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ISOPERIMETRIC INEQUALITIES UNDER BOUNDED INTEGRAL NORMS OF RICCI CURVATURE AND MEAN CURVATURE

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ABSTRACT. We obtain isoperimetric inequalities under bounded integral norms of Ricci curvature and mean curvature. Also we generalize the results to metric-measure spaces.

1. INTRODUCTION

The classical isoperimetric inequality says that for a given bounded domain D in $\mathbb{R}^n$,

$$n^n \omega_n \mathrm{vol}(D)^{n-1} \leq A(\partial D)^n,$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$ and A is $(n-1)$-dimensional volume.

For an n-dimensional Riemannian manifold M^n, the ν-isoperimetric constant $I_\nu(M^n)$ is defined as follows:

$$I_\nu(M^n) = \inf_{D \subset M^n} \frac{A(\partial D)}{\mathrm{vol}(D)^{1-\frac{1}{\nu}}}. \tag{1.1}$$

If $I_\nu(M^n) > 0$, then an isoperimetric inequality is induced. In the case of $\nu = \infty$, $I_\infty(M^n)$ is Cheeger's constant. If $I_\infty(M^n) > 0$, we obtain a linear isoperimetric inequality.

In many cases, the isoperimetric inequalities for Riemannian manifolds are locally obtained [Ch2]. For example, Croke's inequality says that if $r < \frac{1}{2}\mathrm{inj}_{M^n}$, then

$$A(\partial D) \geq c(n)\mathrm{vol}(D)^{\frac{n-1}{n}}$$

for $D \subset B(p, r)$ and a constant $c(n)$, where inj_{M^n} is the injectivity radius of M^n and $B(p, r)$ is the r-ball centered at p. Buser's inequality is a linear isoperimetric inequality as follows: If $\mathrm{Ric}_{M^n} \geq -(n-1)k$ for $k \geq 0$, then there exists a constant $c(n, k, r) > 0$ such that

$$\min\{\mathrm{vol}(D_1), \mathrm{vol}(D_2)\} \leq c(n, k, r)A(\Gamma)$$

for a dividing hypersurface Γ and open subsets $D_1, D_2 \subset B(p, r)$ satisfying that $B(p, r) \setminus \Gamma = D_1 \cup D_2$.

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Recently, there have been many attempts in Riemannian geometry to replace curvature bounds with integral norms of curvatures. In [Ga], Gallot obtained an isoperimetric inequality with an integral norm of Ricci curvature for a closed manifold. He obtained a lower bound of $A(\partial\Omega)$ with $\min\{\mathrm{vol}(\Omega), \mathrm{vol}(M^n \setminus \Omega)\}^{1-\frac{1}{p}}$ for $p > n$ under a condition on the integral norm of Ricci curvature. In [PW], Petersen and Wei generalized the Bishop-Gromov volume comparison estimate with an integral norm of Ricci curvature. In [Pa1], local linear isoperimetric inequalities were obtained with an integral norm of Ricci curvature by using a technique in [PW]. In [DWZ], the authors obtained a local isoperimetric inequality under a sufficiently small integral norm of Ricci curvature.

In this paper, we will consider (nonlocal) isoperimetric inequalities for manifolds with boundary. It should be noted that for manifolds with boundary, the mean curvature of ∂M^n plays an essential role in isoperimetric inequality. For example, let $M_\epsilon = S^2 - B(p,\epsilon)$ and $D = S^2 - B(p, \epsilon+\delta)$. Then the mean curvature of ∂M_ϵ diverges and $\frac{A(\partial D)}{\mathrm{vol}(D)^{1-\frac{1}{\nu}}} \to 0$ as $\epsilon, \delta \to 0$. Hence $I_\nu(M_\epsilon) \to 0$ as $\epsilon \to 0$.

There have been some studies on isoperimetric inequality for manifolds with boundary. In 1959, Reid generalized the classical isoperimetric inequality to region D in a surface in $\mathbb{R}^3$ [R]. Precisely, suppose $D \subset \mathbb{R}^3$ is a C^2 image of a region in the plane bounded by a simple closed curve. Let H be the mean curvature vector on D and X be the position vector to D. If the origin is an arbitrary point on ∂D, then

$$\text{Length}(\partial D)^2 \geq 4\pi\Big(\text{Area}(D) + \int_D X \cdot H\Big). \tag{1.2}$$

In 1961, Hsiung [Hs] proved that the inequality proved by Reid still holds when M^n is a 2-dimensional manifold imbedded in Euclidean space provided ∂M^n is diffeomorphic to a circle. In 1972, Hanes [Ha] generalized the inequalities of Reid and Hsiung to an n-dimensional manifold with boundary embedded in Euclidean $(n+p)$-space, where the boundary is not necessarily diffeomorphic to the sphere.

However, in an inequality such as (1.2), the mean curvature of ∂D is needed to obtain an isoperimetric inequality for $\mathrm{vol}(D)$ and $A(\partial D)$. If $I_\nu(M^n) > 0$, then an isoperimetric inequality is obtained for any domain $D \subset M^n$ without conditions on the mean curvature of ∂D. We will obtain isoperimetric inequalities for a domain D in an n-dimensional Riemannian manifold M^n with boundary under bounded integral norms of Ricci curvature of M^n and mean curvature of ∂M^n. We will not use the mean curvature of ∂D.

We will use the following notation. Let M^n be an n-dimensional Riemannian manifold with smooth boundary and D be a domain in M^n. Let h be the mean curvature of ∂M^n with respect to the inward normal vector field. Let $\rho(q) = \max\{(-\mathrm{Ric}_N(v,v))_+ \mid |v| = 1,\ v \in T_q M^n\}$ and $h_+ = \max\{0, h\}$.

We define integral norms $\mathcal{R}_p$ and $\mathcal{H}_p$ as follows:

$$\mathcal{R}_p = \int_{M^n} \rho^p dV,$$
$$\mathcal{H}_p = \int_{\partial M^n} h_+^p d\mathrm{vol}_{\partial M^n},$$

where dV is the volume form of M^n and $d\mathrm{vol}_{\partial M^n}$ is the volume form of ∂M^n induced from the volume form of M^n.

We will prove the following two isoperimetric inequalities with $\mathcal{R}_p$ and $\mathcal{H}_q$ for M^n. Our isoperimetric inequalities are not local, so we do not assume any conditions on the shape of M^n such as $M^n = B(p,r)$. In the first theorem, $p = q = n-1$. It is possible that $\partial D \cap \partial M^n \neq \emptyset$. (For example, $D = M^n$.) Let $\mathrm{diam}(M^n)$ be the diameter of M^n.

