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LEFSCHETZ PROPERTIES FOR ARTINIAN GORENSTEIN ALGEBRAS PRESENTED BY QUADRICS

RODRIGO GONDIM AND GIUSEPPE ZAPPALÀ

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Abstract. We introduce a family of Artinian Gorenstein algebras, whose combinatorial structure characterizes the ones presented by quadrics. Under certain hypotheses these algebras have non-unimodal Hilbert vector. In particular we provide families of counterexamples to the conjecture that Artinian Gorenstein algebras presented by quadrics should satisfy the weak Lefschetz property.

1. Introduction

It is very useful in algebraic geometry and in commutative algebra to produce geometric or algebraic objects from combinatoric ones. Toric varieties ([CLS]) and toric ideals ([Stu]) are the more stablished of these associations. We can also cite tropical varieties ([S]), Stanley-Reisner theory ([St, St2]), Artinian algebras given by posets ([HMMNWW]), as incarnations of this fruitful interaction among algebra, geometry and combinatorics. In this paper we propose a new construction in this direction, associating simplicial complexes to certain Bigraded Artinian Gorenstein algebras.

The combinatoric structure of the simplicial complex, in our association, characterizes the algebras presented by quadrics. To be more precise, it determines the quotient algebra and its ideal on a natural embedding. In particular, the Hilbert vector of such Artinian algebra is determined by the face vector of the complex. In this way we construct a concrete family of algebras presented by quadrics and whose Hilbert vector is non-unimodal. These algebras provide counterexamples for two conjectures posed in [MN1, MN2].

The kind of algebra we introduce is closely related to Stanley-Reisner theory ([St, St2]). The starting point of both constructions is a homogeneous simplicial complex with $m \geq 2$ vertices and dimension $d-1 \geq 1$, associated to a set of square free monomials in m variables of degree d. Each monomial represents a facet of the simplicial complex. The very distinct point of view is that our construction is also related to Nagata's idealization ([HMMNWW]). This point of view is related to the vanishing of Hessian determinant (see [Go, GRu, CRS, MW]).

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A standard graded $\mathbb{K}$-algebra is said to be presented by quadrics if it is isomorphic to the quotient of a polynomial ring over $\mathbb{K}$ by a homogeneous ideal generated by quadratic forms. Also called quadratic algebras, they are related to Koszul algebras and Gröbner basis (see for instance [Co]). From a more geometric point of view quadratic ideals appear as homogeneous ideals of very positive embeddings of any smooth projective varieties. As pointed out in [MN2], Artinian Gorenstein algebras presented by quadrics are also related to Eisenbud-Green-Harris conjectures motivated by the Cayley-Bacharach theorem ([EGH]).

The Lefschetz properties, on the other side, have attracted a great deal of attention over the years of research in different subjects including commutative algebra, algebraic geometry and combinatorics; see [St, St2, MN1, HMMNWW, HMNW]. The present work lies in the border of these three areas.

In [MN2], the authors studied Artinian Gorenstein algebras presented by quadrics. They provided some constructions of such algebras and described their possible Hilbert vectors in low codimension. In [MN1] and [MN2] the authors proposed two conjectures.

Conjecture 1.1 (Migliore-Nagel injective conjecture)**.** *For any Artinian Gorenstein algebra of socle degree at least three, presented by quadrics, defined over a field $\mathbb{K}$ of characteristic zero there exists $L \in A_1$, such that, the multiplication map $\bullet L : A_1 \to A_2$ is injective.*

Conjecture 1.2 (Migliore-Nagel WLP conjecture)**.** *Any Artinian Gorenstein algebra presented by quadrics, over a field $\mathbb{K}$ of characteristic zero, has the Weak Lefschetz Property, that is, there exists $L \in A_1$ such that all the maps $\bullet L : A_i \to A_{i+1}$ have maximal rank.*

In [MN2] the authors proved the WLP conjecture for complete intersection of quadratic forms and presented computational evidence for the conjectures in low codimension. We want to stress the fact that as soon as the codimension increases with respect to the socle degree surprising phenomena begin to appear. For instance, in codimension ≤ 2 every Artinian algebra has the strong Lefschetz property (see [HMNW]) and in codimension ≤ 3 every Artinian Gorenetein algebra has unimodal Hilbert vector (see [St3]). On the contrary, in high codimension the Hilbert vector of an Artinian Gorensten algebra does not need to be unimodal. In [BL], the authors studied Artinian Gorenstein algebras whose Hilbert vector are non-unimodal. They appear in codimension ≥ 5.

