the fact that a Lipschitz homotopy group is nontrivial provides a lot of information about the structure of Lipschitz maps into the Heisenberg group.

In order to show that $\pi_k^{\text{Lip}}(\mathbb{H}^n) \neq 0$, we need to construct a Lipschitz mapping $f : \mathbb{S}^k \rightarrow \mathbb{H}^n$ that has no Lipschitz extension $F : \mathbb{B}^{k+1} \rightarrow \mathbb{H}^n$; see comments following Definition 4.1 in [4]. So far, the only known method of proving that $\pi_k^{\text{Lip}}(\mathbb{H}^n) \neq 0$ is based on the idea described below.

The following fact is well known; see [2, 4, 5].

Lemma 1.2. *For $n \geq 1$, there is a smooth and horizontal embedding of the sphere $\Phi : \mathbb{S}^n \rightarrow \mathbb{H}^n$; that is, $\Phi : \mathbb{S}^n \rightarrow \mathbb{R}^{2n+1}$ is a smooth embedding and the tangent space to the embedded sphere $\Phi(\mathbb{S}^n)$ is contained in the horizontal distribution of \mathbb{H}^n . It follows that the embedding $\Phi : \mathbb{S}^n \rightarrow \mathbb{H}^n$ is bi-Lipschitz.*

Now if $\pi_k(\mathbb{S}^n) \neq 0$ and $f \in C^\infty(\mathbb{S}^k, \mathbb{S}^n)$, $0 \neq [f] \in \pi_k(\mathbb{S}^n)$, then, in some cases, one can prove that if Φ is any map that satisfies the conclusion of Lemma 1.2, then $\Phi \circ f : \mathbb{S}^k \rightarrow \mathbb{H}^n$ does not admit a Lipschitz extension $F : \mathbb{B}^{k+1} \rightarrow \mathbb{H}^n$. This was the method used in all of the known cases of $\pi_k^{\text{Lip}}(\mathbb{H}^n) \neq 0$, and it is also a method used in this paper. For this reason I conjectured that if $\pi_k(\mathbb{S}^n) \neq 0$, then $\pi_k^{\text{Lip}}(\mathbb{H}^n) \neq 0$. While the conjecture is still open, in many cases the approach to a proof would have to be very different from the one described here, because of the following result of Wenger and Young [12, Theorem 1].

Theorem 1.3. *If $f : \mathbb{S}^k \rightarrow \mathbb{S}^n$ is Lipschitz and $n + 2 \leq k < 2n - 1$, then $\Phi \circ f : \mathbb{S}^k \rightarrow \mathbb{H}^n$ can be extended to a Lipschitz map $F : \mathbb{B}^{k+1} \rightarrow \mathbb{H}^n$.*

The result is sharp: it does not extend to the case of $k = 2n - 1$ when n is even. Indeed, if n is even and $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{S}^n$ has a nonzero Hopf invariant, then as was shown in [7], the map $\Phi \circ f : \mathbb{S}^{2n-1} \rightarrow \mathbb{H}^n$ does not admit a Lipschitz extension $F : \mathbb{B}^{2n} \rightarrow \mathbb{H}^n$ and hence $\pi_{2n-1}^{\text{Lip}}(\mathbb{H}^n) \neq 0$. Also the proof of Theorem 1.1 shows that Theorem 1.3 cannot be extended to the case $k = n + 1$, which is somewhat surprising in view of [12, Theorem 2].

Note that Theorem 1.3 shows that the method described above of proving that $\pi_k^{\text{Lip}}(\mathbb{H}^n) \neq 0$ cannot be used in the range $n + 2 \leq k < 2n - 1$.

2. PROOF OF THEOREM 1.1

Let $\Phi : \mathbb{S}^n \rightarrow \mathbb{H}^n$ be a smooth and horizontal embedding of the sphere into the Heisenberg group as in Lemma 1.2. Recall that

$$\pi_{n+1}(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}_2 & \text{if } n \geq 3. \end{cases}$$

Let $f \in C^\infty(\mathbb{S}^{n+1}, \mathbb{S}^n)$ be a smooth map such that $0 \neq [f] \in \pi_{n+1}(\mathbb{S}^n)$. We will show that the map

$$g = \Phi \circ f : \mathbb{S}^{n+1} \rightarrow \mathbb{H}^n$$

does not admit a Lipschitz extension

$$G : \mathbb{B}^{n+2} \rightarrow \mathbb{H}^n,$$

which will prove that $0 \neq [g] \in \pi_{n+1}^{\text{Lip}}(\mathbb{H}^n)$. Suppose to the contrary that $G : \mathbb{B}^{n+2} \rightarrow \mathbb{H}^n$ is a Lipschitz extension of g . Since the Heisenberg group is purely unrectifiable in dimensions $k \geq n + 1$ [1, 9] (see [3] for a different proof), we conclude that $\mathcal{H}^{n+2}(G(\mathbb{B}^{n+2})) = 0$, where \mathcal{H}^{n+2} stands for the Hausdorff dimension with respect to the Carnot-Carathéodory metric in \mathbb{H}^n . To arrive at a contradiction it suffices to show that $\mathcal{H}^{n+2}(G(\mathbb{B}^{n+2})) > 0$.