Theorem 1.1. *Let M^n be an n-dimensional Riemannian manifold with smooth boundary. If* $\mathrm{diam}(M^n) = R$, *then for a domain $D \subset M^n$ with smooth boundary, we have*

$$\mathrm{vol}(D) \leq e^{\beta_1 R} + A(\partial D)\frac{e^{\beta_1 R} - 1}{\beta_1},$$

where $\beta_1 = \mathcal{R}_{n-1}^{\frac{1}{n-1}} R + \mathcal{H}_{n-1}^{\frac{1}{n-1}} R^{\frac{1}{n-1}}$.

If M^n is mean convex everywhere, then we have

$$\mathrm{vol}(D) \leq e^{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^2} + A(\partial D)\Big(\frac{e^{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^2} - 1}{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R}\Big).$$

For $p > \frac{n}{2}$, we obtain the following inequality with $\mathcal{R}_p$ and $\mathcal{H}_{2p-1}$:

Theorem 1.2. *Let M^n be an n-dimensional Riemannian manifold with smooth boundary. If* $\mathrm{diam}(M^n) = R$, *then for a domain $D \subset M^n$ with smooth boundary, we have*

$$A(\partial D) \geq \frac{1}{R}\mathrm{vol}(D) - \Big(2^p C_0(n,p)^{2p}\mathcal{R}_p + 2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}\Big)^{\frac{1}{2p}}\mathrm{vol}(D)^{1-\frac{1}{2p}}$$

for $2p > n$ and $C_0(n,p) = \big(\frac{1}{n-1} - \frac{1}{2p-1}\big)^{-\frac{1}{2}}$.

If M^n is mean convex everywhere, then we have

$$A(\partial D) \geq \frac{1}{R}\mathrm{vol}(D) - \big(2^p C_0(n,p)^{2p}\mathcal{R}_p\big)^{\frac{1}{2p}}\mathrm{vol}(D)^{1-\frac{1}{2p}}.$$

If we want a linear isoperimetric inequality from Theorem 1.2, we can modify Theorem 1.2 as follows:

$$\mathrm{vol}(D) \leq 2RA(\partial D) + (2R)^{2p}\Big(2^p C_0(n,p)^{2p}\mathcal{R}_p + 2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}\Big).$$

Although $\mathcal{H}_{2p-1}$ is used in Theorem 1.2 for $2p-1 > n-1$, if $\mathrm{Ric}_{M^n} \geq 0$ and M^n is mean convex everywhere, then we obtain from Theorem 1.2 that

$$\mathrm{vol}(D) \leq RA(\partial D)$$

since $\mathcal{R}_p = 0$ and $\mathcal{H}_{2p-1} = 0$. On the other hand, we obtain from Theorem 1.1 that

$$\mathrm{vol}(D) \leq RA(\partial D) + 1.$$

Hence Theorem 1.2 could give a sharper upper bound for the volume in the case of almost nonnegative Ricci curvature. In particular, if the boundary is a minimal surface, we have $\mathcal{H}_q = 0$. It is known that event horizons of black holes are minimal surfaces which can be considered as the boundary of the universe. Then we can apply the above theorems to estimate the volume of our universe with the area of event horizons.

In the proof of Theorem 1.1, we estimate $\mathrm{vol}(D)$ by integrating the $(n-1)$-dimensional volume of $S_t = \{x \in D \mid d(x, \partial M^n) = t\}$. On the other hand, in the proof of Theorem 1.2, we integrate the volume form along a geodesic γ_q

from $q \in \partial M^n$ and then integrate it on ∂M^n to estimate $\mathrm{vol}(D)$. Therefore, unlike Theorem 1.2, an isoperimetric inequality can be obtained for S_t and $D_t = \{x \in D \mid d(x, \partial M^n) \leq t\}$ in (2.15) in the proof of Theorem 1.1.

Current theories of physics postulate the presence of scalar fields in addition to the metric [GW]. Hence, many results in Riemannian geometry have been generalized to the metric-measure space $(M^n, g, e^{-f}dV)$ with the Bakry-Emery Ricci tensor (for example, [L], [WW]). Bakry and Emery studied this tensor and its relationship to diffusion processes [BE]. Wei and Wylie obtained a volume comparison and a mean curvature comparison in [WW]. In Section 4, we will extend our results to metric-measure spaces by using integral norms of the Bakry-Emery Ricci tensor.

2. Proof of Theorem 1.1

We will use the following notation. Let

$$M_t = \{x \in M^n \mid d(x, \partial M^n) \geq t\},$$

$$\partial M_t = \{x \in M^n \mid d(x, \partial M^n) = t\}.$$

Let γ_q be the normal geodesic such that $\gamma_q(0) = q$ and $\gamma_q'(0)$ is perpendicular to ∂M^n for $q \in \partial M^n$. Let

$$t_q = \max\{t \mid d(\gamma_q(t), \partial M^n) = t\}.$$

Then we have

$$M^n = \bigcup_{q \in \partial M^n} \{\gamma_q(t) \mid t \leq t_q\}.$$

Let g be the metric of M^n. We denote by g_t the induced metric of ∂M_t from g. Let $d\mathrm{vol}_t$ be the volume form of ∂M_t induced from g_t. Then the volume form $d\mathrm{vol}_{\partial M^n}$ of ∂M^n is $d\mathrm{vol}_0$, and the volume form of M^n satisfies $dV = dt \wedge d\mathrm{vol}_t$. By identifying $\gamma_q(t) \in \partial M_t$ with $q \in \partial M^n$ for $t \leq t_q$, we define $\omega(t, q)$ and $h(t, q)$ as follows:

$$\begin{gathered} d\mathrm{vol}_t = \omega(t, \cdot) d\mathrm{vol}_{\partial M^n}, \\ (\Leftrightarrow d\mathrm{vol}_t|_{\gamma_t(q)} = \omega(t, q) F^* d\mathrm{vol}_{\partial M^n}|_q), \end{gathered} \tag{2.1}$$

where F is the projection from $\gamma_q(t)$ to q, and

$$\frac{\partial}{\partial t}\omega(t, q) = h(t, q)\omega(t, q), \tag{2.2}$$

where h is the mean curvature of ∂M_t. We abbreviate $\omega(t, q)$ and $h(t, q)$ to $\omega(t)$ and $h(t)$, respectively. Then h satisfies the Riccati equation for $t \leq t_q$, so we have

$$h' + \frac{h^2}{n-1} \leq -\mathrm{Ric}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}), \tag{2.3}$$

where $\frac{\partial}{\partial t} = \gamma_q'(t)$, which is the gradient of the distance function $d(\cdot, \partial M^n)$, so $|\frac{\partial}{\partial t}| = 1$. Let

$$\psi = \begin{cases} h_+ & \text{if } t \leq t_q, \\ 0 & \text{if } t > t_q, \end{cases} \tag{2.4}$$

and recall that $\rho = \max\{(-\mathrm{Ric}(v,v))_+ \mid |v|=1,\ v\in T_qM^n\}$. Then