We recall that the Lefschetz properties for standard graded Artinian $\mathbb{K}$-algebras are algebraic abstractions motivated by the Hard Lefschetz Theorem on the cohomology rings of smooth complex projective varieties; see for instance the survey [La] for the theorem and [Ru] for an overview. The Poincaré duality for these cohomology rings inspired the definition of Poincaré duality algebras which, in this context, is equivalent to the Gorenstein hypothesis (see [MW] and [Ru]). In [Wa2] and [MW] the authors used Macaulay-Matlis duality in characteristic zero to present the Artinian Gorenstein algebra as $A = Q/\operatorname{Ann}_Q(f)$ where $f \in R = \mathbb{K}[x_1, \dots, x_n]$ a polynomial ring and $Q = \mathbb{K}[X_1, \dots, X_N]$ the associated ring of differential operators.

Our strategy to construct standard graded Artinian Gorenstein algebras presented by quadrics is to deal with the simplest ones, which are those whose defining ideal contains the complete intersection $(x_1^2, \dots, x_n^2)$. This assumption forces all monomials that occur in f to be square free. As a matter of fact we deal with bihomogeneous forms of bidegree $(1, d-1)$ of special type, here called of monomial square free type (see Definition 2.8). We associate to any bihomogeneous form of monomial square free type, bijectively, a pure simplicial complex whose combinatoric structure determines a set of generators of the annihilator ideal; see Theorem 3.2. This combinatorial object also characterizes the associated algebra presented by quadrics (see Theorem 3.5, which summarizes the main results of the work). Inspired by the famous Turan's graph theorem (see [Tu]) that characterizes maximal graphs not containing a complete subgraph K_l, we introduce a simplicial complex, here called the Turan complex, whose associated algebra is always presented by quadrics and such that, in very large codimension with respect to the socle degree, the Hilbert vector of A is totally non-unimodal, that is, $\dim A_1 > \dim A_2 > \ldots > \dim A_{\lfloor \frac{d}{2} \rfloor}$.

We now describe the contents of the paper in more detail. In the first section we recall the basic definitions and constructions of standard graded Artinian Gorenstein algebras. We deal also with the bigraded case which is of particular interest. We recall the Lefschetz properties and Macaulay-Matlis duality.

The second section is devoted to the main results and constructions. Theorem 3.2 describes the annihilator of a standard bigraded form of bidegree $(1, d-1)$ of monomial square free type, showing that it is a binomial ideal whose generators are determined by the combinatoric of the associated simplicial complex. Theorem 3.2, Theorem 3.5 and Corollary 3.8 are the main results. Theorem 3.5 characterizes when such algebras are presented by quadrics. We introduce the Turan complex and in Corollary 3.8, we produce counterexamples to both Migliore-Nagel conjectures in any socle degree $d \geq 4$ and sufficient large codimension. It is surprising that if the codimension is very large with respect to the socle degree, the Hilbert vector of Turan algebras, that are quadratic and binomial, is totally non-unimodal. In fact monomial and closely related ideals, in characteristic zero, are expected to have the Weak Lefschetz Property (see [MMN]).

2. Combinatorics, Lefschetz properties and Macaulay-Matlis duality

2.1. **Combinatorics.**

Definition 2.1. Let $V = \{u_1, \ldots, u_m\}$ be a finite set. A simplicial complex Δ with vertex set V is a collection of subsets of V, *i.e.* a subset of the power set 2^V, such that for all $A \in \Delta$ and for all subset $B \subseteq A$ we have $B \in \Delta$. The members of Δ referred to as faces and maximal faces (with respect to the inclusion) are the facets. If $A \in \Delta$ and $|A| = k$, it is called a $(k-1)$-face, or a face of dimension $k-1$. If all the facets have the same dimension d the complex is said to be homogeneous of (pure) dimension d. We say that Δ is a simplex if $\Delta = 2^V$.