We will need the following result of Gromov [6, Theorem 3.1.A]; see also [10, Lemma 22 and Theorem 6].

Lemma 2.1. *If a set $E \subset \mathbb{H}^n$ has topological dimension at least $n + 1$, then $\mathcal{H}^{n+2}(E) > 0$.*

In fact the statement of Gromov's result only says that the Hausdorff dimension of E is greater than or equal to $n + 2$, so one cannot conclude from the statement of Gromov's result that $\mathcal{H}^{n+2}(E) > 0$. However, the proof of Gromov's result clearly shows that $\mathcal{H}^{n+2}(E) > 0$; see the proof of Lemma 22 in [10].

Thus it suffices to show that the topological dimension of $G(\mathbb{B}^{n+2})$ is greater than or equal to $n + 1$. Suppose to the contrary that $\dim_{\text{top}} G(\mathbb{B}^{n+2}) \leq n$. We will need the following classical result [8, Theorem VI.4].

Lemma 2.2. *A separable metric space X has topological dimension less than or equal to n if and only if for each closed set $C \subset X$ and a continuous map $h : C \rightarrow \mathbb{S}^n$, there is a continuous extension $H : X \rightarrow \mathbb{S}^n$ of h .*

Let $h = \Phi^{-1} : \Phi(\mathbb{S}^n) \rightarrow \mathbb{S}^n$. Since $\Phi(\mathbb{S}^n)$ is a compact subset of $G(\mathbb{B}^{n+2})$ and $\dim_{\text{top}} G(\mathbb{B}^{n+2}) \leq n$, there is a continuous extension $H : G(\mathbb{B}^{n+2}) \rightarrow \mathbb{S}^n$ of h . Note that $H \circ G : \mathbb{B}^{n+2} \rightarrow \mathbb{S}^n$ is a continuous extension of

$$H \circ G \Big|_{\mathbb{S}^{n+1}} = f : \mathbb{S}^{n+1} \rightarrow \mathbb{S}^n,$$

which contradicts the assumption that $0 \neq [f] \in \pi_{n+1}(\mathbb{S}^n)$. The proof is complete. \square

REFERENCES

- [1] Luigi Ambrosio and Bernd Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), no. 3, 527–555, DOI 10.1007/s002080000122. MR1800768
- [2] Zoltán M. Balogh and Katrin S. Fässler, *Rectifiability and Lipschitz extensions into the Heisenberg group*, Math. Z. **263** (2009), no. 3, 673–683, DOI 10.1007/s00209-008-0437-z. MR2545863
- [3] Zoltán M. Balogh, Piotr Hajlasz, and Kevin Wildrick, *Weak contact equations for mappings into Heisenberg groups*, Indiana Univ. Math. J. **63** (2014), no. 6, 1839–1873, DOI 10.1512/iumj.2014.63.5411. MR3298724
- [4] Noel DeJarnette, Piotr Hajlasz, Anton Lukyanenko, and Jeremy T. Tyson, *On the lack of density of Lipschitz mappings in Sobolev spaces with Heisenberg target*, Conform. Geom. Dyn. **18** (2014), 119–156, DOI 10.1090/S1088-4173-2014-00267-7. MR3226622
- [5] Tobias Ekholm, John Etnyre, and Michael Sullivan, *Non-isotopic Legendrian submanifolds in \mathbb{R}^{2n+1}* , J. Differential Geom. **71** (2005), no. 1, 85–128. MR2191769
- [6] Mikhael Gromov, *Carnot-Carathéodory spaces seen from within*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–323. MR1421823
- [7] Piotr Hajlasz, Armin Schikorra, and Jeremy T. Tyson, *Homotopy groups of spheres and Lipschitz homotopy groups of Heisenberg groups*, Geom. Funct. Anal. **24** (2014), no. 1, 245–268, DOI 10.1007/s00039-014-0261-z. MR3177382
- [8] Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941. MR0006493

- [9] Valentino Magnani, *Unrectifiability and rigidity in stratified groups*, Arch. Math. (Basel) **83** (2004), no. 6, 568–576, DOI 10.1007/s00013-004-1057-4. MR2105335
- [10] P. Pansu, *Submanifolds and differential forms on Carnot manifolds, after M. Gromov and M. Rumin*, preprint, 2016, arXiv:1604.06333.
- [11] Stefan Wenger and Robert Young, *Lipschitz extensions into jet space Carnot groups*, Math. Res. Lett. **17** (2010), no. 6, 1137–1149, DOI 10.4310/MRL.2010.v17.n6.a12. MR2729637
- [12] Stefan Wenger and Robert Young, *Lipschitz homotopy groups of the Heisenberg groups*, Geom. Funct. Anal. **24** (2014), no. 1, 387–402, DOI 10.1007/s00039-014-0264-9. MR3177387

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, 301 THACKERAY HALL, PITTSBURGH, PENNSYLVANIA 15260

E-mail address: hajlasz@pitt.edu