$$\psi' + \frac{\psi^2}{n-1} \le \rho. \tag{2.5}$$

In order to obtain the integral norm of mean curvature on ∂M_t, we use similar arguments as in [Pa2]. Since $\int_{\partial M_t} f d\mathrm{vol}_t = \int_{\partial M^n} f\omega d\mathrm{vol}_{\partial M^n}$, we have $\frac{d}{dt}\int_{\partial M_t} f d\mathrm{vol}_t = \int_{\partial M_t}(f'+fh)d\mathrm{vol}_t$. From (2.5) and $\psi \ge 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\partial M_t}\psi^{n-1}d\mathrm{vol}_t &= \int_{\partial M_t}((n-1)\psi^{n-2}\psi' + \psi^{n-1}h)d\mathrm{vol}_t\\ &\le \int_{\partial M_t}\psi^{n-2}((n-1)\psi'+\psi^2)d\mathrm{vol}_t\\ &\le (n-1)\int_{\partial M_t}\psi^{n-2}\rho(q)d\mathrm{vol}_t.\end{aligned}\tag{2.6}$$

Let

$$\mathcal{R}_{n-1}(t) = \int_{M^n\setminus M_t}\rho^{n-1}dV,$$
$$\mathcal{H}_{n-1}(t) = \int_{\partial M_t} h_+^{n-1}d\mathrm{vol}_t.$$

Then $\mathcal{H}_{n-1} = \mathcal{H}_{n-1}(0)$ and $\mathcal{R}_{n-1} = \mathcal{R}_{n-1}(R)$. Since we have

$$\int_{\partial M_t}\psi^{n-2}\rho(q)d\mathrm{vol}_t \le \Big(\int_{\partial M_t}\psi^{n-1}d\mathrm{vol}_t\Big)^{\frac{n-2}{n-1}}\Big(\int_{\partial M_t}\rho(q)^{n-1}d\mathrm{vol}_t\Big)^{\frac{1}{n-1}},$$

we obtain from (2.6) that

$$\mathcal{H}'_{n-1}(t) \le (n-1)\mathcal{H}_{n-1}(t)^{\frac{n-2}{n-1}}\Big(\int_{\partial M_t}\rho(q)^{n-1}d\mathrm{vol}_t\Big)^{\frac{1}{n-1}}. \tag{2.7}$$

Then

$$\frac{\mathcal{H}'_{n-1}(t)}{(n-1)\mathcal{H}_{n-1}(t)^{\frac{n-2}{n-1}}} \le \Big(\int_{\partial M_t}\rho(q)^{n-1}d\mathrm{vol}_t\Big)^{\frac{1}{n-1}}. \tag{2.8}$$

Integrating the above for t, we obtain that

$$\begin{aligned}\mathcal{H}_{n-1}(t)^{\frac{1}{n-1}} - \mathcal{H}_{n-1}(0)^{\frac{1}{n-1}} &= \int_0^t \frac{\mathcal{H}'_{n-1}(u)}{(n-1)\mathcal{H}_{n-1}(u)^{\frac{n-2}{n-1}}}du\\ &\le \int_0^t\Big(\int_{\partial M_u}\rho(q)^{n-1}d\mathrm{vol}_u\Big)^{\frac{1}{n-1}}du\\ &\le \Big(\int_{M^n\setminus M_t}\rho(q)^{n-1}dV\Big)^{\frac{1}{n-1}}t^{\frac{n-2}{n-1}}\\ &= \mathcal{R}_{n-1}(t)^{\frac{1}{n-1}}t^{\frac{n-2}{n-1}}.\end{aligned}\tag{2.9}$$

Since $t\le R$, we obtain that

$$\begin{aligned}\mathcal{H}_{n-1}(t)^{\frac{1}{n-1}} &\le \mathcal{R}_{n-1}(t)^{\frac{1}{n-1}}t^{\frac{n-2}{n-1}} + \mathcal{H}_{n-1}(0)^{\frac{1}{n-1}}\\ &\le \mathcal{R}_{n-1}^{\frac{1}{n-1}}R^{\frac{n-2}{n-1}} + \mathcal{H}_{n-1}^{\frac{1}{n-1}}.\end{aligned}\tag{2.10}$$

Now we will use the following divergence theorem [HS]. Let

$$S_t = D\cap\partial M_t.$$

The divergence of X on S_t is defined as follows:

$$\mathrm{div}_{S_t}(X) = \mathrm{tr}\nabla X = \sum_{i=1}^{n-1} \langle \nabla_{e_i} X, e_i \rangle$$

for an orthonormal basis $\{e_i\}$ on TS_t, where $\nabla X : TS_t \to TM^n$ is the map $Y \mapsto \nabla_Y X$ for a covariant differentiation ∇ on M^n. Then for a vector field X on S_t, we have

$$\int_{S_t} \mathrm{div}_{S_t}(X) d\mathrm{vol}_t = -\int_{S_t} \langle X, H \rangle d\mathrm{vol}_t + \int_{\partial S_t} \langle X, U \rangle d\theta_{\partial S_t}, \tag{2.11}$$

where H is the mean curvature vector field of S_t, U is the outward normal vector field on ∂S_t which is tangent to S_t, and $d\theta_{\partial S_t}$ is the volume form of ∂S_t. (See Figure 1.) Let X be a variational vector field of variation S_t. Then the projection of X to the normal direction to S_t is $\frac{\partial}{\partial t}$. We may assume that X is normal to ∂S_t; i.e. we may consider $\langle X, \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t} + \langle X, U \rangle U = \frac{\partial}{\partial t} + \langle X, U \rangle U$ instead of X.

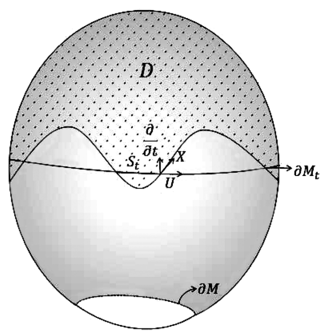

FIGURE 1. Variation vector field X of S_t

Let $V(t) = A(S_t) = \int_{S_t} d\mathrm{vol}_t$. Since

$$\frac{\partial}{\partial t} A(S_t) = \int_{S_t} \mathrm{div}_{S_t}(X) d\mathrm{vol}_t,$$

we obtain from (2.10) that

$$\begin{aligned} V'(t) &= \int_{S_t} h d\mathrm{vol}_t + \int_{\partial S_t} \langle X, U \rangle d\theta_{\partial S_t} \\ &\le \Big(\int_{S_t} \psi^{n-1} d\mathrm{vol}_t \Big)^{\frac{1}{n-1}} \Big(\int_{S_t} d\mathrm{vol}_t \Big)^{\frac{n-2}{n-1}} + \int_{\partial S_t} |X| d\theta_{\partial S_t} \\ &\le \mathcal{H}_{n-1}(t)^{\frac{1}{n-1}} V(t)^{\frac{n-2}{n-1}} + \int_{\partial S_t} |X| d\theta_{\partial S_t} \end{aligned} \tag{2.12}$$

by (2.11) and $H = -\mathrm{tr}\nabla \frac{\partial}{\partial t} = -h \frac{\partial}{\partial t}$.