In our context we identify the faces of a simplicial complex with monomials in the variables $\{u_1, \ldots, u_m\}$. Let $\mathbb{K}$ be any field and let $R = \mathbb{K}[u_1, \ldots, u_m]$ be the polynomial ring. To any finite subset $F \subset \{u_1, \ldots, u_m\}$ we associate the monomial $m_F = \prod_{u_i \in F} u_i$. In this way there is a natural bijection between the simplicial complex Δ and the set of the monomials m_F, where F a facet of Δ.

2.2. **The Lefschetz properties.** Let $\mathbb{K}$ be an infinite field and $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates.

Definition 2.2. Let A be a standard graded $\mathbb{K}$-algebra. We say that A is presented by quadrics if $A \simeq R/I$, where $R = \mathbb{K}[x_1, \ldots, x_n]$ and the homogeneous ideal I has a set of generators consisting of quadratic forms.

Let $A = R/I$ be an Artinian standard graded R-algebra; then A has a decomposition $A = \bigoplus_{i=0}^{d} A_i$, as a sum of finite dimensional $\mathbb{K}$-vector spaces with $A_d \neq 0$.

Let $A = R/I$ be an Artinian standard graded R-algebra. A form $F \in R_d$ induces a $\mathbb{K}$-vector spaces map $\varphi_{i,F} : A_i \to A_{i+d}$, defined by $\varphi_{i,F}(\alpha) = F\alpha$, for every $\alpha \in A_i$.

Definition 2.3. We say that A has the *Strong Lefschetz Property* (SLP) if there exists a linear form $L \in R_1$ such that $\operatorname{rk} \varphi_{i,L^k} = \min\{\dim_{\mathbb{K}} A_i, \dim_{\mathbb{K}} A_{i+k}\}$, for every i, k.

Definition 2.4. We say that A has the *Weak Lefschetz Property* (WLP) if there exists a linear form $L \in R_1$ such that $\operatorname{rk} \varphi_{i,L} = \min\{\dim_{\mathbb{K}} A_i, \dim_{\mathbb{K}} A_{i+1}\}$, for every i.

Definition 2.5. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $A = R/I$ be an Artinian standard graded R-algebra, with $I_1 = 0$. The integer n is said to be the codimension of A. If $A_d \neq 0$ and $A_i = 0$ for all $i > d$, then d is called the socle degree of A. The Hilbert vector of A is $h_A = \operatorname{Hilb}(A) = (1, h_1, h_2, \ldots, h_d)$, where $h_k = \dim A_k$. We say that h_A is unimodal if there exists k such that $1 \leq h_1 \leq \ldots \leq h_k \geq h_{k+1} \geq h_d$.

Remark 2.6. We recall that an Artinian algebra $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, is a *Gorenstein algebra* if and only if $\dim_{\mathbb{K}} A_d = 1$ and the bilinear pairing

$$A_i \times A_{d-i} \to A_d$$

inducted by the multiplication is non-degenerated for $0 \leq i \leq d$. So we have an isomorphism $A_i \simeq \operatorname{Hom}_{\mathbb{K}}(A_{d-i}, A_d)$ for $i = 0, \ldots, d$. In particular, $\dim_{\mathbb{K}} A_i = \dim_{\mathbb{K}} A_{d-i}$, for $i = 0, \ldots, d$. Moreover, for every $L \in R_1$, $\operatorname{rk} \varphi_{i,L} = \operatorname{rk} \varphi_{d-i-1,L}$, for $0 \leq i \leq d$.

Since A is generated in degree 0 as an R-module, if $\varphi_{i,L}$ is surjective, then $\varphi_{j,L}$ is surjective for every $j \geq i$. Therefore, if A is a Gorenstein Artinian algebra, if $\varphi_{i,L}$ is injective, then $\varphi_{j,L}$ is injective for every $j \leq i$. Of course SLP implies WLP. Notice also that the WLP implies the unimodality of the Hilbert vector of A. Unimodality in the Gorenstein case implies that $\dim A_{k-1} \leq \dim A_k$ for all $k \leq \frac{d}{2}$. The converse of these implications are not true (see [Go]).

2.3. **Macaulay-Matlis duality.** Now we assume that $\operatorname{char} \mathbb{K} = 0$. Let us regard the polynomial algebra R as a module over the algebra $Q = \mathbb{K}[X_1, \ldots, X_n]$ via the identification $X_i = \partial/\partial x_i$. If $f \in R$ we set

$$\operatorname{Ann}_Q(f) = \{p(X_1, \ldots, X_n) \in Q \mid p(\partial/\partial x_1, \ldots, \partial/\partial x_n) f = 0\}.$$

By Macaulay-Matlis duality we have a bijection:

$$\begin{array}{ccc} \{\text{Homogeneous ideals of } R\} & \leftrightarrow & \{\text{Graded } R-\text{submodules of } Q\} \\ \operatorname{Ann}_Q(M) & \leftarrow & M \\ I & \rightarrow & I^{-1}. \end{array}$$

Let $I \subset Q$ be a homogeneous ideal. It is well known that $A = Q/I$ is a Gorenstein standard graded Artinian algebra if and only if there exists a form $f \in R$ such that $I = \mathrm{Ann}_Q(f)$ (for more details see, for instance, [MW]).

In the sequel we always assume that $\mathrm{char}(\mathbb{K}) = 0$, $A = Q/I$, $I = \mathrm{Ann}_Q(f)$ and $I_1 = 0$. When we need to assume that $\mathbb{K}$ is algebraically closed it will be explicit. All arguments work over $\mathbb{C}$.

We deal with standard bigraded Artinian Gorenstein algebras $A = \bigoplus_{i=0}^{d} A_i$, $A_d \neq 0$, with $A_k = \bigoplus_{i=0}^{k} A_{(i,k-1)}$, $A_{(d_1,d_2)} \neq 0$ for some d_1, d_2 such that $d_1 + d_2 = d$. We call (d_1, d_2) the socle bidegree of A. Since $A_k^* \simeq A_{d-k}$ and since duality is compatible with direct sum, we get $A^*_{(i,j)} \simeq A_{(d_1-i,d_2-j)}$.

Let $R = \mathbb{K}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be the polynomial ring viewed as a standard bigraded ring in the sets of variables $\{x_1, \ldots, x_n\}$ and $\{u_1, \ldots, u_m\}$ and let $Q = \mathbb{K}[X_1, \ldots, X_n, U_1, \ldots, U_m]$ be the associated ring of differential operators.

We want to stress that the bijection given by Macaulay-Matlis duality preserves bigrading, that is, there is a bijection:

$$\begin{array}{ccc} \{\text{Bihomogeneous ideals of } R\} & \leftrightarrow & \{\text{Bigraded } R-\text{submodules of } Q\} \\ \mathrm{Ann}_Q(M) & \leftarrow & M \\ I & \rightarrow & I^{-1}. \end{array}$$

If $f \in R_{(d_1,d_2)}$ is a bihomogeneous polynomial of total degree $d = d_1 + d_2$, then $I = \mathrm{Ann}_Q(f) \subset Q$ is a bihomogeneous ideal and $A = Q/I$ is a standard bigraded Artinian Gorenstein algebra of socle bidegree (d_1, d_2) and codimension $r = m + n$ if we assume, without lost of generality, that $I_1 = 0$.

Remark 2.7. If $f \in R_{(d_1,d_2)}$ is a bihomogeneous polynomial of bidegree (d_1, d_2), consider the associated bigraded algebra A of socle bidegree (d_1, d_2). Notice that for all $\alpha \in Q_{(i,j)}$ with $i > d_1$ or $j > d_2$ we get $\alpha(f) = 0$; therefore, under these conditions $I_{(i,j)} = Q_{(i,j)}$. As a consequence, we have the following decomposition for all A_k:

$$A_k = \bigoplus_{i+j=k, i \leq d_1, j \leq d_2} A_{(i,j)}.$$

Furthermore, for $i < d_1$ and $j < d_1$, the evaluation map $Q_{(i,j)} \to A_{(d_1-i,d_2-j)}$ given by $\alpha \mapsto \alpha(f)$ provides the following short exact sequence:

$$0 \to I_{(i,j)} \to Q_{(i,j)} \to A_{(d_1-i,d_2-j)} \to 0.$$

One of our goals is to produce bigraded algebras of socle bidegree $(1, d-1)$ presented by quadrics. In order to achieve this objective we study the ideal of a particular family.