If ∂D is transversal to ∂M_t, then $|X| < \infty$ and $V'(t) < \infty$. Hence if ∂D is transversal to ∂M_t for any $t \in (0, R)$, then $d\mathrm{vol}_{\partial D} = |X| d\theta_{\partial S_t} dt$ since $X \perp \partial S_t$, where $d\mathrm{vol}_{\partial D}$ is the volume form of ∂D. Then by integrating (2.12), we obtain (2.15). Even if ∂D is not transversal to ∂M_t, we can obtain (2.15) as follows.

Let $B = \{t \in (0,R) \mid V'(t) = \infty\}$ and $(0,R) \setminus B = \bigcup_j I_j$ for open intervals I_j. Integrating (2.12) over t, we have that

$$(2.13)\quad \begin{aligned} V(t) - V(0) \leq & \int_0^t \mathcal{H}_{n-1}(u)^{\frac{1}{n-1}} V(u)^{\frac{n-2}{n-1}} du + \sum_j \int_{I_j} \int_{\partial S_u} |X| d\theta_{\partial S_u} du \\ & + \sum_{t \in B} V(t+) - V(t-), \end{aligned}$$

where $V(t+) = \lim_{h \to 0+} V(t+h)$ and $V(t-) = \lim_{h \to 0+} V(t-h)$.

The second term $\sum_j \int_{I_j} \int_{\partial S_u} |X| d\theta_{\partial S_u} du$ of the right hand side is the volume of $\partial D \setminus \bigcup_{t \in B \cup \{0\}} \partial M_t$ since $d\text{vol}_{\partial D} = |X| d\theta_{\partial S_t} dt$. For the third term, let $S_{t+} = \lim_{h \to 0+} S_{t+h} \subset \partial M_t$ and $S_{t-} = \lim_{h \to 0+} S_{t-h} \subset \partial M_t$ for $t \in B$. (See Figure 2.) Let $x \in \partial M_t$ be an interior point of D. If $y \in B(x,h) \cap M_t$, then $y \in \partial M_{t+h'}$ for $0 \leq h' \leq h$. (Recall that $M_t = \{x \in M^n \mid d(x, \partial M^n) \geq t\}$.) Also we let $z = \partial M_{t-h} \cap \gamma_x$ for the shortest geodesic γ_x from ∂M^n to x. Then $y, z \to x$ as $h \to 0$ and $y \in \partial M_{t+h'} \cap D = S_{t+h'}$ and $z \in \partial M_{t-h} \cap D = S_{t-h}$. So we have $x \in S_{t+} \cap S_{t-}$ for an interior point x of D. Hence $(S_{t+} \setminus S_{t-}) \cup (S_{t-} \setminus S_{t+}) \subset \partial D \cap \partial M_t$. Since $|V(t+) - V(t-)|$ is the volume of $(S_{t+} \setminus S_{t-}) \cup (S_{t-} \setminus S_{t+}) \subset \partial D \cap \partial M_t$, $\sum_{t \in B} |V(t+) - V(t-)|$ is the volume of $\partial D \cap \bigcup_{t \in B} \partial M_t$.

Since $\bigcup_{u \in (0,R)} \partial S_u \subset \partial D \setminus \partial M^n$ and $\partial D \cap \bigcup_{t \in B} \partial M_t \subset \partial D \setminus \partial M^n$, the sum of the second and the third terms in the right hand side of (2.13) is smaller than or equal to $A(\partial D \setminus \partial M^n)$. So we obtain that

$$(2.14)\quad \begin{aligned} V(t) - V(0) &\leq \left(\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^{\frac{n-2}{n-1}} + \mathcal{H}_{n-1}^{\frac{1}{n-1}}\right) R^{\frac{1}{n-1}} \left(\int_0^t V(u) du\right)^{\frac{n-2}{n-1}} \\ &\quad + A(\partial D \setminus \partial M^n) \\ &\leq \left(\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^{\frac{n-2}{n-1}} + \mathcal{H}_{n-1}^{\frac{1}{n-1}}\right) R^{\frac{1}{n-1}} \left(\int_0^t V(u) du\right)^{\frac{n-2}{n-1}} \\ &\quad + A(\partial D) - A(\partial D \cap \partial M^n). \end{aligned}$$

Since $A(\partial D \cap \partial M^n) = V(0)$, we have

$$(2.15)\quad V(t) \leq \left(\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^{\frac{n-2}{n-1}} + \mathcal{H}_{n-1}^{\frac{1}{n-1}}\right) R^{\frac{1}{n-1}} \left(\int_0^t V(u) du\right)^{\frac{n-2}{n-1}} + A(\partial D).$$

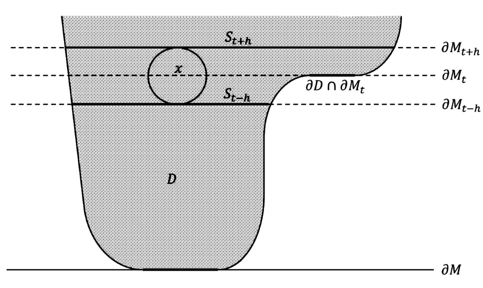

FIGURE 2. Singular point of V'

If we let $Y = \int_0^t V$ and $\beta_1 = \mathcal{R}_{n-1}^{\frac{1}{n-1}} R + \mathcal{H}_{n-1}^{\frac{1}{n-1}} R^{\frac{1}{n-1}}$, then we obtain the following differential inequality:

$$Y' - \beta_1 Y^{\frac{n-2}{n-1}} \leq A(\partial D). \tag{2.16}$$

Assume that $Y(t) \geq 1$ for $t \geq t_0$. Then for $t \geq t_0$, we obtain the following linear differential inequality:

$$Y' - \beta_1 Y \leq A(\partial D).$$

Then

$$\begin{aligned} Y(R) &\leq \frac{e^{\beta_1(R-t_0)} - 1}{\beta_1} A(\partial D) + e^{\beta_1(R-t_0)} \\ &\leq \frac{e^{\beta_1 R} - 1}{\beta_1} A(\partial D) + e^{\beta_1 R}. \end{aligned}$$