Definition 2.8. With the previous notation, all bihomogeneous polynomials of bidegree $(1, d-1)$ can be written in the form

$$f = x_1 g_1 + \ldots + x_n g_n,$$

where $g_i \in \mathbb{K}[u_1, \ldots, u_m]_{d-1}$. We say that f is of *monomial square free type* if all g_i are square free monomials. The associated algebra, $A = Q/\mathrm{Ann}_Q(f)$, is bigraded, has socle bidegree $(1, d-1)$ and we assume that $I_1 = 0$, so $\mathrm{codim}\, A = m + n$.

The combinatoric structure inward bihomogeneous polynomials of monomial square free type allows us to give necessary and sufficient conditions in order for the associated algebra to be presented by quadrics. On the other hand we construct, in sufficiently large codimension, Artinian Gorenstein algebras presented by quadrics failing the WLP.

3. The main results

Let Δ be a homogeneous simplicial complex of dimension $d-2$ whose facets are given by the monomials $g_i \in \mathbb{K}[u_1,\dots,u_m]_{d-1}$ (see Section 2.1). Let $f \in \mathbb{K}[x_1,\dots,x_n,$
$u_1,\dots,u_m]_{(1,d-1)}$ be the bihomogeneous form of monomial square free type associated to Δ, that is, $f = f_\Delta = \sum_{i=1}^{n} x_i g_i$ (see Difinition 2.8). The vertex set of Δ is also called 0-skeleton and we write $V = \{u_1,\dots,u_m\}$. We identify the 1-skeleton with a simple graph $\Delta_1 = (V,E)$, hence the 1-faces are called edges. Since, by differentiation, $X_i(f) = g_i$, we can identify each facet g_i with the differential operator X_i. We denote by e_k the number of $(k-1)$-faces, hence $e_1 = m$ and $e_{d-1} = n$ and we put $e_0 := 1$ and $e_j := 0$ for $j \geq d-1$. Let $A = Q/\operatorname{Ann}_Q(f)$ be the associated algebra, and we suppose that $I_1 = 0$. We identify the faces of Δ with the dual differential operators by $u_i \leftrightarrow U_i$. If $p \in \mathbb{K}[u_1,\dots,u_m]$ is a square free monomial, we denote by $P \in \mathbb{K}[U_1,\dots,U_m]$ the dual differential operator $P = p(U_1,\dots,U_m)$. Notice that $P(p) = 1$.

Definition 3.1. Let Δ be a homogeneous simplicial complex of dimension $d-2$. The associated algebra is $A_\Delta = Q/\operatorname{Ann}(f_\Delta)$.

Theorem 3.2. *Let Δ be a homogeneous simplicial complex of dimension $d-2$ and let A_Δ be the associated algebra. Then*

(1) $A = \bigoplus_{k=0}^{d} A_k$ *where* $A_k = A_{(0,k)} \oplus A_{(1,k-1)}$.
(2) $A_{(0,k)}$ *has a basis identified with the $(k-1)$-faces of Δ, hence* $\dim A_{(0,k)} = e_k$.
(3) *By duality, $A^*_{(1,k-1)} \simeq A_{(0,d-k)}$, and a basis for $A_{(1,k-1)}$ can be chosen by taking, for each $(d-k-1)$-face of Δ, a monomial $X_i\tilde{G}_i$ such that $X_i\tilde{G}_i(f)$ represents it.*
(4) *The Hilbert vector of A is given by $h_k = \dim A_k = e_k + e_{d-k}$.*
(5) *Furthermore, $I = \operatorname{Ann}_Q(f)$ is generated by*
   (a) $(X_1,\dots,X_n)^2$; $U_1^2,\dots,U_m^2$.
   (b) *The monomials in I representing minimal non-faces of Δ;*
   (c) *The monomials X_iF_i where f_i does not represent a subface of g_i;*
   (d) *The binomials $X_i\tilde{G}_i - X_j\tilde{G}_j$ where $g_i = \tilde{g}_i g_{ij}$ and $g_j = \tilde{g}_j g_{ij}$ and g_{ij} represents a common subface of g_i, g_j.*

Proof. It is easy to see that $A_{(0,k)}$ is generated by the monomials of degree k that represent $(k-1)$-faces, since they are the only ones that do not annihilate f. Now we show that they are linearly independent over $\mathbb{K}$. For any $(k-1)$-face ω, let Ω be the associated monomial of $Q_{(0,k)}$, and let $\Omega_1,\dots,\Omega_s$ be all of them.

Since $\Omega(f) = \sum_{i=1}^{n} x_i \Omega(g_i)$, if we take any linear combination

$$0 = \sum_{j=1}^{s} c_j \Omega_j(f) = \sum_{j=1}^{s} c_j \sum_{i=1}^{n} x_i \Omega_j(g_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{s} c_j \Omega_j(g_i).$$

We get $\sum_{j=1}^{s} c_j \Omega_j(g_i) = 0$ for all $i = 1, \ldots, n$. For a fixed i, $\Omega_j(g_i)$ are distinct monomials or zero, but for each j there is an i such that $\Omega_j(g_i) \neq 0$; therefore $c_j = 0$ for all $j = 1, \ldots, s$. The other assertions about A are now clear.

Notice that $I_{(i,j)} = Q_{(i,j)}$ for all $i \geq 2$ and it is generated by $I_{(2.0)} = (X_1, \ldots, X_n)^2$. Now we describe $I_{(0,k)}$ and $I_{(1,k-1)}$. Consider the exact sequence given by evaluation:

$$0 \to I_{(0,k)} \to Q_{(0,k)} \to A_{(1,d-1-k)} \to 0.$$

Since $\dim A_{(1,d-1-k)} = \dim A^*_{(1,d-1-k)} = \dim A_{(0,k)} = e_k$, we get $\dim I_{(0,k)} = \dim Q_{(0,k)} - e_k$. Since $\dim A_{(0,k)} = e_k$ and it has a basis given by the $(k-1)$-faces of Δ and since all the other $\dim Q_{(0,k)} - e_k$ monomials are linearly independent elements of $I_{(0,k)}$, they form a basis for it. Consider the sequence given by evaluation:

$$0 \to I_{(1,k-1)} \to Q_{(1,k-1)} \to A_{(0,d-k)} \to 0.$$

We have $\dim I_{(1,k-1)} = \dim Q_{(1,k-1)} - e_{d-k}$. Let us write $Q_{(1,k-1)} = \bar{I}_{(1,k-1)} \oplus \tilde{Q}_{(1,k-1)}$ where $\bar{I}_{(1,k-1)}$ is the $\mathbb{K}$-vector space spanned by the monomials $X_i F_i$ where F_i does not represent a subface of G_i. Of course $\bar{I}_{(1,k-1)} \subset I_{(1,k-1)}$ and $\tilde{Q}_{(1,k-1)}$ is spanned by all the monomials $X_i \tilde{G}_i$ where $\tilde{G}_i$ is a subface of G_i. The exact sequence given by evaluation restricted to $\tilde{Q}_{(1,k-1)}$ becomes

$$0 \to \tilde{I}_{(1,k-1)} \to \tilde{Q}_{(1,k-1)} \to A_{(0,d-k)} \to 0.$$

Hence, $I_{(1,k-1)} = \tilde{I}_{(1,k-1)} \oplus \bar{I}_{(1,k-1)}$, since $X_i \tilde{G}_i(f)$ is a face of Δ, $\tilde{I}_{(1,k-1)}$ is generated by the binomials $X_i \tilde{G}_i - X_j \tilde{G}_j$ such that $X_i \tilde{G}_i(f) = g_{ij} = X_j \tilde{G}_j(f)$ where g_{ij} is a common subface of g_i, g_j, $g_i = \tilde{g}_i g_{ij}$ and $g_j = \tilde{g}_j g_{ij}$. The result follows. ☐

Definition 3.3. Let Δ be a homogeneous simplicial complex of dimension $d-2$. We say that Δ is facet connected if for any pair of facets F, F' of Δ there exists a sequence of facets, $F_0 = F, F_1, \ldots, F_s = F'$ such that $F_i \cap F_{i+1}$ is a $(d-3)$-face. We say that Δ is a flag complex if every collection of pairwise adjacent vertices spans a simplex.

Remark 3.4. The difinition of a flag complex Δ is equivalent to saying that for all complete subgraphs $H = K_l \subset \Delta_1$ for $l \geq 3$, there exists an $(l-1)$-face $F \in \Delta_l$ such that H is the first skeleton of F. In particular, if Δ is a flag complex, then Δ_1 does not contain any K_{d-1}.