Otherwise, $Y(t) \leq 1$ for any $t > 0$. Since $\mathrm{vol}(D) \leq Y(R)$ and $\frac{e^{\beta_1 R}-1}{\beta_1} A(\partial D) + e^{\beta_1 R} \geq 1$, we obtain that

$$\mathrm{vol}(D) \leq e^{\beta_1 R} + A(\partial D) \frac{e^{\beta_1 R} - 1}{\beta_1}. \tag{2.17}$$

If M^n is mean convex everywhere, then $\mathcal{H}_{n-1} = 0$ and $\beta_1 = \mathcal{R}_{n-1}^{\frac{1}{n-1}} R$. So

$$\mathrm{vol}(D) \leq e^{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^2} + A(\partial D)\Big(\frac{e^{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R^2} - 1}{\mathcal{R}_{n-1}^{\frac{1}{n-1}} R}\Big). \tag{2.18}$$

Remark 2.1. We can consider a functional inequality from our isoperimetric inequality. For a smooth function $f : M^n \to \mathbb{R}$, let $\Omega(t) = \{x \mid |f(x)| > t\}$. By the co-area formula and Cavalieri's principle [Ch2], we have that

$$\int_{M^n} |\nabla f| dV = \int_0^{|f|_\infty} A(|f|^{-1}(t)) dt,$$

$$\int_{M^n} |f| dV = \int_0^{|f|_\infty} \mathrm{vol}(\Omega(t)) dt.$$

Since $\partial\Omega(t) \subset |f|^{-1}(t) \cup \partial M^n$, we have $A(\partial\Omega(t)) \leq A(|f|^{-1}(t)) + A(\partial M^n)$. Then we obtain the following functional inequality from Theorem 1.1:

$$\int_{M^n} |f| dV \leq \frac{e^{\beta_1 R} - 1}{\beta_1} \int_{M^n} |\nabla f| dV + |f|_\infty \Big(\frac{e^{\beta_1 R} - 1}{\beta_1} A(\partial M^n) + e^{\beta_1 R}\Big). \tag{2.19}$$

3. Proof of Theorem 1.2

We use the same notation as in (2.1), (2.2) and (2.4). We will follow a similar procedure as in [Pa1]. With an integral norm of Ricci curvature, we obtain the following comparisons. Since $\omega' = h\omega$, we have

$$\frac{d}{dt}\omega \leq \psi\omega.$$

Integrating the above for t, we obtain that

$$\omega(r) - \omega(r_1) \leq \int_{r_1}^{r} \psi\omega ds \tag{3.1}$$

for $r_1 \leq r$. Then we have for $r_2 > r_1$,

$$\int_{r_1}^{r_2} \omega(s)ds \leq (r_2 - r_1)(\omega(r_1) + \int_{r_1}^{r_2} \psi\omega ds). \tag{3.2}$$

Let $\{\gamma_q(t) \mid t \leq t_q\} \cap D = \bigcup_{j_q} \gamma_q[\alpha_{j_q}, \beta_{j_q}]$. From (3.2), we obtain that

$$\begin{aligned} \int_{\alpha_{j_q}}^{\beta_{j_q}} \omega(s)ds &\leq (\beta_{j_q} - \alpha_{j_q})(\omega(\alpha_{j_q}) + \int_{\alpha_{j_q}}^{\beta_{j_q}} \psi\omega ds) \\ &\leq R(\omega(\alpha_{j_q}) + \int_{\alpha_{j_q}}^{\beta_{j_q}} \psi\omega ds). \end{aligned} \tag{3.3}$$

By (3.3), we have

$$\begin{aligned} \mathrm{vol}(D) &= \int_{\partial M^n} \sum_{j_q} \int_{\alpha_{j_q}}^{\beta_{j_q}} \omega(s)ds \ d\mathrm{vol}_{\partial M^n} \\ &\leq R\Big(\int_{\partial M^n} \sum_{j_q} \omega(\alpha_{j_q}) d\mathrm{vol}_{\partial M^n} + \int_{\partial M^n} \sum_{j_q} \int_{\alpha_{j_q}}^{\beta_{j_q}} \psi\omega ds \ d\mathrm{vol}_{\partial M^n}\Big). \end{aligned} \tag{3.4}$$

On the right hand side,

$$\begin{aligned} \int_{\partial M^n} \sum_{j_q} \int_{\alpha_{j_q}}^{\beta_{j_q}} \psi\omega ds \ d\mathrm{vol}_{\partial M^n} &\leq \Big(\int_{\partial M^n} \sum_{j_q} \int_{\alpha_{j_q}}^{\beta_{j_q}} \psi^{2p}\omega ds \ d\mathrm{vol}_{\partial M^n}\Big)^{\frac{1}{2p}} \mathrm{vol}(D)^{1-\frac{1}{2p}} \\ &\leq \Big(\int_D \psi^{2p} dV\Big)^{\frac{1}{2p}} \mathrm{vol}(D)^{1-\frac{1}{2p}}. \end{aligned} \tag{3.5}$$

By (3.4), we have

$$\begin{aligned} \mathrm{vol}(D) &\leq R\Big(\int_{\partial M^n} \sum_{j_q} \omega(\alpha_{j_q}) d\mathrm{vol}_{\partial M^n} + \Big(\int_{M^n} \psi^{2p} dV\Big)^{\frac{1}{2p}} \mathrm{vol}(D)^{1-\frac{1}{2p}}\Big) \\ &\leq R\Big(A(\partial D) + \Big(\int_{M^n} \psi^{2p} dV\Big)^{\frac{1}{2p}} \mathrm{vol}(D)^{1-\frac{1}{2p}}\Big). \end{aligned} \tag{3.6}$$

Now we estimate $\int \psi^{2p}$ by using similar arguments as in [PW], [Pa2]. In (2.5), multiplying $\psi^{2p-2}\omega$ and integrating, we have

$$\int_0^R \psi' \psi^{2p-2} \omega dt + \int_0^R \frac{\psi^{2p}}{n-1}\omega \leq \int_0^R \rho\psi^{2p-2}\omega dt. \tag{3.7}$$

By integration by parts, we obtain that

$$\int_0^R \psi' \psi^{2p-2} \omega dt \geq -\frac{\psi^{2p-1}}{2p-1}\omega(0) - \frac{1}{2p-1}\int_0^R \psi^{2p}\omega dt.$$

Inserting into (3.7), we obtain that

$$\Big(\frac{1}{n-1} - \frac{1}{2p-1}\Big) \int_0^R \psi^{2p}\omega dt \leq \int_0^R \rho\psi^{2p-2}\omega dt + \frac{\psi^{2p-1}}{2p-1}\omega(0). \tag{3.8}$$

Integrating on ∂M^n, then we have

$$\begin{aligned}\Big(\frac{1}{n-1}-\frac{1}{2p-1}\Big)\int_{M^n}\psi^{2p}dV &\le \int_{M^n}\rho\psi^{2p-2}dV+\frac{\mathcal{H}_{2p-1}}{2p-1}\\ &\le (\int_{M^n}\rho^p dV)^{\frac{1}{p}}(\int_{M^n}\psi^{2p}dV)^{1-\frac{1}{p}}+\frac{\mathcal{H}_{2p-1}}{2p-1}.\end{aligned}\tag{3.9}$$