Theorem 3.5. *Let Δ be a homogeneous simplicial complex of dimension $d-2 \geq 1$ and let A_Δ be the associated algebra. A is presented by quadrics if and only if Δ is a facet connected flag complex.*

Proof. Suppose that Δ is a facet connected flag complex and let $I = \mathrm{Ann}_Q(f_\Delta)$. By applying Theorem 3.2 to I, it is enough to consider the monomials in the U_i that does not represent a face of Δ, monomials $X_i F_i$ where F_i is a monomial in the U_j that does not represent a subface of G_i and the binomials $X_i \tilde{G}_i - X_j \tilde{G}_j$ where

$X_i\tilde{G}_i(f) = g_{ij} = X_j\tilde{G}_j(f)$ is a common subface of g_i, g_j. Let $M = U_1^{e_1}\dots U_m^{e_m}$ be a monomial such that $M(f) = 0$, since $U_i^2 \in I$ we can consider M square free and suppose that it does not represent a face of Δ. In this case, the first skeleton of M represents a complete graph K_l with the same vertex set of G, since, by hypothesis, for $3 \leq l \leq d-2$ all $K_l \subset G$ comes from an l-face of Δ and since G does not contain a K_{d-1} as subgraph, there exists U_iU_j in M such that $U_iU_j(f) = 0$ and $M = U_iU_j\tilde{M} \in I_2Q$.

Let $\Omega = X_iM$ with $M = U_1^{e_1}\dots U_m^{e_m}$ being a monomial such that $\Omega(f) = M(g_i) = 0$. We can suppose that M is square free and it does not represent a subface of g_i, hence there is a U_j in M that does not belong to G_i, yielding $\Omega = X_iU_j\tilde{M} \in I_2Q$.

To finish the proof, consider the binomials $X_i\tilde{G}_i - X_j\tilde{G}_j$ where $X_i\tilde{G}_i(f) = g_{ij} = X_j\tilde{G}_j(f)$ and g_{ij} is a common subface of g_i, g_j. If $\tilde{G}_i$ and $\tilde{G}_j$ are subfaces of the facets G_i, G_j respectively and if $g_{ij} \subset G_i \cap G_j$ and the intersection is a $(d-3)$-face, then there are only two vertexes they do not share, say u_i, u_j, $\tilde{G}_i = U_iG_{ij}$ and $\tilde{G}_j = U_jG_{ij}$ and finally $X_i\tilde{G}_i - X_j\tilde{G}_j = (X_iU_i - X_jU_j)G_{ij} \in I_2Q$. In the general case, by the facet connection of Δ, there exists a sequence of facets $G_{i_0} = G_i, G_{i_1}, \dots, G_{i_s} = G_j$ such that the intersection of two consecutive facets is a $(d-3)$-face, hence $X_i\tilde{G}_i - X_{i_1}\tilde{G}_{i_1}, X_{i_1}\tilde{G}_{i_1} - X_{i_2}\tilde{G}_{i_2}, \dots, X_{i_s}\tilde{G}_{i_s} - X_j\tilde{G}_j \in I_2Q$. Summing up we get the desired result.

Conversely, if Δ is not facet connected, let g_j, g_j be two facets that cannot be facet connected and let $g_{ij} = \gcd(g_ig_j)$. By Theorem 3.2 it is easy to see that $X_i\tilde{G}_i - X_j\tilde{G}_j$ is a minimal generator of I where $g_i = \tilde{g}_ig_{ij}$ and $g_j = \tilde{g}_jg_{ij}$. If Δ is not a flag complex, then there is a complete subgraph $K_s \subset G$ that does not come from an s-face of Δ. In this case, if we choose s to be minimal, then by Theorem 3.2 the monomial $M = \displaystyle\prod_{v\in V(K_s)} v$ is a minimal generator of I. $\square$

We introduce the following complexes inspired by the famous Turan's Graph Theorem characterizing maximal graphs not containing a complete subgraph K_{d-1} as the $(d-2)$-partite complete graph $K(a_1, \dots, a_{d-1})$ with $|a_i - a_j| \leq 1$ (cf. [Tu]).

Definition 3.6. Let $2 \leq a_1 \leq \dots \leq a_{d-1}$ be integers. The Turan complex of order $a_1, \dots, a_{d-1}$, $\mathcal{K} = \mathcal{TK}(a_1, \dots, a_{d-1})$, is the homogeneous simplicial complex whose facets set is the cartesian product $\pi = \displaystyle\prod_{i=1}^{d-1}\{1, 2, \dots, a_i\}$. The associated algebra is called the Turan algebra of order $(a_1, \dots, a_{d-1})$ and denoted by $TA(a_1, \dots, a_{d-1})$.

Theorem 3.7. *Every Turan algebra $TA(a_1, \dots, a_{d-1})$ is presented by quadrics. Its Hilbert vector is given by $h_k = s_{k-1} + s_{d-k-1}$ where $s_k = s_k(a_1, \dots, a_{d-1})$ is the elementary symmetric polynomial of order k.*

Proof. By Theorem 3.5, the first claim is equivalent to proving that every Turan complex is a facet connected flag complex. Let $2 \leq a_1 \leq \dots \leq a_{d-1}$ be integers and consider the Turan complex $\mathcal{K} = \mathcal{TK}(a_1, \dots, a_{d-1})$.

To show that $\mathcal{K}$ is facet connected, let us consider F, F' two of its facets. $F = \{x_1, \dots, x_{d-1}\}$ and $F' = \{y_1, \dots, y_{d-1}\}$ with $x_i, y_i \in \{1, \dots, a_i\}$. Consider the following sequence of facets in $\mathcal{K}$: $F_0 = F$, $F_1 = (F \cup y_1) \setminus x_1$. We have that

$F_0 \cap F_1$ is a $(d-3)$ face; and we construct inductively, for $k \in 1, \ldots, d-1$, $F_k = (F_{k-1} \cup y_k) \setminus x_k$. It is easy to see that $F_k \cap F_{k-1}$ is a $(d-3)$-face and that $F_{d-1} = F'$, therefore, $\mathcal{K}$ is facet connected as claimed.

To show that $\mathcal{K}$ is a flag complex. First notice that $\mathcal{K}$ does not contain a complete graph K_{d-1} in its first skeleton, by the $d-1$-coloration. Let us consider any complete subgraph of the first skeleton $H = K_l \subset \mathcal{K}_1$ with $3 \leq l \leq d-2$. We can suppose without loss of generality that the vertex set of H is $V = \{x_1, \ldots, x_l\}$ with $x_i \leq a_i$. By definition of $\mathcal{K}$, there is a a facet of $\mathcal{K}$ whose vertex set contains V. By the definition of simplicial complex, there is a face of $\mathcal{K}$ such that the first skeleton is H and the result follows.

The second claim follows from the fact that the number of $(k-1)$-faces of a Turan complex is $e_k = s_k$ where $s_k = s_k(a_1, \ldots, a_{d-1})$ is the symmetric function of order k. By Theorem 3.5, the Hilbert vector of the Turan algebra $TA(a_1, \ldots, a_{d-1})$ is given by $h_k = s_k + s_{d-k}$. □

We now present a family of counterexamples to Migliore-Nagel conjectures that occur in large codimension with respect to the socle degree.

Corollary 3.8. *Let $A = TA(a_1, \ldots, a_{d-1})$ be the Turan algebra of order $(a_1, \ldots, a_{d-1})$ with $a_1 \approx \ldots \approx a_{d-1}$ large enough. Then* $\operatorname{Hilb}(A)$ *is totally non-unimodal, that is,*

$$\dim A_1 > \dim A_2 > \ldots > \dim A_{\lfloor \frac{d}{2} \rfloor}.$$

Proof. If $a_1 \approx \ldots \approx a_{d-1} \approx a$ are large enough, then, by a trivial Calculus I argument, we get for $2 \leq k+1 \leq \lfloor \frac{d}{2} \rfloor$, $k < d-k$ and $d-k > d-k-1 \geq k+1$:

$$\dim A_k \approx \binom{d-1}{k} a^k + \binom{d-1}{d-k} a^{d-k} > \binom{d-1}{k+1} a^{k+1} + \binom{d-1}{d-k-1} a^{d-k-1} \approx \dim A_{k+1}.$$

In this case, the Hilbert vector $\operatorname{Hilb}(A)$ is totally non-unimodal. □

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Universidade Federal Rural de Pernambuco, av. Don Manoel de Medeiros s/n, Dois Irmos - Recife - PE 52171-900, Brazil
E-mail address: rodrigo.gondim@ufrpe.br

Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria 5, 95125 Catania, Italy
E-mail address: zappalag@dmi.unict.it