If we let $\Psi=\int_{M^n}\psi^{2p}dV$, then

$$\Psi\le C_0(n,p)^2(\mathcal{R}_p^{\frac{1}{p}}\Psi^{1-\frac{1}{p}}+\frac{\mathcal{H}_{2p-1}}{2p-1}).$$

If $\Psi\ge\epsilon$, then $\Psi\ge\epsilon^{\frac{1}{p}}\Psi^{1-\frac{1}{p}}$ and

$$\Psi\le C_0(n,p)^2(\epsilon^{-\frac{1}{p}}\mathcal{R}_p^{\frac{1}{p}}\Psi+\frac{\mathcal{H}_{2p-1}}{2p-1}).$$

If we let

$$\epsilon=2^pC_0(n,p)^{2p}\mathcal{R}_p,$$

then $C_0(n,p)^2\epsilon^{-\frac{1}{p}}\mathcal{R}_p^{\frac{1}{p}}=\frac{1}{2}$. So we have $\Psi\le 2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}$. Consequently, we obtain that

$$\begin{aligned}\int_{M^n}\psi^{2p}dV &\le \max\{\epsilon,2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}\}\\ &\le 2^pC_0(n,p)^{2p}\mathcal{R}_p+2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}.\end{aligned}\tag{3.10}$$

From (3.6) and (3.10), we obtain that

$$A(\partial D)\ge\frac{1}{R}\mathrm{vol}(D)-(2^pC_0(n,p)^{2p}\mathcal{R}_p+2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1})^{\frac{1}{2p}}\mathrm{vol}(D)^{1-\frac{1}{2p}}.$$

In order to obtain a linear isoperimetric inequality, we apply the same technique as above. If $\mathrm{vol}(D)\ge\epsilon$, then $\mathrm{vol}(D)\ge\epsilon^{\frac{1}{2p}}\mathrm{vol}(D)^{1-\frac{1}{2p}}$ and

$$A(\partial D)\ge\frac{1}{R}\mathrm{vol}(D)-(2^pC_0(n,p)^{2p}\mathcal{R}_p+2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1})^{\frac{1}{2p}}\epsilon^{-\frac{1}{2p}}\mathrm{vol}(D).\tag{3.11}$$

If we let

$$\epsilon=(2R)^{2p}(2^pC_0(n,p)^{2p}\mathcal{R}_p+2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}),$$

then

$$A(\partial D)\ge\frac{1}{2R}\mathrm{vol}(D).$$

Consequently, we obtain that

$$\begin{aligned}\mathrm{vol}(D)&\le\max\{2RA(\partial D),\epsilon\}\\ &\le 2RA(\partial D)+(2R)^{2p}(2^pC_0(n,p)^{2p}\mathcal{R}_p+2C_0(n,p)^2\frac{\mathcal{H}_{2p-1}}{2p-1}).\end{aligned}\tag{3.12}$$

4. Isoperimetric inequalities for metric-measure space

Let $(M^n, g, e^{-f}dV)$ be an n-dimensional metric-measure space with smooth boundary. The Bakry-Emery Ricci tensor is $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f$, and the weighted mean curvature is $h_f = h - \frac{\partial f}{\partial t}$. Let $d\nu = e^{-f}dV$ and $d\mu_t = e^{-f}d\mathrm{vol}_t$. In particular, we denote $d\mu_0$ by $d\mu$. We define integral norms $\mathcal{R}_{f,p}$ and $\mathcal{H}_{f,p}$ as follows:

$$\mathcal{R}_{f,p} = \int_{M^n} \rho_f^p d\nu,$$
$$\mathcal{H}_{f,p} = \int_{M^n} (h_f)_+^p d\mu$$

where $\rho_f = \max\{\ (-\mathrm{Ric}_f(v,v))_+ \mid |v| = 1, v \in T_qN\}$.

We will prove the following theorems similarly as Theorems 1.1 and 1.2.

Theorem 4.1. *If* $\mathrm{diam}(M^n) = R$ *and* $|\nabla f| \leq L_1$*, then for a domain* $D \subset M^n$ *with smooth boundary,*

$$\nu(D) \leq e^{\beta_2 R} + \mu(\partial D)\frac{e^{\beta_2 R} - 1}{\beta_2},$$

where $\beta_2 = e^{\frac{2L_1}{n-1}R}\Big(\mathcal{H}_{f,n-1}^{\frac{1}{n-1}} R^{\frac{1}{n-1}} + \mathcal{R}_{f,n-1}^{\frac{1}{n-1}} R\Big)$.

Theorem 4.2. *If* $\mathrm{diam}(M^n) = R$ *and* $(\int_{M^n} |\nabla f|^{2p} d\nu)^{\frac{1}{2p}} \leq L_2$*, then for a domain* $D \subset M^n$ *with smooth boundary and* $2p > n$*,*

$$\nu(D) \leq R\Big(\mu(\partial D) + \Big(\epsilon + 2C_0(n,p)^2 \frac{\mathcal{H}_{f,2p-1}}{2p-1}\Big)^{\frac{1}{2p}} \nu(D)^{1-\frac{1}{2p}}\Big), \tag{4.1}$$

where $\epsilon = \Bigg(\frac{-\frac{2C_0(n,p)^2 L_2}{n-1} + \sqrt{(\frac{2C_0(n,p)^2 L_2}{n-1})^2 + 2C_0(n,p)^2 \mathcal{R}_{f,p}^{\frac{1}{p}}}}{2C_0(n,p)^2 \mathcal{R}_{f,p}^{\frac{1}{p}}}\Bigg)^{-2p}$.

Proof of Theorem 4.1. We use similar notation as previously. Let $d\mu_t = \omega_f d\mu$ similarly as in (2.1) and

$$\psi = \begin{cases} (h_f)_+ & \text{if } t \leq t_q, \\ 0 & \text{if } t > t_q. \end{cases} \tag{4.2}$$

Since $h = h_f + \frac{\partial f}{\partial t}$, we have the following inequality from the Riccati equation:

$$\begin{aligned} h_f' + \frac{h_f^2}{n-1} &\leq -\mathrm{Ric}_f(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) - \frac{2h_f \frac{\partial f}{\partial t}}{n-1} - \frac{(\frac{\partial f}{\partial t})^2}{n-1} \\ &\leq -\mathrm{Ric}_f(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) - \frac{2h_f \frac{\partial f}{\partial t}}{n-1}. \end{aligned} \tag{4.3}$$

Recall that $\rho_f = \max\{(-\mathrm{Ric}_f(v,v))_+ \mid |v| = 1,\ v \in T_qN\}$. Then

$$\psi' + \frac{\psi^2}{n-1} \leq \rho_f + \frac{2|\nabla f|}{n-1}\psi. \tag{4.4}$$

Similarly as in the proof of Theorem 1.1, since $\frac{\partial}{\partial t}\omega_f = h_f\omega_f$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\partial M_t}\psi^{n-1}d\mu_t &= \int_{\partial M_t}((n-1)\psi^{n-2}\psi' + \psi^{n-1}h_f)d\mu_t \\ &\le \int_{\partial M_t}\psi^{n-2}((n-1)\psi' + \psi^2)d\mu_t \\ &\le (n-1)\int_{\partial M_t}\psi^{n-2}\rho_f(q)d\mu_t + 2\int_{\partial M_t}|\nabla f|\psi^{n-1}d\mu_t \\ &\le (n-1)\Big(\int_{\partial M_t}\psi^{n-1}d\mu_t\Big)^{\frac{n-2}{n-1}}\Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}} \\ &\quad + 2L_1\int_{\partial M_t}\psi^{n-1}d\mu_t.\end{aligned}\tag{4.5}$$

We define $\mathcal{R}_{f,n-1}(t), \mathcal{H}_{f,n-1}(t)$ similarly as previously:

$$\mathcal{R}_{f,n-1}(t) = \int_{M^n\setminus M_t}\rho_f^{n-1}d\nu,$$

$$\mathcal{H}_{f,n-1}(t) = \int_{\partial M_t}h_+^{n-1}d\mu_t.$$

So we obtain that

$$\mathcal{H}'_{f,n-1}(t) \le (n-1)\Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}}\mathcal{H}_{f,n-1}(t)^{\frac{n-2}{n-1}} + 2L_1\mathcal{H}_{f,n-1}(t).\tag{4.6}$$

Then we have

$$\frac{\mathcal{H}'_{f,n-1}(t)}{(n-1)\mathcal{H}_{f,n-1}(t)^{\frac{n-2}{n-1}}} \le \Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}} + \frac{2L_1}{n-1}\mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}},\tag{4.7}$$

which implies that

$$\Big(\mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}}\Big)' - \frac{2L_1}{n-1}\mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}} \le \Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}}.\tag{4.8}$$

Letting $Y = \mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}}$, we have

$$\begin{aligned}(e^{-\frac{2L_1}{n-1}t}Y)' &= e^{-\frac{2L_1}{n-1}t}(Y' - \frac{2L_1}{n-1}Y) \le e^{-\frac{2L_1}{n-1}t}\Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}} \\ &\le \Big(\int_{\partial M_t}\rho_f(q)^{n-1}d\mu_t\Big)^{\frac{1}{n-1}}.\end{aligned}\tag{4.9}$$

Hence

$$e^{-\frac{2L_1}{n-1}t}\mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}} \le \mathcal{H}_{f,n-1}^{\frac{1}{n-1}} + \mathcal{R}_{f,n-1}^{\frac{1}{n-1}}R^{\frac{n-2}{n-1}},\tag{4.10}$$

so we obtain that

$$\mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}} \le e^{\frac{2L_1}{n-1}R}\Big(\mathcal{H}_{f,n-1}^{\frac{1}{n-1}} + \mathcal{R}_{f,n-1}^{\frac{1}{n-1}}R^{\frac{n-2}{n-1}}\Big).\tag{4.11}$$

In the proof of Theorem 1.1, we use that

$$\begin{aligned}\frac{\partial}{\partial t}A(S_t) &= \int_{S_t}\mathrm{div}_{S_t}(X)d\mathrm{vol}_t \\ &= \int_{S_t}hd\mathrm{vol}_t + \int_{\partial S_t}\langle X, U\rangle d\theta_{\partial S_t}.\end{aligned}$$

For the metric-measure space, we prove the following lemma:

Lemma 4.3. *For the measures $d\mu_t = e^{-f}\mathrm{dvol}_t$ and $d\mu_{\partial S_t} = e^{-f}d\theta_{\partial S_t}$, we have*

$$\frac{\partial}{\partial t}\mu_t(S_t) = \int_{S_t} h_f d\mu_t + \int_{\partial S_t} \langle X, U\rangle d\mu_{\partial S_t}.$$

Proof. We have that

$$\begin{aligned}\frac{\partial}{\partial t}\mu_t(S_t) &= \int_{S_t} L_X(e^{-f}\mathrm{dvol}_t)\\ &= \int_{S_t} (\mathrm{div}_{S_t}(X) - X[f])e^{-f}\mathrm{dvol}_t \qquad (4.12)\\ &= \int_{S_t} (\mathrm{div}_{S_t}(X) - X[f])d\mu_t,\end{aligned}$$

where L_X is the Lie derivative. Since $\mathrm{div}_{S_t}(Z) = \sum_{i=1}^{n-1}\langle \nabla_{e_i} Z, e_i\rangle$ for $e_i \in TS_t$, we have

$$\int_{S_t} \mathrm{div}_{S_t}(e^{-f}X)\mathrm{dvol}_t = \int_{S_t} (\mathrm{div}_{S_t}(X) - X^T[f])d\mu_t, \tag{4.13}$$

where $X = \frac{\partial}{\partial t} + \langle X, U\rangle U$ and $X^T = \langle X, U\rangle U$ as we saw in Section 2. Since $X[f] - X^T[f] = \frac{\partial}{\partial t} f$, we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\mu_t(S_t) &= \int_{S_t} \mathrm{div}_{S_t}(e^{-f}X)\mathrm{dvol}_t - \int_{S_t} \frac{\partial f}{\partial t} d\mu_t\\ &= -\int_{S_t} \langle e^{-f}X, H\rangle \mathrm{dvol}_t + \int_{\partial S_t} \langle e^{-f}X, U\rangle d\theta_{\partial S_t} - \int_{S_t} \frac{\partial f}{\partial t} d\mu_t\\ &= -\int_{S_t} \langle X, H\rangle d\mu_t + \int_{\partial S_t} \langle X, U\rangle d\mu_{\partial S_t} - \int_{S_t} \frac{\partial f}{\partial t} d\mu_t \qquad (4.14)\\ &= \int_{S_t} (h - \frac{\partial f}{\partial t}) d\mu_t + \int_{\partial S_t} \langle X, U\rangle d\mu_{\partial S_t}\\ &= \int_{S_t} h_f d\mu_t + \int_{\partial S_t} \langle X, U\rangle d\mu_{\partial S_t},\end{aligned}$$

which completes the proof of Lemma 4.3. □

Similarly as in the proof of Theorem 1.1, we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\mu_t(S_t) &\leq \int_{S_t} h_f d\mu_t + \int_{\partial S_t} |X| d\mu_{\partial S_t}\\ &\leq \mathcal{H}_{f,n-1}(t)^{\frac{1}{n-1}}\mu_t(S_t)^{\frac{n-2}{n-1}} + \int_{\partial S_t} |X| d\mu_{\partial S_t}.\end{aligned} \tag{4.15}$$

Since $d\mu_{\partial S_t} = e^{-f}\mathrm{dvol}_{\partial S_t}$ and $d\mu_{\partial D} = e^{-f}\mathrm{dvol}_{\partial D}$, we obtain from (4.11) that

$$\begin{aligned}\mu_t(S_t) \leq\ & e^{\frac{2L_1}{n-1}R}\Big(\mathcal{H}_{f,n-1}^{\frac{1}{n-1}} R^{\frac{1}{n-1}} + \mathcal{R}_{f,n-1}^{\frac{1}{n-1}} R\Big)\Big(\int_0^t \mu_t(S_t)\Big)^{\frac{n-2}{n-1}}\\ &+ \mu_{\partial D}(\partial D)\end{aligned} \tag{4.16}$$

similarly as (2.15) in the proof of Theorem 1.1.

Now it remains only to follow the proof of Theorem 1.1 with (4.16) instead of (2.13), where we will use $\beta_2 = e^{\frac{2L_1}{n-1}R}\Big(\mathcal{H}_{f,n-1}^{\frac{1}{n-1}} R^{\frac{1}{n-1}} + \mathcal{R}_{f,n-1}^{\frac{1}{n-1}} R\Big)$ instead of β_1. □

Proof of Theorem 4.2. We will follow the proof of Theorem 1.2. Since $\frac{d}{ds}\omega_f \le \psi\omega_f$, we have

$$\int_{r_1}^{r_2} \omega_f(s)ds \le (r_2 - r_1)(\omega_f(r_1) + \int_0^r \psi\omega_f) \tag{4.17}$$

for $r_2 > r_1$. Multiplying $\psi^{2p-2}\omega_f$ in (4.4) and integrating, we have

$$\begin{aligned} &\int_0^R \psi'\psi^{2p-2}\omega_f dt + \int_0^R \frac{\psi^{2p}}{n-1}\omega_f \\ &\le \int_0^R \rho_f\psi^{2p-2}\omega_f dt + \int_0^R \frac{2|\nabla f|}{n-1}\psi^{2p-1}\omega_f dt. \end{aligned} \tag{4.18}$$

Similarly as in (3.8), we obtain that

$$\begin{aligned} \Big(\frac{1}{n-1} - \frac{1}{2p-1}\Big)\int_0^R \psi^{2p}\omega_f dt &\le \int_0^R \rho_f\psi^{2p-2}\omega_f dt \\ &+ \frac{2}{n-1}\int_0^R |\nabla f|\psi^{2p-1}\omega_f dt + \frac{\psi^{2p-1}}{2p-1}\omega_f(0). \end{aligned} \tag{4.19}$$

Integrating on ∂M^n, we have

$$\begin{aligned} &\Big(\frac{1}{n-1} - \frac{1}{2p-1}\Big)\int_{M^n} \psi^{2p} d\nu \\ &\le \int_{M^n} \rho_f\psi^{2p-2}d\nu + \frac{2}{n-1}\int_{M^n} |\nabla f|\psi^{2p-1}d\nu + \frac{\mathcal{H}_{f,2p-1}}{2p-1} \\ &\le (\int_{M^n} \rho_f^p d\nu)^{\frac{1}{p}}(\int_{D_T} \psi^{2p}d\nu)^{1-\frac{1}{p}} \\ &+ \frac{2}{n-1}(\int_{M^n} |\nabla f|^{2p}d\nu)^{\frac{1}{2p}}(\int_{M^n} \psi^{2p}d\nu)^{\frac{2p-1}{2p}} + \frac{\mathcal{H}_{f,2p-1}}{2p-1} \\ &\le (\int_{M^n} \rho_f^p d\nu)^{\frac{1}{p}}(\int_{M^n} \psi^{2p}d\nu)^{1-\frac{1}{p}} + \frac{2L_2}{n-1}(\int_{M^n} \psi^{2p}d\nu)^{\frac{2p-1}{2p}} + \frac{\mathcal{H}_{f,2p-1}}{2p-1}. \end{aligned} \tag{4.20}$$

If we let $\Psi = \int_{M^n} \psi^{2p}d\nu$, then

$$\Psi \le C_0(n,p)^2(\mathcal{R}_{f,p}^{\frac{1}{p}}\Psi^{1-\frac{1}{p}} + \frac{2L_2}{n-1}\Psi^{1-\frac{1}{2p}} + \frac{\mathcal{H}_{f,2p-1}}{2p-1}).$$

If $\Psi \ge \epsilon$, then $\Psi \ge \epsilon^{\frac{1}{2p}}\Psi^{1-\frac{1}{2p}} \ge \epsilon^{\frac{1}{p}}\Psi^{1-\frac{1}{p}}$ and

$$\Psi \le C_0(n,p)^2(\epsilon^{-\frac{1}{p}}\mathcal{R}_{f,p}^{\frac{1}{p}}\Psi + \epsilon^{-\frac{1}{2p}}\frac{2L_2}{n-1}\Psi + \frac{\mathcal{H}_{f,2p-1}}{2p-1}).$$

If

$$\epsilon = \Big(\frac{-\frac{2C_0(n,p)^2L}{n-1} + \sqrt{(\frac{2C_0(n,p)^2L_2}{n-1})^2 + 2C_0(n,p)^2\mathcal{R}_{f,p}^{\frac{1}{p}}}}{2C_0(n,p)^2\mathcal{R}_{f,p}^{\frac{1}{p}}}\Big)^{-2p},$$

then $C_0(n,p)^2(\epsilon^{-\frac{1}{p}}\mathcal{R}_{f,p}^{\frac{1}{p}} + \epsilon^{-\frac{1}{2p}}\frac{2L_2}{n-1}) = \frac{1}{2}$. We obtain that

$$\begin{aligned} \int_{M^n} \psi^{2p}d\nu &\le \max\{\epsilon, 2C_0(n,p)^2\frac{\mathcal{H}_{f,2p-1}}{2p-1}\} \\ &\le \epsilon + 2C_0(n,p)^2\frac{\mathcal{H}_{f,2p-1}}{2p-1}. \end{aligned} \tag{4.21}$$

Now we only need to follow the proof of Theorem 1.2. Then

$$\nu(D) \leq R\Big(\mu(\partial D) + \Big(\epsilon + 2C_0(n,p)^2 \frac{\mathcal{H}_{f,2p-1}}{2p-1}\Big)^{\frac{1}{2p}} \nu(D)^{1-\frac{1}{2p}}\Big). \tag{4.22}$$

□

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